

On the Energy and Nullity of Uniform Semigraphs

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Abstract. The energy of a semigraph is defined as the sum of the moduli of the eigenvalues of its adjacency matrix. The study of semigraph energy provides a useful spectral framework for analysing stability, connectivity and redundancy in ordered higher-order networks. The nullity of a graph encodes both algebraic and structural information. We extend this concept to semigraphs, an ordered-tuple generalisation of graphs, and derive closed-form formulas for the nullity and energy of (i) path-uniform semigraphs P_{k+1}^n , (ii) cycle-uniform semigraphs C_k^n , (iii) star-type semigraphs $S_{2,n}^3$ and their 3-uniform counterparts S_n^3 , and (iv) rooted 3-uniform semigraph trees T_n^3 . We also compare these formulas with the spectra of the corresponding ordinary path, cycle and star cores, showing how the ordered middle vertices produce additional zero eigenvalues while the end vertices determine a weighted core spectrum. General lower and upper bounds are established in terms of pure middle and isolated vertices; in particular, every non-trivial semigraph has nullity at most $n-2$. The results provide spectral tools for studying controllability, energy and structural redundancy in ordered multi-vertex network models.

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1 Introduction

Graph theory studies the structure of pairwise relations among objects. Spectral graph theory, which analyses eigenvalues and eigenvectors of matrices associated with graphs, has become a central tool in mathematics, computer science, chemistry and network science. The spectra of matrices such as the adjacency, Laplacian, signless Laplacian and normalized Laplacian matrices are used to describe structural features of graphs [2, 4]. One important spectral parameter is the nullity [1], namely the algebraic multiplicity of the eigenvalue 0 in the adjacency spectrum. In chemical graph theory, and in particular in the Hückel molecular orbital description of alternant hydrocarbons, positive nullity is associated with the presence of non-bonding orbitals and may indicate radical character or reduced stability in that specific model [5, 8]. This chemical interpretation is model-dependent; throughout this paper, nullity is used as an algebraic indicator of spectral redundancy. In engineering and network science, nullity is also connected with controllability questions, because zero modes can indicate redundant or difficult-to-control directions in a linearized network model [10].

Semigraphs were introduced by Sampathkumar [11] as an ordered-tuple generalisation of ordinary graphs. Unlike hypergraphs [3], whose edges are unordered sets, semigraph edges are ordered tuples. Hence an edge distinguishes its end vertices from its middle vertices. This distinction is important in systems where the position of a vertex inside a multi-vertex interaction changes its role. For example, an ordered chain of processors in a communication route can be modelled by a path-uniform semigraph in which the end vertices transmit between modules while the middle vertices act as relay or processing stages. A cycle-uniform semigraph models a feedback loop with ordered internal stages, such as cyclic routing, cyclic supply chains or ordered chemical rings. Star-type semigraphs model hub-and-spoke systems in which a central vertex interacts with several ordered branches, and rooted 3-uniform semigraph trees model a common source connected to several ordered two-step branches.

These examples explain why the families considered in this paper are natural first test cases. They are the semigraph analogues of the ordinary path, cycle, star and rooted star-like tree, which are standard

building blocks in graph theory. The ordered middle vertices do not merely increase the number of vertices. They create zero rows in the semigraph adjacency matrix and hence contribute forced zero modes, while the end vertices determine a smaller weighted core matrix. Therefore the comparison with ordinary graphs has two components: a scaling of the nonzero core spectrum and an increase in nullity caused by the ordered middle vertices.

The main contributions of this paper are as follows. We use the adjacency-matrix framework for semigraphs to extend the notion of nullity from ordinary graphs to semigraphs, retaining the classical definition when all edges have length two. We derive closed-form expressions for the nullity and energy of the path-uniform semigraphs P_{k+1}^n , cycle-uniform semigraphs C_k^n , star-type semigraphs $S_{2,n}^3$, 3-uniform star semigraphs S_n^3 , and rooted 3-uniform semigraph trees T_n^3 . We also compare these formulas with the spectra of the corresponding ordinary graph cores, which helps to identify the contribution of the ordered middle vertices to the nullity and energy. Related non-uniform path and cycle semigraphs have been studied in [9]. In the present paper, we focus on uniform families in order to obtain simple closed forms, compare them directly with ordinary graph cores, and extend the analysis to star-type and rooted-tree semigraphs. Finally, we prove general bounds for semigraph nullity and indicate how large nullity may be interpreted in terms of redundancy, possible non-bonding modes, and controllability.

2 Preliminaries

We recall the notation and standard terminology needed in the sequel. Unless otherwise stated, the notation follows Sampathkumar's semigraph framework and the adjacency convention used in semigraph spectral studies.

Definition 2.1. *Semigraph*[11] *A semigraph is an ordered pair $G = (V, E)$, where V is a non-empty vertex set and E is a collection of ordered tuples of distinct vertices, each tuple having length at least two. The following conditions are imposed:*

1. *two different edges have at most one common vertex;*

2. the tuples $(u_1, u_2, \dots, u_k)$ and $(w_1, w_2, \dots, w_r)$ represent the same edge exactly when $r = k$ and the two orders agree either directly or in reverse, that is, $u_j = w_j$ for all j , or $u_j = w_{r+1-j}$ for all j .

For an edge $e = (w_1, w_2, \dots, w_s)$, the vertices w_1 and w_s are called end vertices and the remaining vertices, if any, are called middle vertices.

Definition 2.2. Vertex classes and adjacency[6] Two vertices of a semigraph are adjacent if they occur together in some edge. A vertex is called a pure end vertex if it appears only as an end vertex of every edge containing it, and a pure middle vertex if it appears only in middle positions. A vertex which is a middle vertex in one edge and an end vertex in another edge is called a middle-end vertex.

Definition 2.3. Adjacency matrix[11] Let $G = (V, E)$ be a semigraph with $V = \{1, 2, \dots, k\}$. Its adjacency matrix is the $k \times k$ matrix $A(G) = [a_{ij}]$ obtained as follows. For an edge $e = (i_1, i_2, \dots, i_s)$ and for $r = 1, 2, \dots, s$, put

$$a_{i_1 i_r} = r - 1, \quad a_{i_s i_r} = s - r.$$

All entries not determined by these rules are zero.

Definition 2.4. Star semigraph of type I[12, 14] The semigraph $S_{2,n}^3$ consists of one 3-edge with centre vertex v_1 , together with n additional 2-edges incident with the same centre vertex.

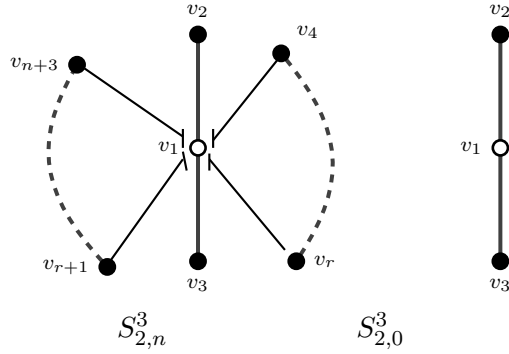


Fig. 1. Type-I star semigraphs considered in this work [12, 14].

Definition 2.5. *3-Uniform star semigraph of type II* [12] The notation S_n^3 denotes the star semigraph in which all edges have length 3 and the common vertex v_0 occurs as the middle vertex of each edge.

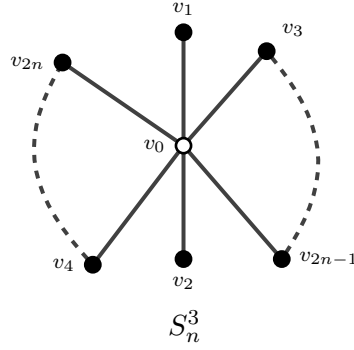


Fig. 2. A 3-uniform star semigraph of type II [12].

Definition 2.6. *Rooted 3-uniform semigraph tree* [14, 13] The rooted 3-uniform semigraph tree T_n^3 has vertex set $\{v_1, v_2, \dots, v_{2n+1}\}$ and edge set

$$\{(v_1, v_{2i}, v_{2i+1}) : 1 \leq i \leq n\}.$$

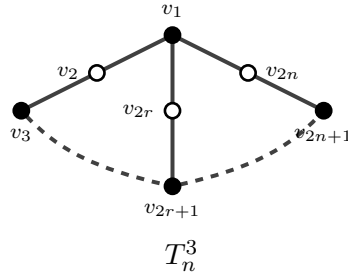


Fig. 3. Rooted 3-uniform semigraph tree [13, 14].

Definition 2.7. *Spectrum of the adjacency matrix* [11] If the distinct eigenvalues of $A(G)$ are $\lambda_1, \lambda_2, \dots, \lambda_t$ with algebraic multiplicities $m_1, m_2, \dots, m_t$, then the adjacency spectrum of G is written as

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_t \\ m_1 & m_2 & \cdots & m_t \end{pmatrix}.$$

Definition 2.8. Nullity of a graph[1] *For an ordinary graph G , the nullity $\eta(G)$ is the algebraic multiplicity of the eigenvalue 0 in the adjacency spectrum of G .*

Definition 2.9. Nullity of a semigraph[9] *For a semigraph G , we define $\eta(G)$ to be the algebraic multiplicity of the eigenvalue 0 in the spectrum of its adjacency matrix. This agrees with the usual graph nullity when every edge has length two.*

Definition 2.10. Energy of a semigraph[7] *Let $A(G)$ be the adjacency matrix of a semigraph G on n vertices, and let $\lambda_1, \lambda_2, \dots, \lambda_n$ be its eigenvalues, counted with algebraic multiplicity. The energy of G is*

$$E(G) = \sum_{i=1}^n |\lambda_i|.$$

These definitions fix the notation used below for the explicit computations of nullity and energy.

3 Nullity and Energy of Semigraphs

We first introduce the path-uniform and cycle-uniform semigraphs and then compute the nullity and energy of these families, of star semigraphs of type I , of 3-uniform star semigraphs of type II , and of rooted 3-uniform semigraph trees.

3.1 Nullity and energy of path-uniform semigraphs P_{k+1}^n

A path-uniform semigraph P_{k+1}^n has $kn+1$ vertices and k ordered edges, each of length $n+1$. Consecutive edges have exactly one common end vertex. In this notation the parameter n is the end-to-end weight induced by an ordered edge of length $n+1$.

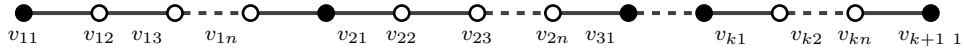


Fig. 4. Path-uniform semigraph P_{k+1}^n .

From definition 2.3 adjacency matrix of path-uniform semigraph P_{k+1}^n is given by $A_{P_{k+1}^n}$.

$$\begin{array}{c}
 v_{11} \\ v_{12} \\ v_{13} \\ \vdots \\ v_{1n} \\ v_{21} \\ v_{22} \\ \vdots \\ v_{2n} \\ v_{31} \\ v_{32} \\ \vdots \\ v_{3n} \\ v_{41} \\ \vdots \\ v_{k1} \\ v_{k2} \\ v_{k3} \\ \vdots \\ v_{kn} \\ v_{(k+1)1}
 \end{array}
 \begin{pmatrix}
 v_{11} & v_{12} & \cdots & v_{1n} & v_{21} & v_{22} & \cdots & v_{2n} & v_{31} & \cdots & v_{3n} & \cdots & v_{k1} & v_{k2} & \cdots & v_{kn} & v_{(k+1)1} \\
 0 & 1 & \cdots & n-1 & n & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
 n & n-1 & \cdots & 1 & 0 & 1 & \cdots & n-1 & n & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
 0 & 0 & \cdots & 0 & n & n-1 & \cdots & 1 & 0 & \cdots & n-1 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & n & n-1 & \cdots & 1 & 0
 \end{pmatrix}$$

Theorem 3.1. *The characteristic polynomial of the adjacency matrix of the path-uniform semigraph P_{k+1}^n is*

$$\chi_{P_{k+1}^n}(\lambda) = \lambda^{(n-1)k} \sum_{j=0}^{\lfloor (k+1)/2 \rfloor} (-1)^j \binom{k+1-j}{j} n^{2j} \lambda^{k+1-2j}.$$

Proof. Consider the characteristic polynomial $\chi_{P_{k+1}^n}(\lambda) = \det(\lambda I - A_{P_{k+1}^n})$. Since the adjacency matrix has $(n-1)k$ zero rows, the characteristic polynomial takes the form

$$\begin{aligned}
 \chi_{P_{k+1}^n}(\lambda) &= \det(\lambda I - A_{P_{k+1}^n}) \\
 &= \lambda^{(n-1)k} \left[\lambda^{k+1} + c_0 \lambda^k + c_1 \lambda^{k-1} + c_2 \lambda^{k-2} + \cdots + c_{k-1} \lambda + c_k \right].
 \end{aligned}$$

Delete these zero rows and corresponding columns. Suppose the resul-

tant matrix is

$$A_{P_{k+1}} = \begin{bmatrix} 0 & n & 0 & 0 & \cdots & 0 & 0 \\ n & 0 & n & 0 & \cdots & 0 & 0 \\ 0 & n & 0 & n & \cdots & 0 & 0 \\ 0 & 0 & n & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & n \\ 0 & 0 & 0 & 0 & \cdots & n & 0 \end{bmatrix}_{(k+1) \times (k+1)}$$

The characteristic equation of $A_{P_{k+1}}$ is

$$\begin{aligned} \chi_{P_{k+1}}(\lambda) &= \det(\lambda I - A_{P_{k+1}}) \\ &= \begin{vmatrix} \lambda & -n & 0 & 0 & \cdots & 0 & 0 \\ -n & \lambda & -n & 0 & \cdots & 0 & 0 \\ 0 & -n & \lambda & -n & \cdots & 0 & 0 \\ 0 & 0 & -n & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \lambda & -n \\ 0 & 0 & 0 & 0 & \cdots & -n & \lambda \end{vmatrix}_{(k+1) \times (k+1)} \end{aligned}$$

Using cofactor expansion along the first column,

$$\begin{aligned} \chi_{P_{k+1}}(\lambda) &= \lambda \begin{vmatrix} \lambda & -n & 0 & 0 & \cdots & 0 & 0 \\ -n & \lambda & -n & 0 & \cdots & 0 & 0 \\ 0 & -n & \lambda & -n & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \lambda & -n \\ 0 & 0 & 0 & 0 & \cdots & -n & \lambda \end{vmatrix}_{k \times k} \\ &\quad + n \begin{vmatrix} -n & 0 & 0 & 0 & \cdots & 0 & 0 \\ -n & \lambda & -n & 0 & \cdots & 0 & 0 \\ 0 & -n & \lambda & -n & \cdots & 0 & 0 \\ 0 & 0 & 0 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -n & \lambda \end{vmatrix}_{k \times k} . \end{aligned}$$

Observe that the second term is the characteristic polynomial of $-n\chi_{P_{k-1}}(\lambda)$. Therefore,

$$\chi_{P_{k+1}}(\lambda) = \lambda\chi_{P_k}(\lambda) - n^2\chi_{P_{k-1}}(\lambda)$$

Claim: Characteristic polynomial of P_{k+1} is

$$\chi_{P_{k+1}}(\lambda) = \sum_{j=0}^{\lfloor (k+1)/2 \rfloor} (-1)^j \binom{k+1-j}{j} n^{2j} \lambda^{k+1-2j}.$$

We proceed by method of induction.

For the base case $k = 0, 1$ the characteristic polynomials are

$$\chi_{P_1}(\lambda) = |\lambda| = \lambda, \quad \chi_{P_2}(\lambda) = \begin{vmatrix} \lambda & -n \\ -n & \lambda \end{vmatrix} = \lambda^2 - n^2$$

Both match the claimed equation.

So the formula is true for $k = 0, 1$.

Assume the result holds for every $r \leq k$. Next, we show it holds for $r = k + 1$. We have,

$$\begin{aligned} \chi_{P_{k+1}}(\lambda) &= \lambda\chi_{P_k}(\lambda) - n^2\chi_{P_{k-1}}(\lambda) \\ &= \lambda \sum_{j=0}^{\lfloor k/2 \rfloor} (-1)^j \binom{k-j}{j} n^{2j} \lambda^{k-2j} \\ &\quad - n^2 \sum_{j=0}^{\lfloor (k-1)/2 \rfloor} (-1)^j \binom{k-1-j}{j} n^{2j} \lambda^{k-1-2j} \\ &= \sum_{j=0}^{\lfloor k/2 \rfloor} (-1)^j \binom{k-j}{j} n^{2j} \lambda^{k+1-2j} \\ &\quad - \sum_{j=0}^{\lfloor (k-1)/2 \rfloor} (-1)^j \binom{k-1-j}{j} n^{2(j+1)} \lambda^{k+1-2(j+1)}. \end{aligned}$$

In the second sum, set $i = j + 1$. Then $j = 0, \dots, \lfloor (k-1)/2 \rfloor$ becomes $i = 1, \dots, \lfloor (k+1)/2 \rfloor$, and its general term is $(-1)^{i-1} \binom{k-i}{i-1} n^{2i} \lambda^{k+1-2i}$.

Hence

$$\begin{aligned}
\chi_{P_{k+1}}(\lambda) &= \sum_{i=0}^{\lfloor (k+1)/2 \rfloor} \left[(-1)^i \binom{k-i}{i} - (-1)^{i-1} \binom{k-i}{i-1} \right] n^{2i} \lambda^{k+1-2i} \\
&= \sum_{i=0}^{\lfloor (k+1)/2 \rfloor} (-1)^i \left[\binom{k-i}{i} + \binom{k-i}{i-1} \right] n^{2i} \lambda^{k+1-2i} \\
&= \sum_{i=0}^{\lfloor (k+1)/2 \rfloor} (-1)^i \binom{k+1-i}{i} n^{2i} \lambda^{k+1-2i},
\end{aligned}$$

by using Pascal's identity. Hence our claim holds.

Therefore, characteristic polynomial of P_{k+1}^n is

$$\begin{aligned}
\chi_{P_{k+1}^n}(\lambda) &= \det(\lambda I - A_{P_{k+1}^n}) \\
&= \lambda^{(n-1)k} \sum_{j=0}^{\lfloor (k+1)/2 \rfloor} (-1)^j \binom{k+1-j}{j} n^{2j} \lambda^{k+1-2j}.
\end{aligned}$$

□

Remark 3.2. [1] The spectrum of the usual path P_n is given by

$$\text{Spectrum}(P_n) = \left\{ 2 \cos \left(\frac{\pi r}{n+1} \right) : 1 \leq r \leq n \right\}$$

Proposition 3.3. *The spectrum of the path-uniform semigraph P_{k+1}^n is composed of the following eigenvalues with their corresponding multiplicities:*

1. *The eigenvalue 0 with multiplicity*

$$\eta(P_{k+1}^n) = \begin{cases} (n-1)k & \text{if } k+1 \text{ is even,} \\ (n-1)k+1 & \text{if } k+1 \text{ is odd.} \end{cases}$$

2. *The remaining eigenvalues are given by the set*

$$\left\{ 2n \cos \left(\frac{j\pi}{k+2} \right) \mid j = 1, 2, \dots, k+1 \right\}$$

each having multiplicity 1.

Also, the energy of the path-uniform semigraph P_{k+1}^n is $2n \sum_{j=1}^{k+1} \left| \cos \left(\frac{j\pi}{k+2} \right) \right|$.

Proof. From the proof of the previous theorem, the characteristic polynomial of P_{k+1}^n is given by

$$\chi_{P_{k+1}^n}(\lambda) = \lambda^{(n-1)k} \chi_{P_{k+1}}(\lambda)$$

$\lambda = 0$ is an eigenvalue with a multiplicity of at least $(n-1)k$. The remaining $k+1$ eigenvalues are the roots of $\chi_{P_{k+1}}(\lambda)$.

The adjacency matrix $A_{P_{k+1}}$ corresponding to nonzero rows is given by:

$$\begin{aligned} A_{P_{k+1}} &= \begin{pmatrix} 0 & n & 0 & \cdots & 0 \\ n & 0 & n & \cdots & 0 \\ 0 & n & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & n \\ 0 & 0 & \cdots & n & 0 \end{pmatrix}_{(k+1) \times (k+1)} \\ &= n \cdot \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \cdots & 0 \\ 0 & 1 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}_{(k+1) \times (k+1)} \\ &= n \cdot A(P_{k+1}), \end{aligned}$$

where $A(P_{k+1})$ is the standard adjacency matrix of the path graph P_{k+1} . Eigenvalues of $A(P_{k+1})$ are $\left\{ 2 \cos \left(\frac{j\pi}{k+2} \right) \mid j = 1, \dots, k+1 \right\}$. Therefore, the eigenvalues of $A_{P_{k+1}}$ are:

$$\text{Spectrum}(A_{P_{k+1}}) = \left\{ 2n \cos \left(\frac{j\pi}{k+2} \right) \mid j = 1, 2, \dots, k+1 \right\}.$$

To find the total nullity $\eta(P_{k+1}^n)$, we must add the number of zero eigenvalues from this set to the initial $(n-1)k$ zeros. Consider the following two cases based on the nature of $k+1$:

Case 1: $k+1$ is even. So k is odd implies $k+2$ is also odd. The value $\frac{k+2}{2}$ is not an integer. Since j must be an integer, there is no value of j in the range $\{1, \dots, k+1\}$ that satisfies $j = \frac{k+2}{2}$. Therefore,

none of the eigenvalues of $A(P_{k+1})$ are zero. Thus the total nullity is $\eta(P_{k+1}^n) = (n-1)k$.

Case 2: $k+1$ is odd. So k is even implies $k+2$ is also even. The value $j = \frac{k+2}{2}$ is an integer. So the eigenvalue $2n \cos\left(\frac{j\pi}{k+2}\right) = 2n \cos(\pi/2) = 0$. Exactly one eigenvalue of $A(P_{k+1})$ is zero. Thus the total nullity is $\eta(P_{k+1}^n) = (n-1)k + 1$.

Since zero eigenvalues do not contribute to the energy, we have

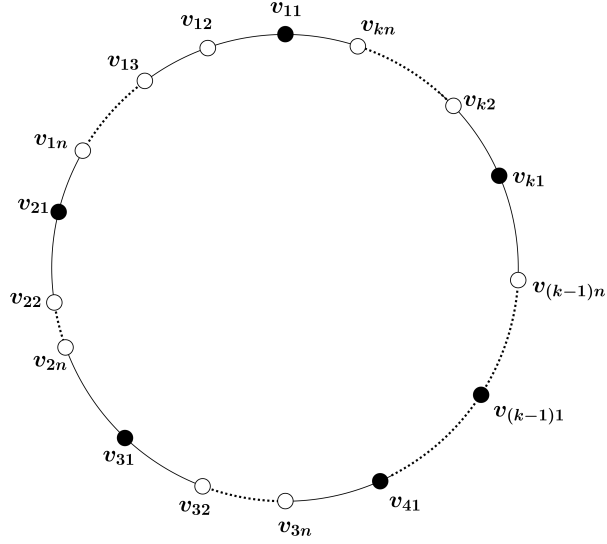
$$E(P_{k+1}^n) = 2n \sum_{j=1}^{k+1} \left| \cos \frac{j\pi}{k+2} \right|.$$

□

Remark 3.4 (Comparison with the ordinary path). The nonzero part of the spectrum of P_{k+1}^n is exactly the spectrum of the ordinary path P_{k+1} multiplied by n . Consequently, $E(P_{k+1}^n) = nE(P_{k+1})$, where $E(P_{k+1}) = 2 \sum_{j=1}^{k+1} \left| \cos\left(\frac{j\pi}{k+2}\right) \right|$ is the ordinary graph energy of P_{k+1} . Thus the semigraph operation has two spectral effects: it scales the nonzero path spectrum by the effective end-to-end weight n , and it adds $(n-1)k$ forced zero eigenvalues coming from the pure middle vertices. In a network interpretation, this means that adding ordered relay vertices increases redundancy without changing the oscillatory shape of the path spectrum; in a chemical interpretation, the added zero modes can be viewed as non-bonding modes localized on the ordered middle positions.

3.2 Nullity and energy of cycle-uniform semigraphs C_k^n

The cycle-uniform semigraph C_k^n has kn vertices and k ordered edges, each of length $n+1$, arranged cyclically so that consecutive edges share one end vertex.


 Fig. 5. Cycle Uniform Semigraph C_k^n

From Definition 2.3, the adjacency matrix of the cycle-uniform semigraph C_k^n is given by $A_{C_k^n}$

$$A_{C_k^n} = \begin{pmatrix}
 v_{11} & v_{12} & \cdots & v_{1n} & v_{21} & v_{22} & \cdots & v_{2n} & v_{31} & \cdots & v_{3n} & \cdots & v_{k1} & v_{k2} & \cdots & v_{k(n-1)} & v_{kn} \\
 v_{11} & 0 & 1 & \cdots & n-1 & n & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & n & n-1 & \cdots & 2 & 1 \\
 v_{12} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
 v_{13} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 v_{1n} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
 v_{21} & n & n-1 & \cdots & 1 & 0 & 1 & \cdots & n-1 & n & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
 v_{22} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 v_{2n} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
 v_{31} & 0 & 0 & \cdots & 0 & n & n-1 & \cdots & 1 & 0 & \cdots & n-1 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
 v_{32} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 v_{3n} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
 v_{41} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 v_{k1} & n & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 1 & \cdots & n-2 & n-1 \\
 v_{k2} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
 v_{k3} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 v_{kn} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0
 \end{pmatrix}$$

Theorem 3.5. *The spectrum of Cycle Uniform Semigraph C_k^n associated with adjacency matrix are*

$$\begin{pmatrix} 0 & 2n \cos\left(\frac{2\pi j}{k}\right) \\ (n-1)k & 1 \end{pmatrix}$$

where $j = 0, 1, 2, \dots, k-1$ and $k \neq 4m$, where m is a positive integer.

$$\begin{pmatrix} 0 & 2n \cos\left(\frac{2\pi j}{k}\right) \\ (n-1)k+2 & 1 \end{pmatrix}$$

where $j = 0, 1, 2, \dots, k-1$, $j \neq \frac{k}{4}, \frac{3k}{4}$, and $k = 4m$, where m is a positive integer.

Proof. Let the characteristic polynomial be $P_{C_k^n}(\lambda) = \det(\lambda I - A_{C_k^n})$. The adjacency matrix has $(n-1)k$ zero rows. Thus the characteristic equation takes the form

$$\begin{aligned} P_{C_k^n}(\lambda) &= \det(\lambda I - A_{C_k^n}) \\ &= \lambda^{(n-1)k} \left[\lambda^k + c_1 \lambda^{k-1} + c_2 \lambda^{k-2} + c_3 \lambda^{k-3} + \dots + c_{k-1} \lambda + c_k \right]. \end{aligned}$$

Deleting the zero rows and corresponding columns in the matrix $A_{C_k^n}$, the resultant matrix is

$$nA_{C_k} = \begin{bmatrix} 0 & n & 0 & 0 & \dots & 0 & n \\ n & 0 & n & 0 & \dots & 0 & 0 \\ 0 & n & 0 & n & \dots & 0 & 0 \\ 0 & 0 & n & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & n \\ n & 0 & 0 & 0 & \dots & n & 0 \end{bmatrix}_{k \times k}.$$

where

$$A_{C_k} = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix}_{k \times k}.$$

Proceeding as in [2], for a positive integer $k \geq 2$, the eigenvalues of Q_k (Q_k is a full cycle permutation matrix of order k , the $(i, i+1)$ -element of Q_k is 1, $i = 1, 2, 3, \dots, k-1$, the $(k, 1)$ -element of Q_k is 1, and the remaining elements of Q_k are zero) are $1, \omega, \omega^2, \omega^3, \dots, \omega^{k-1}$, where

$$\omega = \exp\left(\frac{2\pi i}{k}\right)$$

is the primitive k th root of unity.

So $A_{C_k} = Q_k + Q'_k$, where Q'_k denotes the transpose of Q_k . Thus $A_{C_k} = Q_k + Q_k^{k-1}$ and the eigenvalues of A_{C_k} are $\omega^j + \omega^{k-j}$, where $j = 0, 1, 2, \dots, k-1$.

$$\begin{aligned} \omega^j + \omega^{k-j} &= \omega^j + \omega^{-j} \\ &= \exp\left(\frac{2\pi i j}{k}\right) + \exp\left(\frac{-2\pi i j}{k}\right) \\ &= 2 \cos\left(\frac{2\pi j}{k}\right). \end{aligned}$$

Therefore the eigenvalues of nA_{C_k} are

$$2n \cos\left(\frac{2\pi j}{k}\right), \quad j = 0, 1, 2, \dots, k-1.$$

Now,

$$2n \cos\left(\frac{2\pi j}{k}\right) = 0$$

if and only if

$$j = \frac{k}{4} \quad \text{or} \quad j = \frac{3k}{4},$$

which occurs precisely when $k = 4m$, where m is a positive integer. Hence, if $k \neq 4m$, the only zero eigenvalues are the $(n-1)k$ zero eigenvalues arising from the zero rows. If $k = 4m$, two additional zero eigenvalues arise from the reduced cycle matrix. Therefore the spectrum of C_k^n is

$$\begin{pmatrix} 0 & 2n \cos\left(\frac{2\pi j}{k}\right) \\ (n-1)k & 1 \end{pmatrix}$$

where $j = 0, 1, 2, \dots, k-1$ and $k \neq 4m$, where m is a positive integer;

$$\begin{pmatrix} 0 & 2n \cos\left(\frac{2\pi j}{k}\right) \\ (n-1)k+2 & 1 \end{pmatrix}$$

where $j = 0, 1, 2, \dots, k-1$, $j \neq \frac{k}{4}, \frac{3k}{4}$, and $k = 4m$, where m is a positive integer. $\square$

Corollary 3.6. *The nullity and energy of the adjacency matrix of the cycle uniform semigraph are*

$$\eta(C_k^n) = \begin{cases} (n-1)k, & \text{if } k \not\equiv 0 \pmod{4}, \\ (n-1)k+2, & \text{if } k \equiv 0 \pmod{4}, \end{cases}$$

and

$$E(C_k^n) = 2n \sum_{j=0}^{k-1} \left| \cos\left(\frac{2\pi j}{k}\right) \right|,$$

respectively.

Proof. From Theorem 3.5, we get

$$\eta(C_k^n) = \begin{cases} (n-1)k, & \text{if } k \not\equiv 0 \pmod{4}, \\ (n-1)k+2, & \text{if } k \equiv 0 \pmod{4}. \end{cases}$$

The energy is defined as

$$E(G) = \sum_i |\lambda_i|,$$

where λ_i are the eigenvalues of the adjacency matrix associated with the cycle uniform semigraph. Since zero eigenvalues do not contribute to the energy, from Theorem 3.5 we get

$$\begin{aligned} E(C_k^n) &= \sum_{\lambda \in \text{Spec}(C_k^n)} |\lambda| \\ &= \sum_{j=0}^{k-1} \left| 2n \cos\left(\frac{2\pi j}{k}\right) \right| \\ &= 2n \sum_{j=0}^{k-1} \left| \cos\left(\frac{2\pi j}{k}\right) \right|. \end{aligned}$$

Hence the proof. $\square$

3.3 Nullity and energy of star semigraphs of type I $S_{2,n}^3$, 3-uniform star semigraphs of type II S_n^3 and rooted 3-uniform semigraph trees T_n^3

In this section, we find the nullity and energy of star semigraphs of type I $S_{2,n}^3$, 3-uniform star semigraphs of type II S_n^3 and rooted 3-uniform semigraph trees T_n^3 with respect to the adjacency matrix in Definition 2.3. These three semigraph families are taken from the work of Shinde and coauthors [12, 13].

3.3.1 Nullity and energy of star semigraphs of type I

The adjacency matrix of the star semigraph of type I $S_{2,n}^3$ is as follows from definition 2.3

$$\begin{array}{c} v_1 \quad v_2 \quad v_3 \quad v_4 \quad v_5 \quad \cdots \quad v_{n+3} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ \vdots \\ v_{n+3} \end{array} \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 2 & 0 & 0 & \cdots & 0 \\ 1 & 2 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \end{array} \quad_{(n+3) \times (n+3)}$$

Theorem 3.7. *For $n \geq 1$, the characteristic polynomial of the adjacency matrix of the Star semigraph type I $S_{2,n}^3$ is*

$$P_{S_{2,n}^3}(\lambda) = \lambda^{n-1}(\lambda^2 - 4)(\lambda^2 - n).$$

For the degenerate case $n = 0$, we have $P_{S_{2,0}^3}(\lambda) = \lambda(\lambda^2 - 4)$.

Proof. Let the characteristic polynomial be

$$P_{S_{2,n}^3}(\lambda) = \det(\lambda I - A) = \begin{vmatrix} \lambda & 0 & 0 & -1 & \cdots & -1 & -1 & -1 \\ -1 & \lambda & -2 & 0 & \cdots & 0 & 0 & 0 \\ -1 & -2 & \lambda & 0 & \cdots & 0 & 0 & 0 \\ -1 & 0 & 0 & \lambda & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ -1 & 0 & 0 & 0 & \cdots & 0 & \lambda & 0 \\ -1 & 0 & 0 & 0 & \cdots & 0 & 0 & \lambda \end{vmatrix}_{(n+3) \times (n+3)}.$$

Using cofactor expansion along the first row, the contribution of the first entry is

$$\lambda \begin{vmatrix} \lambda & -2 & 0 & 0 & \cdots & 0 & 0 \\ -2 & \lambda & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \lambda & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \lambda & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \lambda \end{vmatrix} = \lambda^{n+1}(\lambda^2 - 4).$$

The remaining n nonzero entries in the first row correspond to the pendant vertices $v_4, v_5, \dots, v_{n+3}$. These cofactor terms are not zero. By symmetry, all these n cofactor contributions are equal. For any one such pendant column, expanding the corresponding minor along the row of the same pendant vertex gives the contribution

$$-\lambda^{n-1}(\lambda^2 - 4).$$

Hence the total contribution of the n pendant columns is

$$-n\lambda^{n-1}(\lambda^2 - 4).$$

Therefore, for $n \geq 1$,

$$\begin{aligned} P_{S_{2,n}^3}(\lambda) &= \lambda^{n+1}(\lambda^2 - 4) - n\lambda^{n-1}(\lambda^2 - 4) \\ &= \lambda^{n-1}(\lambda^2 - 4)(\lambda^2 - n). \end{aligned}$$

When $n = 0$, there are no pendant 2-edges. The adjacency matrix reduces to the matrix corresponding to the single 3-edge, and hence

$$P_{S_{2,0}^3}(\lambda) = \lambda(\lambda^2 - 4).$$

□

Corollary 3.8. *For $n \geq 1$, the nullity and energy of the adjacency matrix of Star Semigraph-type I are $\eta(S_{2,n}^3) = n - 1$ and $E(S_{2,n}^3) = 4 + 2\sqrt{n}$, respectively. For $n = 0$, we have $\eta(S_{2,0}^3) = 1$ and $E(S_{2,0}^3) = 4$.*

Proof. From Theorem 3.7, for $n \geq 1$,

$$P_{S_{2,n}^3}(\lambda) = \lambda^{n-1}(\lambda^2 - 4)(\lambda^2 - n).$$

Thus, the eigenvalues are 0 with multiplicity $n - 1$, together with $-2, 2, -\sqrt{n}, \sqrt{n}$, counted with algebraic multiplicity. Therefore, $\eta(S_{2,n}^3) = n - 1$. Also,

$$E(S_{2,n}^3) = |-2| + |2| + |-\sqrt{n}| + |\sqrt{n}| = 4 + 2\sqrt{n}.$$

For $n = 0$, the characteristic polynomial is $P_{S_{2,0}^3}(\lambda) = \lambda(\lambda^2 - 4)$, so the eigenvalues are 0, $-2, 2$. Hence

$$\eta(S_{2,0}^3) = 1 \quad \text{and} \quad E(S_{2,0}^3) = 4.$$

□

3.3.2 Nullity and energy of 3-uniform star semigraphs of type II

The adjacency matrix of the 3-uniform star semigraph of type II, S_n^3 , is as follows from Definition 2.3:

$$A = \begin{matrix} & \begin{matrix} v_0 & v_1 & v_2 & v_3 & v_4 & \cdots & v_{2n-3} & v_{2n-2} & v_{2n-1} & v_{2n} \end{matrix} \\ \begin{matrix} v_0 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ \vdots \\ v_{2n-3} \\ v_{2n-2} \\ v_{2n-1} \\ v_{2n} \end{matrix} & \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 2 & \cdots & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & \cdots & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 2 & 0 \end{pmatrix} \end{matrix} \quad (2n+1) \times (2n+1)$$

Theorem 3.9. *The characteristic polynomial of the adjacency matrix of S_n^3 is $P_{S_n^3}(\lambda) = \lambda(\lambda^2 - 4)^n$.*

Proof. Characteristic polynomial of S_n^3 is

$$P_{S_n^3}(\lambda) = \det(\lambda I - A)$$

$$P_{S_n^3}(\lambda) = \begin{vmatrix} \lambda & 0 & 0 & 0 & \cdots & 0 & 0 \\ -1 & \lambda & -2 & 0 & \cdots & 0 & 0 \\ -1 & -2 & \lambda & 0 & \cdots & 0 & 0 \\ -1 & 0 & 0 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & 0 & 0 & 0 & \cdots & \lambda & -2 \\ -1 & 0 & 0 & 0 & \cdots & -2 & \lambda \end{vmatrix} \quad (2n+1) \times (2n+1)$$

Using cofactor expansion along the first row of $\lambda I - A$, we obtain

$$P_{S_n^3}(\lambda) = \lambda \begin{vmatrix} \lambda & -2 & 0 & 0 & \cdots & 0 & 0 \\ -2 & \lambda & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \lambda & -2 & \cdots & 0 & 0 \\ 0 & 0 & -2 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \lambda & -2 \\ 0 & 0 & 0 & 0 & \cdots & -2 & \lambda \end{vmatrix}_{(2n) \times (2n)}.$$

The remaining matrix is block diagonal with n identical blocks $\begin{pmatrix} \lambda & -2 \\ -2 & \lambda \end{pmatrix}$, each having determinant $\lambda^2 - 4$. Hence

$$P_{S_n^3}(\lambda) = \lambda(\lambda^2 - 4)^n.$$

□

Corollary 3.10. *The nullity and energy of the adjacency matrix of the star semigraph of type II, S_n^3 , are*

$$\eta(S_n^3) = 1, \quad E(S_n^3) = 4n.$$

Proof. By Theorem 3.9, the spectrum of S_n^3 is

$$\begin{pmatrix} -2 & 2 & 0 \\ n & n & 1 \end{pmatrix}.$$

Therefore $\eta(S_n^3) = 1$ and $E(S_n^3) = 2n + 2n = 4n$. □

3.3.3 Nullity and energy of rooted 3-uniform semigraph trees

The adjacency matrix of the rooted 3-uniform semigraph tree T_n^3 is as follows from definition 2.3

$$A = \begin{matrix} & v_1 & v_2 & v_3 & v_4 & v_5 & \cdots & v_{2n-2} & v_{2n-1} & v_{2n} & v_{2n+1} \\ \begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ \vdots \\ v_{2n-2} \\ v_{2n-1} \\ v_{2n} \\ v_{2n+1} \end{matrix} & \begin{pmatrix} 0 & 1 & 2 & 1 & 2 & \cdots & 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \end{pmatrix} \end{pmatrix}_{(2n+1) \times (2n+1)}$$

Theorem 3.11. *The characteristic polynomial of adjacency matrix of the Rooted 3-Uniform Semigraph tree T_n^3 is*

$$P_{T_n^3}(\lambda) = \lambda^{2n-1}(\lambda^2 - 4n).$$

Proof. Characteristic polynomial of T_n^3 is

$$P_{T_n^3}(\lambda) = \det(\lambda I - A).$$

Since n rows of A are all zero, the characteristic polynomial takes the form

$$\lambda^n(\lambda^{n+1} + c_0\lambda^n + c_1\lambda^{n-1} + \cdots + c_n).$$

Deleting the zero rows and the corresponding columns, we get the reduced matrix

$$2M = \begin{bmatrix} 0 & 2 & 2 & 2 & \cdots & 2 & 2 \\ 2 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 2 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 2 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 2 & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}_{(n+1) \times (n+1)}.$$

By [?], the characteristic polynomial of M is

$$\begin{aligned} P_M(\lambda) &= \lambda^{n+1} - n\lambda^{n-1} \\ &= \lambda^{n-1}(\lambda^2 - n). \end{aligned}$$

Thus the roots of $P_M(\lambda)$ are $\lambda = 0$ with multiplicity $n - 1$, together with $\sqrt{n}$ and $-\sqrt{n}$. Therefore, the roots corresponding to $2M$ are $\lambda = 0$ with multiplicity $n - 1$, together with $2\sqrt{n}$ and $-2\sqrt{n}$. Hence the characteristic polynomial of the reduced matrix $2M$ is

$$P_{2M}(\lambda) = \lambda^{n-1}(\lambda^2 - 4n).$$

Combining this with the factor λ^n arising from the n zero rows of A , we obtain

$$P_{T_n^3}(\lambda) = \lambda^n P_{2M}(\lambda) = \lambda^n \lambda^{n-1}(\lambda^2 - 4n) = \lambda^{2n-1}(\lambda^2 - 4n).$$

□

Corollary 3.12. *The nullity and energy of the Rooted 3-Uniform Semigraph tree are $\eta(T_n^3) = 2n - 1$ and $E(T_n^3) = 4\sqrt{n}$, respectively.*

Proof. The characteristic polynomial of adjacency matrix of T_n^3 is

$$P_{T_n^3}(\lambda) = \lambda^{2n-1}(\lambda^2 - 4n).$$

So the spectrum of T_n^3 associated with adjacency matrix is

$$\begin{pmatrix} 2\sqrt{n} & -2\sqrt{n} & 0 \\ 1 & 1 & 2n-1 \end{pmatrix}.$$

Therefore, the nullity of T_n^3 is $2n - 1$.

Energy of Rooted 3-Uniform Semigraph tree is

$$E(T_n^3) = |-2\sqrt{n}| + |2\sqrt{n}| = 4\sqrt{n}.$$

□

3.4 Comparison with ordinary graph cores and interpretation

The formulas obtained above show that the nonzero part of the spectrum is governed by a smaller weighted core, while the ordered middle vertices contribute additional zero eigenvalues. This comparison with ordinary graph cores explains both the scaling of energy and the increase in nullity in the semigraph setting.

Family	Weighted duction	core/re-	Energy relation	Main source of zero eigenvalues
P_{k+1}^n	$nA(P_{k+1})$		$E(P_{k+1}^n) = nE(P_{k+1})$	Pure middle vertices; one additional core zero when $k+1$ is odd
C_k^n	$nA(C_k)$		$E(C_k^n) = nE(C_k)$	Pure middle vertices; two additional core zeros when $k \equiv 0$ (mod 4)
$S_{2,n}^3$ ($n \geq 1$)	quotient with eigen- values $\pm 2, \pm\sqrt{n}$		$E(S_{2,n}^3) = 4 + 2\sqrt{n}$	$n - 1$ redundant pen- dant directions
S_n^3	n independent 2×2 blocks		$E(S_n^3) = 4n$	one central zero row
T_n^3	$2A(K_{1,n})$		$E(T_n^3) = 4\sqrt{n}$	middle rows and re- dundant leaf direc- tions

Table 1 Comparison between the semigraph families and their weighted ordinary graph cores or reductions.

In this way, large nullity reflects the presence of zero modes arising from ordered middle vertices, repeated pendant directions, or branching structures. In chemical models, these modes may correspond to possible non-bonding modes, while in network models they may indicate redundant or weakly controllable directions. For non-uniform path and cycle semigraphs, this weighted-core approach leads to cores with unequal edge weights and has been developed in [9]. Extending such methods to broader non-uniform semigraph families, especially those with mixed middle-end vertex roles, remains a natural problem for further study.

3.5 Upper and lower bounds for nullity of semigraphs and its energy

This section focuses on establishing fundamental upper and lower limits for the nullity of a semigraph, $\eta(G)$. These bounds are linked to structural properties of the semigraph, such as the presence of pure middle vertices and isolated vertices. Additionally, we establish that the energy of a semigraph is determined by its nonzero eigenvalues.

Theorem 3.13. Lower bound for semigraphs. *Let $G(V, E)$ be a semigraph with p pure middle vertices and q isolated vertices. Then $\eta(G) \geq p + q$.*

Proof. Each pure middle vertex contributes a zero row to the adjacency matrix $A(G)$, since only end vertices contribute nonzero row entries in the adjacency matrix. Suppose there are p pure middle vertices. Also, each isolated vertex contributes a zero row and a zero column. Suppose there are q isolated vertices in $G(V, E)$. Therefore, $A(G)$ has at least $p + q$ zero rows.

If $N = |V(G)|$, then $\text{rank } A(G) \leq N - (p + q)$. Hence, by the rank-nullity theorem,

$$\dim \ker A(G) = N - \text{rank } A(G) \geq p + q.$$

Since the geometric multiplicity of the eigenvalue 0 is at most its algebraic multiplicity, we get $\eta(G) \geq p + q$. $\square$

Theorem 3.14. Upper bound for semigraphs. *Let G be a semigraph with N vertices. Then $\eta(G) \leq N$. Furthermore, if G contains at least one edge, then $\eta(G) \leq N - 2$.*

Proof. The nullity of an $N \times N$ matrix cannot exceed N , so $\eta(G) \leq N$.

Suppose G contains at least one edge $e = (v_1, v_2, \dots, v_s)$, with $s \geq 2$. The vertices v_1 and v_s are end vertices of e . By the definition of the adjacency matrix,

$$a_{v_1 v_s} = s - 1 \quad \text{and} \quad a_{v_s v_1} = s - 1.$$

Since $s \geq 2$, we have $s - 1 \geq 1$. Also,

$$a_{v_1 v_1} = 0 \quad \text{and} \quad a_{v_s v_s} = 0.$$

Consider the principal 2×2 submatrix of $A(G)$ corresponding to v_1 and v_s :

$$A_{v_1, v_s} = \begin{pmatrix} 0 & s-1 \\ s-1 & 0 \end{pmatrix}.$$

This submatrix has determinant $-(s-1)^2 \neq 0$.

Now let

$$\det(\lambda I - A(G)) = \lambda^N + c_1 \lambda^{N-1} + c_2 \lambda^{N-2} + \cdots + c_N.$$

The coefficient c_2 is the sum of all principal 2×2 minors of $A(G)$. Since the diagonal entries of $A(G)$ are zero, each such minor has the form

$$\begin{vmatrix} 0 & a_{uv} \\ a_{vu} & 0 \end{vmatrix} = -a_{uv}a_{vu} \leq 0.$$

The minor corresponding to the end vertices v_1 and v_s is strictly negative. Hence $c_2 \neq 0$.

Therefore, the characteristic polynomial is not divisible by λ^{N-1} or by λ^N . Thus the algebraic multiplicity of the eigenvalue 0 is at most $N-2$. Hence $\eta(G) \leq N-2$. $\square$

Theorem 3.15. *Let G be a semigraph on N vertices with nullity $\eta(G)$. Let the nonzero eigenvalues of its adjacency matrix $A(G)$, counted with algebraic multiplicity, be $\lambda_1, \lambda_2, \dots, \lambda_{N-\eta(G)}$. Then the energy of the semigraph is*

$$E(G) = \sum_{i=1}^{N-\eta(G)} |\lambda_i|.$$

Proof. By definition, the energy of a semigraph is $E(G) = \sum_{i=1}^N |\lambda_i|$. The spectrum of $A(G)$ consists of $N - \eta(G)$ nonzero eigenvalues and $\eta(G)$ zero eigenvalues, counted with algebraic multiplicity. Therefore,

$$E(G) = \sum_{\lambda_i \neq 0} |\lambda_i| + \sum_{\lambda_i = 0} |\lambda_i|.$$

The second term is zero. Hence the energy is determined by the nonzero eigenvalues:

$$E(G) = \sum_{i=1}^{N-\eta(G)} |\lambda_i|.$$

$\square$

4 Conclusion

This paper extends the spectral notion of nullity to semigraphs and derives closed-form formulas for the nullity and energy of several fundamental uniform semigraph families, namely path-uniform semigraphs P_{k+1}^n , cycle-uniform semigraphs C_k^n , star semigraphs of type I , 3-uniform star semigraphs and rooted 3-uniform semigraph trees. The results show that these families have a natural two-level spectral structure: the nonzero eigenvalues arise from a smaller weighted ordinary graph core, while ordered middle vertices and repeated branches contribute additional zero eigenvalues. Thus the comparison with ordinary paths, cycles and stars clarifies both the scaling of energy and the increase in nullity in the semigraph setting. In particular, large nullity indicates that many modes do not contribute to the nonzero part of the spectrum. In chemical models, such modes may be interpreted as possible non-bonding modes, while in higher-order network models they may indicate redundant or weakly controllable directions.

We also established general bounds for semigraph nullity. If p is the number of pure middle vertices and q is the number of isolated vertices, then $\eta(G) \geq p+q$, while every semigraph on N vertices with at least one edge satisfies $\eta(G) \leq N-2$. The energy expression $E(G) = \sum_{i=1}^{N-\eta(G)} |\lambda_i|$ further emphasizes that only the nonzero part of the spectrum contributes to energy. These results suggest several directions for further work, including Laplacian and signless Laplacian versions of semigraph nullity, extremal semigraphs attaining the above bounds, and broader non-uniform semigraph families beyond the path and cycle cases studied in [9]. In such settings, the weighted core is no longer a constant multiple of an ordinary graph and mixed middle-end vertex roles make the spectral reduction more difficult.

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