Characterization of generalized derivations associate with Hochschild 2-cocycles on triangular Banach algebras

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ABSTRACT. Let $\mathcal{A}$ and $\mathcal{B}$ be unital Banach algebras and $\mathcal{M}$ be a left $\mathcal{A}$-module and right $\mathcal{B}$-module. We consider generalized derivations associate with Hochschild 2-cocycles on triangular Banach algebra $\mathcal{T}$ (related to $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{M}$). We characterize this new version of generalized derivations on triangular Banach algebras and we obtain some results for ℓ^1-direct summands of Banach algebras.

1. Introduction

Let $\mathcal{A}$ be a Banach algebra, and let X be a Banach $\mathcal{A}$-bimodule. A derivation is a linear map $D : \mathcal{A} \longrightarrow X$ such that

$$D(ab) = a \cdot D(b) + D(a) \cdot b \qquad (a, b \in \mathcal{A}).$$

For $x \in X$, set $ad_x : a \mapsto a \cdot x - x \cdot a$, $\mathcal{A} \longrightarrow X$. Then ad_x is the inner derivation induced by x.

The linear space of bounded derivations from $\mathcal{A}$ into X denoted by $Z^1(\mathcal{A}, X)$ and the linear subspace of inner derivations denoted by $N^1(\mathcal{A}, X)$. We consider the quotient space $H^1(\mathcal{A}, X) = Z^1(\mathcal{A}, X)/N^1(\mathcal{A}, X)$, called the *first Hochschild cohomology group* of $\mathcal{A}$ with coefficients in X.

Let $\mathcal{A}$ be a Banach algebra, and let X be a Banach $\mathcal{A}$-bimodule. By $B^n(\mathcal{A}, X)$, we mean that the space of bounded n-linear maps form $\mathcal{A}^n$ into X. A 2-linear map $\gamma \in B^2(\mathcal{A}, X)$ is called Hochschild *2-cocycle* if it satisfies in the following equation

$$a \cdot \gamma(b, c) - \gamma(ab, c) + \gamma(a, bc) - \gamma(a, b) \cdot c = 0,$$

for every $a, b, c \in \mathcal{A}$. The space of Hochschild 2-cocycles is a linear subspace of $B^2(\mathcal{A}, X)$, which denoted by $Z^2(\mathcal{A}, X)$. Here in after we used the word 2-cocycle instead Hochschild 2-cocycle. Let $\varphi \in \Delta(\mathcal{A})$, where $\Delta(\mathcal{A})$ is the carrier space of $\mathcal{A}$, then A 2-linear map $\gamma \in B^2(\mathcal{A}, X)$ is called *point 2-cocycle* at φ if it satisfies in the following equation

$$\varphi(a)\gamma(b, c) - \gamma(ab, c) + \gamma(a, bc) - \gamma(a, b)\varphi(c) = 0,$$

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for every $a, b, c \in \mathcal{A}$. For given $T \in B(\mathcal{A}, X)$, we let

$$(\delta^1 T)(a, b) = a \cdot T(b) - T(ab) + T(a) \cdot b,$$

for every $a, b \in \mathcal{A}$ and $\delta^1 : B(\mathcal{A}, X) \longrightarrow B^2(\mathcal{A}, X)$. Then the maps $\{\delta^1 T : T \in B(\mathcal{A}, X)\}$ is a linear subspace of $Z^2(\mathcal{A}, X)$. These maps are called *2-coboundaries*. The collection of all 2-coboundaries is denoted by $N^2(\mathcal{A}, X)$.

An additive map $D : \mathcal{A} \longrightarrow X$ called generalized 2-cocycle derivation if there exists a 2-cocycle γ such that

$$D(xy) = x \cdot D(y) + D(x) \cdot y + \gamma(x, y) \tag{1.4}$$

for every $x, y \in \mathcal{A}$. Similarly, an additive map $D : \mathcal{A} \longrightarrow X$ called generalized Jordan derivation if there exists a 2-cocycle γ such that

$$D(x^2) = x \cdot D(x) + D(x) \cdot y + \gamma(x, x) \tag{1.5}$$

for every $x \in \mathcal{A}$. This definitions introduced by Nakajima in [**5**] and he gave some examples for this new definition. In [**3**] authors considered this new notion for some algebras such as von-Neumann and they showed that generalized Jordan derivation of this type from von-Neumann algebras into themselves is a generalized derivation (under some conditions). Similar result obtained by authors in [**4**] for triangular algebras.

An additive map $D : \mathcal{A} \longrightarrow X$ called generalized 2-coboundry derivation if there exists a 2-coboundry γ such that

$$D(xy) = x \cdot D(y) + D(x) \cdot y + (\gamma F)(x, y) \tag{1.6}$$

for every $x, y \in \mathcal{A}$ and $F \in B(\mathcal{A}, X)$. Let $\mathcal{A}$ and $\mathcal{B}$ be unital Banach algebras with units $e_{\mathcal{A}}$ and $e_{\mathcal{B}}$, respectively. Suppose that $\mathcal{M}$ is a unital Banach $\mathcal{A}, \mathcal{B}$-bimodule. We define triangular Banach algebra

$$\mathcal{T} = \begin{bmatrix} \mathcal{A} & \mathcal{M} \\ & \mathcal{B} \end{bmatrix},$$

with the sum and product being giving by the usual 2×2 matrix operations and internal module actions. The norm on $\mathcal{T}$ is

$$\left\| \begin{bmatrix} a & m \\ & b \end{bmatrix} \right\| = \|a\|_{\mathcal{A}} + \|m\|_{\mathcal{M}} + \|b\|_{\mathcal{B}}.$$

The Banach algebra $\mathcal{T}$ as a Banach space is isomorphic to the ℓ^1-direct sum of $\mathcal{A}, \mathcal{B}$ and $\mathcal{M}$. Forrest and Marcoux introduced and studied derivation of triangular Banach algebras in [**1**].

Let $\mathcal{T}$ be a triangular Banach algebras defined as above, and let $\gamma \in B^2(\mathcal{T}, \mathcal{T})$. Let $\gamma_1 : \mathcal{T} \times \mathcal{T} \longrightarrow \mathcal{A}, \gamma_2 : \mathcal{T} \times \mathcal{T} \longrightarrow \mathcal{M}$ and $\gamma_3 : \mathcal{T} \times \mathcal{T} \longrightarrow \mathcal{B}$ denote the coordinate functions associated to γ. That is

$$\gamma(T_1, T_2) = \begin{bmatrix} \gamma_1(T_1, T_2) & \gamma_2(T_1, T_2) \\ & \gamma_3(T_1, T_2) \end{bmatrix},$$

for $T_1, T_2 \in \mathcal{T}$. Let $\gamma : \mathcal{T} \times \mathcal{T} \longrightarrow \mathcal{T}$ be a 2-cocycle. The coordinate function γ_1 is said to correspond to a 2-cocycle (2-coboundary) on $\mathcal{A} \times \mathcal{A}$ if there exists a 2-cocycle (2-coboundary) $\tau_{\mathcal{A}}$ on $\mathcal{A} \times \mathcal{A}$ such that $\gamma_1(T_1, T_2) = \tau_{\mathcal{A}}(a_1, a_2)$, where $T_i = \begin{bmatrix} a_i & m_i \\ & b_i \end{bmatrix}$, for $i = 1, 2$.

Similarly, γ_3 is said to correspond to a 2-cocycle (2-coboundary) on $\mathcal{B} \times \mathcal{B}$ if there exists a 2-cocycle (2-coboundary) $\tau_{\mathcal{B}}$ on $\mathcal{B} \times \mathcal{B}$ such that $\gamma_3(T_1, T_2) = \tau_{\mathcal{B}}(b_1, b_2)$.

Definition 1.1. Let $\gamma \in B^2(\mathcal{T}, \mathcal{T})$, $\gamma_1 : \mathcal{T} \times \mathcal{T} \longrightarrow \mathcal{A}, \gamma_2 : \mathcal{T} \times \mathcal{T} \longrightarrow \mathcal{M}$ and $\gamma_3 : \mathcal{T} \times \mathcal{T} \longrightarrow \mathcal{B}$ denote the coordinate functions associated to γ. That is

$$\gamma(T_1, T_2) = \begin{bmatrix} \gamma_1(T_1, T_2) & \gamma_2(T_1, T_2) \\ & \gamma_3(T_1, T_2) \end{bmatrix},$$

for $T_1, T_2 \in \mathcal{T}$. Let $\gamma \in B^2(\mathcal{T}, \mathcal{T})$ be a 2-cocycle (2-coboundries). We say that γ_1 corresponds to a 2-cocycle (2-coboundries) on $\mathcal{A} \times \mathcal{A}$ if there exists a 2-cocycle (2-coboundries) $\tau_{\mathcal{A}}$ on $\mathcal{A} \times \mathcal{A}$ such that $\gamma_1(T_1, T_2) = \tau_{\mathcal{A}}(a_1, a_2)$, where $T_i = \begin{bmatrix} a_i & m_i \\ & b_i \end{bmatrix}$, for $i = 1, 2$.

Similarly, we say that γ_3 corresponds to a 2-cocycle (2-coboundries) on $\mathcal{B} \times \mathcal{B}$ if there exists a 2-cocycle (2-coboundries) $\tau_{\mathcal{B}}$ on $\mathcal{B} \times \mathcal{B}$ such that $\gamma_3(T_1, T_2) = \tau_{\mathcal{B}}(b_1, b_2)$. Second order cohomology of triangular Banach algebras studied by Forrest and Marcoux in [**2**].

Lemma 1.2. *Let $\gamma \in B^2(\mathcal{T}, \mathcal{T})$ be a 2-cocycle . Then there are continuous corresponding 2-cocycles on $\mathcal{A} \times \mathcal{A}$ and $\mathcal{B} \times \mathcal{B}$.*

Proof. Define $\tau_{\mathcal{A}} : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$ and $\tau_{\mathcal{B}} : \mathcal{B} \times \mathcal{B} \longrightarrow \mathcal{B}$ as follows

$$\tau_{\mathcal{A}}(a_1, a_2) = e_{\mathcal{A}} \gamma \left(\begin{bmatrix} a_1 & 0 \\ & 0 \end{bmatrix}, \begin{bmatrix} a_2 & 0 \\ & 0 \end{bmatrix} \right) e_{\mathcal{A}},$$

and

$$\tau_{\mathcal{B}}(b_1, b_2) = e_{\mathcal{B}} \gamma \left(\begin{bmatrix} 0 & 0 \\ & b_1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ & b_2 \end{bmatrix} \right) e_{\mathcal{B}}.$$

It is easy to check that $\tau_{\mathcal{A}}$ and $\tau_{\mathcal{B}}$ are 2-cocycle. Continuity of $\tau_{\mathcal{A}}$ and $\tau_{\mathcal{B}}$ is clear. □

Lemma 1.3. *Let $\gamma \in B^2(\mathcal{T}, \mathcal{T})$ be a 2-cocycle. Then there are corresponding 2-cocycles $\tau_{\mathcal{A}}$ and $\tau_{\mathcal{B}}$ on $\mathcal{A} \times \mathcal{A}$ and $\mathcal{B} \times \mathcal{B}$, respectively. Furthermore*

(1) $\tau_{\mathcal{A}}(a, 0) = \tau_{\mathcal{B}}(0, b) = \tau_{\mathcal{A}}(e_{\mathcal{A}}, 0) = \tau_{\mathcal{B}}(0, e_{\mathcal{B}}) = \tau_{\mathcal{A}}(0, 0) = \tau_{\mathcal{B}}(0, 0) = 0.$
(2) $\gamma_2(e_{11}, e_{11}) = 0, \gamma_2(e_{11}, 0) = 0.$
(3) $\gamma_2(e_{22}, e_{22}) = 0.$
(4) $\tau_{\mathcal{A}}(a, e_{\mathcal{A}}) = 0$ *and* $\tau_{\mathcal{A}}(e_{\mathcal{A}}, e_{\mathcal{A}}) = 0.$
(5) $\tau_{\mathcal{B}}(b, e_{\mathcal{B}}) = 0$ *and* $\tau_{\mathcal{B}}(e_{\mathcal{B}}, e_{\mathcal{B}}) = 0.$
(6) $\gamma_2(b_{22}, e_{22}) = 0$ *and* $\gamma_2(e_{11}, a_{11}) = 0.$

where $e_{11} = \begin{bmatrix} e_{\mathcal{A}} & 0 \\ & 0 \end{bmatrix}$, $e_{22} = \begin{bmatrix} 0 & 0 \\ & e_{\mathcal{B}} \end{bmatrix}$, $b_{22} = \begin{bmatrix} 0 & 0 \\ & b \end{bmatrix}$, $a_{11} = \begin{bmatrix} a & 0 \\ & 0 \end{bmatrix}$ *and for* $a \in \mathcal{A}, b \in \mathcal{B}$.

PROOF. Existing of $\tau_{\mathcal{A}}$ and $\tau_{\mathcal{B}}$ clear by Lemma 1.2 . Since $\tau_{\mathcal{A}}$ and $\tau_{\mathcal{B}}$ are 2-linear so (1) is clear. For the rest consider the following

$$\gamma(\begin{bmatrix} e_{\mathcal{A}} & 0 \\ & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ & e_{\mathcal{B}} \end{bmatrix}) = \begin{bmatrix} \tau_{\mathcal{A}}(e_{\mathcal{A}}, 0) & \gamma_2(e_{11}, e_{22}) \\ & \tau_{\mathcal{B}}(0, e_{\mathcal{B}}) \end{bmatrix},$$

and

$$e_{11}\gamma(e_{11}, e_{22}) - \gamma(e_{11}, e_{22}) + \gamma(e_{11}, 0) - \gamma(e_{11}, e_{11})e_{22} = 0.$$

Then,

$$\begin{bmatrix} \tau_{\mathcal{A}}(e_{\mathcal{A}}, 0) & \gamma_2(e_{11}, e_{22}) \\ & 0 \end{bmatrix} - \begin{bmatrix} \tau_{\mathcal{A}}(e_{\mathcal{A}}, 0) & \gamma_2(e_{11}, e_{22}) \\ & \tau_{\mathcal{B}}(0, e_{\mathcal{B}}) \end{bmatrix} + \begin{bmatrix} \tau_{\mathcal{A}}(e_{\mathcal{A}}, 0) & \gamma_2(e_{11}, 0) \\ & 0 \end{bmatrix} + \begin{bmatrix} 0 & \gamma_2(e_{11}, e_{11}) \\ & 0 \end{bmatrix} = 0.$$

This follows that $\gamma_2(e_{11}, e_{11}) + \gamma_2(e_{11}, 0) = 0$. From

$$e_{11}\gamma(e_{11}, e_{11}) - \gamma(e_{11}, e_{11}) + \gamma(e_{11}, e_{11}) - \gamma(e_{11}, e_{11})e_{11} = 0,$$

we conclude that $\gamma_2(e_{11}, e_{11}) = 0$. This implies that $\gamma_2(e_{11}, 0) = 0$. Similarly, one can show that $\gamma_2(e_{22}, e_{22}) = 0$. From

$$a_{11}\gamma(e_{11}, e_{11}) - \gamma(a_{11}, e_{11}) + \gamma(a_{11}, e_{11}) - \gamma(a_{11}, e_{11})e_{11} = 0,$$

we conclude that $\tau_A(a, e_{\mathcal{A}}) = 0$. Since a was arbitrary so $\tau_A(e_{\mathcal{A}}, e_{\mathcal{A}}) = 0$. By the similar methods, we obtain the other cases. □

Let $\mathcal{T}$ be a triangular Banach algebra and let $\mathcal{X}$ be a unital Banach $\mathcal{T}$-bimodule, then we use these notations in this paper: $\mathcal{X}_{\mathcal{A}\mathcal{A}} = e_{\mathcal{A}} \cdot \mathcal{X} \cdot e_{\mathcal{A}}, \mathcal{X}_{\mathcal{B}\mathcal{B}} = e_{\mathcal{B}} \cdot \mathcal{X} \cdot e_{\mathcal{B}}, \mathcal{X}_{\mathcal{A}\mathcal{B}} = e_{\mathcal{A}} \cdot \mathcal{X} \cdot e_{\mathcal{B}}$, and $\mathcal{X}_{\mathcal{B}\mathcal{A}} = e_{\mathcal{B}} \cdot \mathcal{X} \cdot e_{\mathcal{A}}$. If $\mathcal{X}$ replaced by $\mathcal{T}$, we have $\mathcal{X}_{\mathcal{A}\mathcal{A}} = \mathcal{A}, \mathcal{X}_{\mathcal{B}\mathcal{B}} = \mathcal{B}, \mathcal{X}_{\mathcal{A}\mathcal{B}} = \mathcal{M}$, and $\mathcal{X}_{\mathcal{B}\mathcal{A}} = 0$.

LEMMA 1.4. *Let $\mathcal{X}$ be a $\mathcal{T}$-bimodule, $\delta_{\mathcal{A}} : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{X}_{\mathcal{A}\mathcal{A}}$, $\delta_{\mathcal{B}} : \mathcal{B} \times \mathcal{B} \longrightarrow \mathcal{X}_{\mathcal{B}\mathcal{B}}$ be 2-cocycles, and $\mathcal{X}_{\mathcal{A}\mathcal{B}} = 0$. Then there exists a 2-cocycle mapping from $\mathcal{T} \times \mathcal{T}$ into $\mathcal{X}$.*

PROOF. For every $\begin{bmatrix} a_1 & m_1 \\ & b_1 \end{bmatrix}, \begin{bmatrix} a_2 & m_2 \\ & b_2 \end{bmatrix} \in \mathcal{T}$, define $D : \mathcal{T} \times \mathcal{T} \longrightarrow \mathcal{X}$ by

$$D\left(\begin{bmatrix} a_1 & m_1 \\ & b_1 \end{bmatrix}, \begin{bmatrix} a_2 & m_2 \\ & b_2 \end{bmatrix}\right) = \delta_{\mathcal{A}}(a_1, a_2) + \delta_{\mathcal{B}}(b_1, b_2).$$

We claim that D is a 2-cocycle. Because for every $T_1 = \begin{bmatrix} a_1 & m_1 \\ & b_1 \end{bmatrix}, T_2 = \begin{bmatrix} a_2 & m_2 \\ & b_2 \end{bmatrix}, T_3 = \begin{bmatrix} a_3 & m_3 \\ & b_3 \end{bmatrix} \in \mathcal{T}$, we have

$$\begin{aligned} & T_1 \cdot D(T_2,T_3) - D(T_1T_2,T_3) + D(T_1,T_2T_3) - D(T_1,T_2)\cdot T_3 \\ = \;& T_1 \cdot (\delta_{\mathcal{A}}(a_2,a_3) + \delta_{\mathcal{B}}(b_2,b_3)) - \delta_{\mathcal{A}}(a_1a_2,a_3) - \delta_{\mathcal{B}}(b_1b_2,b_3) \\ & + \delta_{\mathcal{A}}(a_1,a_2a_3) + \delta_{\mathcal{B}}(b_1,b_2b_3) - (\delta_{\mathcal{A}}(a_1,a_2) + \delta_{\mathcal{B}}(b_1,b_2))\cdot T_3 \\ = \;& T_1 \cdot \tau(e_{\mathcal{A}})\delta_{\mathcal{A}}(a_2,a_3) - \delta_{\mathcal{A}}(a_1a_2,a_3) + \delta_{\mathcal{A}}(a_1,a_2a_3) - \delta_{\mathcal{A}}(a_1,a_2)\cdot e_{\mathcal{A}} \cdot T_3 \\ & + T_1 \cdot \tau(e_{\mathcal{B}})\delta_{\mathcal{B}}(b_2,b_3) - \delta_{\mathcal{B}}(b_1b_2,b_3) + \delta_{\mathcal{B}}(b_1,b_2b_3) - \delta_{\mathcal{B}}(b_1,b_2)\cdot e_{\mathcal{B}} \cdot T_3 \\ = \;& a_1 \cdot \delta_{\mathcal{A}}(a_2,a_3) - \delta_{\mathcal{A}}(a_1a_2,a_3) + \delta_{\mathcal{A}}(a_1,a_2a_3) - \delta_{\mathcal{A}}(a_1,a_2)\cdot a_3 \\ & + b_1 \cdot \delta_{\mathcal{B}}(b_2,b_3) - \delta_{\mathcal{B}}(b_1b_2,b_3) + \delta_{\mathcal{B}}(b_1,b_2b_3) - \delta_{\mathcal{B}}(b_1,b_2)\cdot b_3 \\ = \;& 0. \end{aligned}$$

This proves our claim. □

LEMMA 1.5. [**2**, Lemma 3.1] *Let $\delta_{\mathcal{A}}$ and $\delta_{\mathcal{B}}$ be 2-coboundaries on $\mathcal{A}\times\mathcal{A}$ and $\mathcal{B}\times\mathcal{B}$, respectively. Then there exists a 2-coboundaries δ on $\mathcal{T}\times\mathcal{T}$ such that δ_1 corresponds to $\delta_{\mathcal{A}}$ and δ_2 corresponds to $\delta_{\mathcal{B}}$, where δ_1 and δ_2 are coordinate functions associated to δ.*

2. Characterization of generalized 2-cocycle derivations

In this section we prove main results of paper. We characterize generalized 2-cocycle derivations on triangular Banach algebras and by taking $\mathcal{M} = 0$ we consider generalized 2-cocycle derivations on $\mathcal{A} \oplus_1 \mathcal{B}$, where $\mathcal{A}$ and $\mathcal{B}$ are Banach algebras.

THEOREM 2.1. *Let $\mathcal{T}$ be a triangular Banach algebra and $\mathcal{A},\mathcal{B}$ be unital Banach algebras with units $e_{\mathcal{A}}$ and $e_{\mathcal{B}}$, respectively, and M be a unitary Banach $\mathcal{A},\mathcal{B}$-bimodule. Let $D : \mathcal{T} \longrightarrow \mathcal{T}$ be a generalized 2-cocycle derivation associate with γ, then there exist element $m_D \in M$, corresponding 2-cocycles $\tau_{\mathcal{A}}$ and $\tau_{\mathcal{B}}$ on $\mathcal{A}\times\mathcal{A}$ and $\mathcal{B}\times\mathcal{B}$, respectively, and mappings $D_{\mathcal{A}} : \mathcal{A} \longrightarrow \mathcal{A}$, $D_{\mathcal{B}} : \mathcal{B} \longrightarrow \mathcal{B}$ and $\tau_{\mathcal{M}} : \mathcal{M} \longrightarrow \mathcal{M}$ such that*

(1) $D\Big(\begin{bmatrix} e_{\mathcal{A}} & 0 \\ & 0 \end{bmatrix}\Big) = \begin{bmatrix} 0 & m_D \\ & 0 \end{bmatrix}$.

(2) $D\Big(\begin{bmatrix} a & 0 \\ & 0 \end{bmatrix}\Big) = \begin{bmatrix} D_{\mathcal{A}}(a) & am_D + \gamma_2(a_{11},e_{11}) \\ & 0 \end{bmatrix}$.

(3) $D\Big(\begin{bmatrix} 0 & 0 \\ & e_{\mathcal{B}} \end{bmatrix}\Big) = \begin{bmatrix} 0 & -m_D - \gamma_2(e_{11},e_{22}) \\ & 0 \end{bmatrix}$.

(4) $D\Big(\begin{bmatrix} 0 & 0 \\ & b \end{bmatrix}\Big) = \begin{bmatrix} 0 & -m_Db - \gamma_2(e_{11},b_{22}) \\ & D_{\mathcal{B}}(b) \end{bmatrix}$

(5) $D\Big(\begin{bmatrix} 0 & m \\ & 0 \end{bmatrix}\Big) = \begin{bmatrix} 0 & \tau_{\mathcal{M}}(m) \\ & 0 \end{bmatrix}$.

(6) $\gamma_1(\begin{bmatrix} a & 0 \\ & 0 \end{bmatrix}, \begin{bmatrix} 0 & m \\ & 0 \end{bmatrix}) = 0,\ \gamma_3(\begin{bmatrix} a & 0 \\ & 0 \end{bmatrix}, \begin{bmatrix} 0 & m \\ & 0 \end{bmatrix}) = 0$ *and*

$$\tau_{\mathcal{M}}(a \cdot m) = a \cdot \tau_{\mathcal{M}}(m) + D_{\mathcal{A}}(a) \cdot m + \gamma_2(\begin{bmatrix} a & 0 \\ & 0 \end{bmatrix}, \begin{bmatrix} 0 & m \\ & 0 \end{bmatrix}).$$

(7) $\gamma_1(\begin{bmatrix} 0 & m \\ & \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ & b \end{bmatrix}) = 0,\ \gamma_3(\begin{bmatrix} 0 & m \\ & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ & b \end{bmatrix}) = 0$ *and*

$$\tau_{\mathcal{M}}(m \cdot b) = \tau_{\mathcal{M}}(m) \cdot b + m \cdot D_{\mathcal{B}}(b) + \gamma_2(\begin{bmatrix} 0 & m \\ & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ & b \end{bmatrix}).$$

Furthermore, $D_{\mathcal{A}} : \mathcal{A} \longrightarrow \mathcal{A}$ and $D_{\mathcal{B}} : \mathcal{B} \longrightarrow \mathcal{B}$ are generalized 2-cocycle derivations associate with 2-cocycles $\tau_{\mathcal{A}}$ and $\tau_{\mathcal{B}}$, respectively.

PROOF. Let γ be the 2-cocycle related to D. Let γ_1, γ_2 and γ_3 be the coordinate functions associated to γ. By Lemma 1.2, there exists corresponding 2-cocycles τ_A and $\tau_{\mathcal{B}}$, respectively on $\mathcal{A} \times \mathcal{A}$ and $\mathcal{B} \times \mathcal{B}$ (continuity of these maps do not need).

(1) Put $e_{11} = \begin{bmatrix} e_{\mathcal{A}} & 0 \\ & 0 \end{bmatrix}$ and let $D(e_{11}) = \begin{bmatrix} p & q \\ & r \end{bmatrix}$. Then by Lemma 1.3

$$\begin{aligned} D(e_{11}) &= D(e_{11}e_{11}) = e_{11}D(e_{11}) + D(e_{11})e_{11} + \gamma(e_{11}, e_{11}) \\ &= \begin{bmatrix} e_{\mathcal{A}} & 0 \\ & 0 \end{bmatrix}\begin{bmatrix} p & q \\ & r \end{bmatrix} + \begin{bmatrix} p & q \\ & r \end{bmatrix}\begin{bmatrix} e_{\mathcal{A}} & 0 \\ & 0 \end{bmatrix} + \begin{bmatrix} \gamma_1(e_{11}, e_{11}) & 0 \\ & \gamma_3(e_{11}, e_{11}) \end{bmatrix} \\ &= \begin{bmatrix} 2p & q \\ & 0 \end{bmatrix} + \begin{bmatrix} \tau_{\mathcal{A}}(e_{\mathcal{A}}, e_{\mathcal{A}}) & 0 \\ & \tau_B(0,0) \end{bmatrix} \\ &= \begin{bmatrix} 2p & q \\ & 0 \end{bmatrix}. \end{aligned}$$

Above relations follow that

$$D(e_{11}) = \begin{bmatrix} 0 & q \\ & 0 \end{bmatrix}.$$

Set $q = m_D$, then

$$D(e_{11}) = \begin{bmatrix} 0 & m_D \\ & 0 \end{bmatrix}.$$

(2) Put $a_{11} = \begin{bmatrix} a & 0 \\ & 0 \end{bmatrix}$ and $e_{11} = \begin{bmatrix} e_{\mathcal{A}} & 0 \\ & 0 \end{bmatrix}$. Let $D(a_{11}) = \begin{bmatrix} p & q \\ & r \end{bmatrix}$, then

$$\begin{aligned} D(a_{11}) &= D(a_{11}e_{11}) = a_{11}D(e_{11}) + D(a_{11})e_{11} + \gamma(a_{11}, e_{11}) \\ &= \begin{bmatrix} a & 0 \\ & 0 \end{bmatrix} \begin{bmatrix} 0 & m_D \\ & 0 \end{bmatrix} + \begin{bmatrix} p & q \\ & r \end{bmatrix} \begin{bmatrix} e_A & 0 \\ & 0 \end{bmatrix} \\ &\quad + \begin{bmatrix} \gamma_1(a_{11}, e_{11}) & \gamma_2(a_{11}, e_{11}) \\ & \gamma_3(a_{11}, e_{11}) \end{bmatrix} \\ &= \begin{bmatrix} p & am_D \\ & 0 \end{bmatrix} + \begin{bmatrix} \tau_{\mathcal{A}}(a, e_{\mathcal{A}}) & \gamma_2(a_{11}, e_{11}) \\ & \tau_B(0,0) \end{bmatrix} \\ &= \begin{bmatrix} p & am_D + \gamma_2(a_{11}, e_{11}) \\ & 0 \end{bmatrix}. \end{aligned}$$

Therefore

$$D(a_{11}) = \begin{bmatrix} D_{\mathcal{A}}(a) & am_D + \gamma_2(a_{11}, e_{11}) \\ & 0 \end{bmatrix},$$

where $D_{\mathcal{A}}(a) = p$.

(3) Set $e_{22} = \begin{bmatrix} 0 & 0 \\ & e_{\mathcal{B}} \end{bmatrix}$ and let $D(e_{22}) = \begin{bmatrix} p & q' \\ & r \end{bmatrix}$. Then

$$\begin{aligned} D(e_{22}) &= D(e_{22}e_{22}) = e_{22}D(e_{22}) + D(e_{22})e_{22} + \gamma(e_{22}, e_{22}) \\ &= \begin{bmatrix} 0 & 0 \\ & e_{\mathcal{B}} \end{bmatrix} \begin{bmatrix} p & q' \\ & r \end{bmatrix} + \begin{bmatrix} p & q' \\ & r \end{bmatrix} \begin{bmatrix} 0 & 0 \\ & e_{\mathcal{B}} \end{bmatrix} + \begin{bmatrix} \gamma_1(e_{22}, e_{22}) & \gamma_2(e_{22}, e_{22}) \\ & \gamma_3(e_{22}, e_{22}) \end{bmatrix} \\ &= \begin{bmatrix} 0 & q' \\ & 2r \end{bmatrix} + \begin{bmatrix} \tau_{\mathcal{A}}(0,0) & \gamma_2(e_{22}, e_{22}) \\ & \tau_{\mathcal{B}}(e_{\mathcal{B}}, e_{\mathcal{B}}) \end{bmatrix} \\ &= \begin{bmatrix} 0 & q' \\ & 2r \end{bmatrix}. \end{aligned}$$

Above relations follow that

$$D(e_{22}) = \begin{bmatrix} 0 & q' \\ & 0 \end{bmatrix}.$$

Then

$$\begin{aligned} 0 &= D(0) = D(e_{11}e_{22}) = e_{11}D(e_{22}) + D(e_{11})e_{22} + \gamma(e_{11}, e_{22}) \\ &= \begin{bmatrix} 0 & q' \\ & 0 \end{bmatrix} + \begin{bmatrix} 0 & m_D \\ & 0 \end{bmatrix} + \begin{bmatrix} 0 & \gamma_2(e_{11}, e_{22}) \\ & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & m_D + q' + \gamma_2(e_{11}, e_{22}) \\ & 0 \end{bmatrix} \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} 0 &= D(0) = D(e_{22}e_{11}) = e_{22}D(e_{11}) + D(e_{22})e_{11} + \gamma(e_{22}, e_{11}) \\ &= \begin{bmatrix} 0 & \gamma_2(e_{22}, e_{11}) \\ & 0 \end{bmatrix}. \end{aligned}$$

This follows that $\gamma_2(e_{22}, e_{11}) = 0$.Then we have $q' = -m_D - \gamma_2(e_{11}, e_{22})$.

(4) Set $e_{22} = \begin{bmatrix} 0 & 0 \\ & e_{\mathcal{B}} \end{bmatrix}$, $b_{22} = \begin{bmatrix} 0 & 0 \\ & b \end{bmatrix}$ and let $D(b_{22}) = \begin{bmatrix} p & q \\ & r \end{bmatrix}$. Then by Lemma 1.3 we have

$$\begin{aligned} D(b_{22}) &= D(b_{22}e_{22}) = b_{22}D(e_{22}) + D(b_{22})e_{22} + \gamma(b_{22}, e_{22}) \\ &= \begin{bmatrix} 0 & 0 \\ & b_{22} \end{bmatrix} \begin{bmatrix} 0 & -m_D - \gamma_2(e_{11}, e_{22}) \\ & 0 \end{bmatrix} + \begin{bmatrix} p & q \\ & r \end{bmatrix} \begin{bmatrix} 0 & 0 \\ & e_{\mathcal{B}} \end{bmatrix} \\ &\quad + \begin{bmatrix} \gamma_1(b_{22}, e_{22}) & \gamma_2(b_{22}, e_{22}) \\ & \gamma_3(b_{22}, e_{22}) \end{bmatrix} \\ &= \begin{bmatrix} 0 & q \\ & r \end{bmatrix} + \begin{bmatrix} \tau_{\mathcal{A}}(0,0) & \gamma_2(b_{22}, e_{22}) \\ & \tau_{\mathcal{B}}(b, e_{\mathcal{B}}) \end{bmatrix} \\ &= \begin{bmatrix} 0 & q \\ & r \end{bmatrix}. \end{aligned}$$

By $0 = D(0) = D(e_{11}b_{22})$ we conclude that $q = -m_D b - \gamma_2(e_{11}, b_{22})$. If we set $D_{\mathcal{B}}(b) = r$ we obtain the desire.

(5) Set $e_{11} = \begin{bmatrix} e_{\mathcal{A}} & 0 \\ & 0 \end{bmatrix}$, $e_{22} = \begin{bmatrix} 0 & 0 \\ & e_B \end{bmatrix}$, $m_{12} = \begin{bmatrix} 0 & m \\ & 0 \end{bmatrix}$ and let $D(m_{12}) = \begin{bmatrix} p & q \\ & r \end{bmatrix}$. Then by Lemma 1.3 we have

$$\begin{aligned} D(m_{12}) &= D(m_{12}e_{22}) = m_{12}D(e_{22}) + D(m_{12})e_{22} + \gamma(m_{12}, e_{22}) \\ &= \begin{bmatrix} 0 & m \\ & 0 \end{bmatrix} \begin{bmatrix} 0 & -m_D - \gamma_2(e_{11}, e_{22}) \\ & 0 \end{bmatrix} + \begin{bmatrix} p & q \\ & r \end{bmatrix} \begin{bmatrix} 0 & 0 \\ & e_{\mathcal{B}} \end{bmatrix} \\ &\quad + \begin{bmatrix} \gamma_1(m_{12}, e_{22}) & \gamma_2(m_{12}, e_{22}) \\ & \gamma_3(m_{12}, e_{22}) \end{bmatrix} \\ &= \begin{bmatrix} 0 & q \\ & r \end{bmatrix} + \begin{bmatrix} \tau_{\mathcal{A}}(0,0) & \gamma_2(m_{12}, e_{22}) \\ & \tau_B(0, e_{\mathcal{B}}) \end{bmatrix} \\ &= \begin{bmatrix} 0 & q \\ & r \end{bmatrix}. \end{aligned}$$

This shows that $\gamma_2(m_{12}, e_{22}) = 0$. On the other hand

$$\begin{aligned}
D(m_{12}) &= D(e_{11}m_{12}) = e_{11}D(m_{12}) + D(e_{11})m_{12} + \gamma(e_{11}, m_{12}) \\
&= \begin{bmatrix} e_{\mathcal{A}} & 0 \\ & 0 \end{bmatrix}\begin{bmatrix} 0 & q \\ & r \end{bmatrix} + \begin{bmatrix} 0 & m_D \\ & 0 \end{bmatrix}\begin{bmatrix} 0 & m \\ & 0 \end{bmatrix} \\
&\quad + \begin{bmatrix} \gamma_1(e_{11}, m_{12}) & \gamma_2(e_{11}, m_{12}) \\ & \gamma_3(e_{11}, m_{12}) \end{bmatrix} \\
&= \begin{bmatrix} 0 & q \\ & 0 \end{bmatrix} + \begin{bmatrix} \tau_{\mathcal{A}}(e_{\mathcal{A}}, 0) & \gamma_2(e_{11}, m_{12}) \\ & \tau_{\mathcal{B}}(0,0) \end{bmatrix} \\
&= \begin{bmatrix} 0 & q + \gamma_2(e_{11}, m_{12}) \\ & 0 \end{bmatrix}.
\end{aligned}$$

This implies that $\gamma_2(e_{11}, m_{12}) = 0$. Thus,

$$D(m_{12}) = \begin{bmatrix} 0 & q \\ & 0 \end{bmatrix}.$$

Now; set $q = \tau_M(m)$. Therefore proof is complete.
(6) Let $a_{11} = \begin{bmatrix} a & 0 \\ & 0 \end{bmatrix}$ and $m_{12} = \begin{bmatrix} 0 & m \\ & 0 \end{bmatrix}$ then by (2) and (5) we have

$$\begin{aligned}
\begin{bmatrix} 0 & \tau_{\mathcal{M}}(a \cdot m) \\ & 0 \end{bmatrix} &= D(a_{11}m_{12}) = a_{11}D(m_{12}) + D(a_{11})m_{12} + \gamma(a_{11}, m_{12}) \\
&= \begin{bmatrix} 0 & a \cdot \tau_{\mathcal{M}}(m) \\ & 0 \end{bmatrix} + \begin{bmatrix} 0 & D_{\mathcal{A}}(a) \cdot m \\ & 0 \end{bmatrix} + \begin{bmatrix} \gamma_1(a_{11}, m_{12}) & \gamma_2(a_{11}, m_{12}) \\ & \gamma_3(a_{11}, m_{12}) \end{bmatrix} \\
&= \begin{bmatrix} \gamma_1(a_{11}, m_{12}) & a \cdot \tau_{\mathcal{M}}(m) + D_{\mathcal{A}}(a) \cdot m + \gamma_2(a_{11}, m_{12}) \\ & \gamma_3(a_{11}, m_{12}) \end{bmatrix}.
\end{aligned}$$

Thus, $\gamma_1(a_{11}, m_{12}) = 0$, $\gamma_3(a_{11}, m_{12}) = 0$ and $\tau_{\mathcal{M}}(a \cdot m) = a \cdot \tau_{\mathcal{M}}(m) + D_{\mathcal{A}}(a) \cdot m + \gamma_2(a_{11}, m_{12})$. By the similar method we can prove the case (7).

Finally we show that $D_{\mathcal{A}}$ is a generalized 2-cocycle derivation associative with 2-cocycle $\tau_{\mathcal{A}}$. Put $a'_{11} = \begin{bmatrix} a' & 0 \\ & 0 \end{bmatrix}$ and $(aa')_{11} = \begin{bmatrix} aa' & 0 \\ & 0 \end{bmatrix}$, then

$$\begin{aligned}
D(\begin{bmatrix} a & 0 \\ & 0 \end{bmatrix}\begin{bmatrix} a' & 0 \\ & 0 \end{bmatrix}) &= D(a_{11}a'_{11}) = a_{11}D(a'_{11}) + D(a_{11})a'_{11} + \gamma(a_{11}, a'_{11}) \\
&= \begin{bmatrix} a & 0 \\ & 0 \end{bmatrix}\begin{bmatrix} D_{\mathcal{A}}(a') & a'm_D + \gamma_2(a'_{11}, e_{11}) \\ & 0 \end{bmatrix} \\
&+ \begin{bmatrix} D_{\mathcal{A}}(a) & am_D + \gamma_2(a_{11}, e_{11}) \\ & 0 \end{bmatrix}\begin{bmatrix} a' & 0 \\ & 0 \end{bmatrix} \\
&+ \begin{bmatrix} \tau_A(a, a') & \gamma_2(a_{11}, a'_{11}) \\ & \tau_B(0,0) \end{bmatrix} \\
&= \begin{bmatrix} aD_{\mathcal{A}}(a') + a'D_{\mathcal{A}}(a) + \tau_{\mathcal{A}}(a, a') & aa'm_D + a\gamma_2(a'_{11}, e_{11}) + \gamma_2(a_{11}, a'_{11}) \\ & 0 \end{bmatrix}
\end{aligned}$$

On the other hand we have

$$\begin{aligned}
D(\begin{bmatrix} a & 0 \\ & 0 \end{bmatrix}\begin{bmatrix} a' & 0 \\ & 0 \end{bmatrix}) &= D(\begin{bmatrix} aa' & 0 \\ & 0 \end{bmatrix}) \\
&= \begin{bmatrix} D_{\mathcal{A}}(aa') & aa'm_D + \gamma_2((aa')_{11}, e_{11}) \\ & 0 \end{bmatrix}.
\end{aligned}$$

Above relations follow that

$$D_{\mathcal{A}}(aa') = aD_{\mathcal{A}}(a') + a'D_{\mathcal{A}}(a) + \tau_{\mathcal{A}}(a, a'),$$

and

$$a\gamma_2(a'_{11}, e_{11}) + \gamma_2(a_{11}, a'_{11}) = \gamma_2(a_{11}a'_{11}, e_{11}).$$

Thus $D_{\mathcal{A}}$ is generalized 2-cocycle derivation. The same way of $D_{\mathcal{A}}$ proves that $D_{\mathcal{B}}$ is a generalized 2-cocycle derivation. □

NOTE 2.2. In the Theorem 2.1, if $D : \mathcal{T} \longrightarrow \mathcal{T}$ is continuous and $\gamma \in Z^2(\mathcal{T}, \mathcal{T})$ then all obtained generalized derivations and 2-cocycles will be continuous. From now on, we suppose that all maps (generalized derivations, module mappings and 2-cocycles) are continuous.

PROPOSITION 2.3. *Let $D_{\mathcal{A}} : \mathcal{A} \longrightarrow A$ and $D_{\mathcal{B}} : \mathcal{B} \longrightarrow \mathcal{B}$ be generalized 2-coclyes derivation associate with 2-cocycles $\tau_{\mathcal{A}}$ and $\tau_{\mathcal{B}}$, respectively, and let $\tau_{\mathcal{M}} : \mathcal{M} \longrightarrow \mathcal{M}$ be a linear map that satisfies in conditions (6) and (7) of Theorem 2.1 such that*

$$\gamma_2(T_1, T_2) = \gamma_2\Big(\begin{bmatrix} a_1 & 0 \\ & 0 \end{bmatrix}, \begin{bmatrix} 0 & m_2 \\ & 0 \end{bmatrix}\Big) + \gamma_2\Big(\begin{bmatrix} 0 & m_1 \\ & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ & b_2 \end{bmatrix}\Big), \tag{2.2}$$

and

$$T'\gamma_2(T_1T_2) - \gamma_2(T'T_1, T_2) + \gamma_2(T', T_1T_2) - \gamma_2(T', T_1)T_2 = \tau_{\mathcal{A}}(a', a_1)\cdot m_2 - m'\cdot\tau_{\mathcal{B}}(b_1, b_2), \tag{2.3}$$

for all $T_1 = \begin{bmatrix} a_1 & m_1 \\ & b_1 \end{bmatrix}, T_2 = \begin{bmatrix} a_2 & m_2 \\ & b_2 \end{bmatrix}, T' = \begin{bmatrix} a' & m' \\ & b' \end{bmatrix} \in \mathcal{T}$. *Then the map*

$$D\Big(\begin{bmatrix} a & m \\ & b \end{bmatrix}\Big) = \begin{bmatrix} D_{\mathcal{A}}(a) & \tau_{\mathcal{M}}(m) \\ & D_{\mathcal{B}}(b) \end{bmatrix},$$

is a generalized 2-cocycle derivation with associate 2-cocycle $\gamma = \begin{bmatrix} \tau_{\mathcal{A}} & \gamma_2 \\ & \tau_{\mathcal{B}} \end{bmatrix}$.

Proof. Clearly by (2.3) γ is a 2-cocycle on $\mathcal{T}\times\mathcal{T}$. Then for every $T_1 = \begin{bmatrix} a_1 & m_1 \\ & b_1 \end{bmatrix}, T_2 = \begin{bmatrix} a_2 & m_2 \\ & b_2 \end{bmatrix} \in \mathcal{T}$, we have

$$\begin{aligned}
&T_1D(T_2) + D(T_1)T_2 + \gamma(T_1, T_2)\\
&\quad= \begin{bmatrix} a_1 & m_1 \\ & b_1 \end{bmatrix}\begin{bmatrix} D_{\mathcal{A}}(a_2) & \tau_{\mathcal{M}}(m_2) \\ & D_{\mathcal{B}}(b_2) \end{bmatrix} + \begin{bmatrix} D_{\mathcal{A}}(a_1) & \tau_{\mathcal{M}}(m_1) \\ & D_{\mathcal{B}}(b_1) \end{bmatrix}\begin{bmatrix} a_2 & m_2 \\ & b_2 \end{bmatrix}\\
&\qquad+ \begin{bmatrix} \tau_{\mathcal{A}}(a_1, a_2) & \gamma_2(T_1, T_2) \\ & \tau_{\mathcal{B}}(b_1, b_2) \end{bmatrix}\\
&\quad= \begin{bmatrix} a_1D_{\mathcal{A}}(a_2) + D_{\mathcal{A}}(a_1)a_2 + \tau_{\mathcal{A}}(a_1, a_2) & \begin{matrix} a_1\cdot\tau_{\mathcal{M}}(m_2) + m_1\cdot D_{\mathcal{B}}(b_2) + D_{\mathcal{A}}(a_1)\cdot m_2 \\ +\tau_{\mathcal{M}}(m_1)\cdot b_2 + \gamma_2(T_1, T_2) \end{matrix} \\ & b_1D_{\mathcal{B}}(b_2) + D_{\mathcal{B}}(b_1)b_2 + \tau_{\mathcal{B}}(b_1, b_2) \end{bmatrix}.
\end{aligned}$$

As well as, together with cases (6) and (7) of Theorem 2.1 and (2.2) we have

$$\begin{aligned}
D(T_1T_2) &= D\Big(\begin{bmatrix} a_1a_2 & a_1\cdot m_2 + m_1\cdot b_2 \\ & b_1b_2 \end{bmatrix}\Big)\\
&= \begin{bmatrix} D_{\mathcal{A}}(a_1a_2) & \tau_{\mathcal{M}}(a_1\cdot m_2 + m_1\cdot b_2) \\ & D_{\mathcal{B}}(b_1b_2) \end{bmatrix}\\
&= \begin{bmatrix} a_1D_{\mathcal{A}}(a_2) + D_{\mathcal{A}}(a_1)a_2 + \tau_{\mathcal{A}}(a_1, a_2) & \begin{matrix} a_1\cdot\tau_{\mathcal{M}}(m_2) + m_1\cdot D_{\mathcal{B}}(b_2) + D_{\mathcal{A}}(a_1)\cdot m_2 \\ +\tau_{\mathcal{M}}(m_1)\cdot b_2 + \gamma_2(T_1, T_2) \end{matrix} \\ & b_1D_{\mathcal{B}}(b_2) + D_{\mathcal{B}}(b_1)b_2 + \tau_{\mathcal{B}}(b_1, b_2) \end{bmatrix}.
\end{aligned}$$

Thus D is a generalized derivation associate with 2-cocycle γ. □

If $\mathcal{M} = 0$, then triangular Banach algebra reformed to $\mathcal{A}\oplus_1\mathcal{B}$ with the following sum and product

$$(a, b) + (a', b') = (a + a, b + b'), \text{ and } (a, b)(a', b') = (aa, bb'),$$

for every $(a,b),(a',b') \in \mathcal{A}\oplus_1\mathcal{B}$. It become a Banach algebra with defined norm as $\|(a,b)\| = \|a\|_{\mathcal{A}} + \|b\|_{\mathcal{B}}$. We set $\mathfrak{A} = \mathcal{A}\oplus_1\mathcal{B}$. Let $\gamma \in Z^2(\mathfrak{A},\mathfrak{A})$, $\gamma_1 : \mathfrak{A}\times\mathfrak{A} \longrightarrow \mathcal{A}$ and $\gamma_2 : \mathfrak{A}\times\mathfrak{A} \longrightarrow \mathcal{B}$ be the coordinate functions associated to γ that is

$$\gamma((a_1,b_1),(a_2,b_2)) = (\gamma_1((a_1,b_2),(a_2,b_2)),\gamma_2((a_1,b_2),(a_2,b_2))),$$

for all $(a_1,b_1),(a_2,b_2)\in\mathfrak{A}$. If γ_1 corresponds to a 2-cocycle on $\mathcal{A}\times\mathcal{A}$ then there exists a 2-cocycle $\tau_{\mathcal{A}}$ on $\mathcal{A}\times\mathcal{A}$ such that $\gamma_1((a_1,b_1),(a_2,b_2)) = \tau_{\mathcal{A}}(a_1,a_2)$. Similarly, if γ_2 corresponds to a 2-cocycle on $\mathcal{B}\times\mathcal{B}$ then there exists a 2-cocycle $\tau_{\mathcal{B}}$ on $\mathcal{B}\times\mathcal{B}$ such that $\gamma_1((a_1,b_1),(a_2,b_2)) = \tau_{\mathcal{B}}(b_1,b_2)$. We reduce the Lemma 1.2 into this Banach algebra as follows.

Lemma 2.4. *Let $\gamma\in B^2(\mathfrak{A},\mathfrak{A})$. Then γ is a 2-cocycle if and only if there are 2-cocycles $\tau_{\mathcal{A}}$ and $\tau_{\mathcal{B}}$ on $\mathcal{A}\times\mathcal{A}$ and $\mathcal{B}\times\mathcal{B}$, respectively, such that $\gamma = (\tau_A,\tau_{\mathcal{B}})$.*

Proof. The case where γ is a 2-cocycle existence of $\tau_{\mathcal{A}}$ and $\tau_{\mathcal{B}}$ are clear by the same reasoning in Lemma 1.2. For converse, let $\tau_{\mathcal{A}}\in Z^2(\mathcal{A},\mathcal{A})$ and $\tau_{\mathcal{B}}\in Z^2(\mathcal{B},\mathcal{B})$. We shall show that $\gamma = (\tau_A,\tau_{\mathcal{B}})$ belongs to $Z^2(\mathfrak{A},\mathfrak{A})$. For every $(a_1,b_1),(a_2,b_2),(a_3,b_3)\in\mathfrak{A}$,

$$\begin{aligned}
&(a_1,b_1)\gamma((a_2,b_2),(a_3,b_3)) - \gamma((a_1,b_1)(a_2,b_2),(a_3,b_3)) + \gamma((a_1,b_1),(a_2,b_2)(a_3,b_3))\\
&\qquad -\gamma((a_1,b_1),(a_2,b_2))(a_3,b_3)\\
&= (a_1,b_1)\gamma((a_2,b_2),(a_3,b_3)) - \gamma((a_1a_2,b_1b_2),(a_3,b_3)) + \gamma((a_1,b_1),(a_2a_3,b_2b_3))\\
&\qquad -\gamma((a_1,b_1),(a_2,b_2))(a_3,b_3)\\
&= (a_1,b_1)(\gamma_1((a_2,b_2),(a_3,b_3)),\gamma_2((a_2,b_2),(a_3,b_3)))\\
&\qquad -(\gamma_1((a_1a_2,b_2b_2),(a_3,b_3)),\gamma_2((a_1a_2,b_2b_2),(a_3,b_3)))\\
&\qquad +(\gamma_1((a_1,b_1),(a_2a_3,b_2b_3)),\gamma_2((a_1,b_1),(a_2a_3,b_2b_3)))\\
&\qquad -(\gamma_1((a_1,b_1),(a_2,b_2)),\gamma((a_1,b_1),(a_2,b_2)))(a_3,b_3)\\
&= (a_1,b_1)(\tau_{\mathcal{A}}((a_2,a_3)),\tau_{\mathcal{B}}((b_2,b_3))) - (\tau_{\mathcal{A}}((a_1a_2,a_3)),\tau_{\mathcal{B}}((b_1b_2,b_3)))\\
&\qquad +(\tau_{\mathcal{A}}((a_1,a_2a_3)),\tau_{\mathcal{B}}((b_1,b_2b_3))) - (\tau_{\mathcal{A}}((a_1,a_2)),\tau_{\mathcal{B}}((b_1,b_2)))(a_3,b_3)\\
&= (a_1\tau_{\mathcal{A}}((a_2,a_3)) - \tau_{\mathcal{A}}((a_1a_2,a_3)), b_1\tau_{\mathcal{B}}((b_2,b_3)) - \tau_{\mathcal{B}}((b_1b_2,b_3)))\\
&\qquad +(\tau_{\mathcal{A}}((a_1,a_2a_3)) - \tau_{\mathcal{A}}((a_1,a_2))a_3, \tau_{\mathcal{B}}((b_1,b_2b_3)) - \tau_{\mathcal{B}}((b_1,b_2))b_3)\\
&= 0.
\end{aligned}$$

Thus γ is a 2-cocycle. □

Proposition 2.5. *Let $\mathfrak{A} = \mathcal{A}\oplus_1\mathcal{B}$, where $\mathcal{A}$ and $\mathcal{B}$ are Banach algebras. If $D : \mathfrak{A}\longrightarrow\mathfrak{A}$ is a generalized derivation associate with 2-cocycle γ, then there are generalized derivation $D_{\mathcal{A}}$ and $D_{\mathcal{B}}$ associate with 2-cocycles $\tau_{\mathcal{A}}$ and $\tau_{\mathcal{B}}$, respectively, such that $\gamma = (\tau_{\mathcal{A}},\tau_{\mathcal{B}})$.*

Proof. Define $D_{\mathcal{A}} : \mathcal{A}\longrightarrow\mathcal{A}$ by $D_{\mathcal{A}}(a) = e_{\mathcal{A}}D((a,0))e_{\mathcal{A}}$ for all $a\in\mathcal{A}$. By Lemma 2.4, there are 2-cocycles $\tau_{\mathcal{A}}$ and $\tau_{\mathcal{B}}$ such that $\gamma((a_1,b_1),(a_2,b_2)) = (\tau_{\mathcal{A}}(a_1,a_2),\tau_{\mathcal{B}}(b_1,b_2))$ for all $(a_1,b_1),(a_2,b_2)\in\mathfrak{A}$.

Then

$$\begin{aligned} D_{\mathcal{A}}(a_1a_2) &= e_{\mathcal{A}}D((a_1a_2,0))e_{\mathcal{A}} = e_{\mathcal{A}}D((a_1,0)(a_2,0))e_{\mathcal{A}} \\ &= e_{\mathcal{A}}(a_1,0)D((a_2,0))e_{\mathcal{A}} + e_{\mathcal{A}}D((a_1,0))(a_2,0)e_{\mathcal{A}} + e_{\mathcal{A}}\gamma((a_1,0),(a_2,0))e_{\mathcal{A}} \\ &= e_{\mathcal{A}}(a_1,0)D((a_2,0))e_{\mathcal{A}} + e_{\mathcal{A}}D((a_1,0))(a_2,0)e_{\mathcal{A}} + e_{\mathcal{A}}(\tau_{\mathcal{A}}(a_1,a_2),0)e_{\mathcal{A}} \\ &= a_1D_{\mathcal{A}}(a_2) + D_{\mathcal{A}}(a_1)a_2 + \tau_{\mathcal{A}}(a_1,a_2), \end{aligned}$$

for every $a_1, a_2 \in \mathcal{A}$. Similarly, if we define $D_{\mathcal{B}} : \mathcal{B} \longrightarrow \mathcal{B}$ by $D_{\mathcal{B}}(b) = e_{\mathcal{B}}D((0,b))e_{\mathcal{B}}$ for all $b \in \mathcal{B}$, then by the same reasoning for proof of $D_{\mathcal{A}}$, $D_{\mathcal{B}}$ become a generalized derivation associate with 2-cocycle $\tau_{\mathcal{B}}$. □

PROPOSITION 2.6. *Let $\mathfrak{A} = \mathcal{A} \oplus_1 \mathcal{B}$, where $\mathcal{A}$ and $\mathcal{B}$ are Banach algebras. If $D_{\mathcal{A}}$ and $D_{\mathcal{B}}$ are generalized derivation associate with 2-cocycles $\tau_{\mathcal{A}}$ and $\tau_{\mathcal{B}}$, respectively, then $D : \mathfrak{A} \longrightarrow \mathfrak{A}$ defined by $D = (D_{\mathcal{A}}, D_{\mathcal{B}})$ is a generalized derivation associate with 2-cocycle $\gamma = (\tau_{\mathcal{A}}, \tau_{\mathcal{B}})$.*

PROOF. By Lemma 2.4, $\gamma = (\tau_{\mathcal{A}}, \tau_{\mathcal{B}})$ is a 2-cocycle. Then

$$\begin{aligned} D((a_1,b_1)(a_2,b_2)) &= D((a_1a_2, b_1b_2)) = (D_{\mathcal{A}}(a_1a_2), D_{\mathcal{B}}(b_1b_2)) \\ &= (a_1D_{\mathcal{A}}(a_2) + D_{\mathcal{A}}(a_1)a_2 + \tau_{\mathcal{A}}(a_1,a_2), b_1D_{\mathcal{B}}(ba_2) + D_{\mathcal{B}}(b_1)b_2 + \tau_{\mathcal{B}}(b_1,b_2)), \end{aligned}$$

and

$$\begin{aligned} &(a_1,b_1)D((a_2,b_2)) + D((a_1,b_1))(a_2,b_2) + \gamma((a_1,b_1),(a_2,b_2)) \\ &\quad = (a_1D_{\mathcal{A}}(a_2), b_1D_{\mathcal{B}}(ba_2)) + (D_{\mathcal{A}}(a_1)a_2, D_{\mathcal{B}}(b_1)b_2) + (\tau_{\mathcal{A}}(a_1,a_2), \tau_{\mathcal{B}}(b_1,b_2)), \end{aligned}$$

for all $(a_1,b_1),(a_2,b_2) \in \mathfrak{A}$. Thus, D is a generalized derivation associate with 2-cocycle γ. □

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