Characterization of Generalized Derivations
Associate with Hochschild 2-Cocycles on
Triangular Banach Algebras

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Abstract. Let $A$ and $B$ be unital Banach algebras and $M$ be a left $A$-module and right $B$-module. We consider generalized derivations associated with Hochschild 2-cocycles on triangular Banach algebra $T$ (related to $A$, $B$ and $M$). We characterize this new version of generalized derivations on triangular Banach algebras and we obtain some results for $l^1$ direct summands of Banach algebras.

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1. Introduction

Let $A$ be a Banach algebra, and let $X$ be a Banach $A$-bimodule. A derivation is a linear map $D : A \rightarrow X$ such that

$$D(ab) = a \cdot D(b) + D(a) \cdot b \quad (a, b \in A).$$

For $x \in X$, set $ad_x : a \mapsto a \cdot x - x \cdot a, A \rightarrow X$. Then $ad_x$ is the inner derivation induced by $x$. 

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The linear space of bounded derivations from $\mathcal{A}$ into $X$ denoted by $Z^1(\mathcal{A}, X)$ and the linear subspace of inner derivations denoted by $N^1(\mathcal{A}, X)$. We consider the quotient space $H^1(\mathcal{A}, X) = Z^1(\mathcal{A}, X)/N^1(\mathcal{A}, X)$, called the first Hochschild cohomology group of $\mathcal{A}$ with coefficients in $X$.

Let $\mathcal{A}$ be a Banach algebra, and let $X$ be a Banach $\mathcal{A}$-bimodule. By $B^n(\mathcal{A}, X)$, we mean that the space of bounded $n$-linear maps form $\mathcal{A}^n$ into $X$. A 2-linear map $\gamma \in B^2(\mathcal{A}, X)$ is called Hochschild 2-cocycle if it satisfies in the following equation

$$a \cdot \gamma(b, c) - \gamma(ab, c) + \gamma(a, bc) - \gamma(a, b) \cdot c = 0,$$

for every $a, b, c \in \mathcal{A}$. The space of Hochschild 2-cocycles is a linear subspace of $B^2(\mathcal{A}, X)$, which denoted by $Z^2(\mathcal{A}, X)$. Here in after we used the word 2-cocycle instead Hochschild 2-cocycle. Let $\varphi \in \Delta \mathcal{A}$, where $\Delta(\mathcal{A})$ is the carrier space of $\mathcal{A}$, then A 2-linear map $\gamma \in B^2(\mathcal{A}, X)$ is called point 2-cocycle at $\varphi$ if it satisfies in the following equation

$$\varphi(a)\gamma(b, c) - \gamma(ab, c) + \gamma(a, bc) - \gamma(a, b)\varphi(c) = 0,$$

for every $a, b, c \in \mathcal{A}$. For given $T \in B(\mathcal{A}, X)$, we let

$$(\delta^1T)(a, b) = a \cdot T(b) - T(ab) + T(a) \cdot b,$$

for every $a, b \in \mathcal{A}$ and $\delta^1 : B(\mathcal{A}, X) \rightarrow B^2(\mathcal{A}, X)$. Then the maps $\{\delta^1T : T \in B(\mathcal{A}, X)\}$ is a linear subspace of $Z^2(\mathcal{A}, X)$. These maps are called 2-coboundaries. The collection of all 2-coboundaries is denoted by $N^2(\mathcal{A}, X)$.

An additive map $D : \mathcal{A} \rightarrow X$ called generalized 2-cocycle derivation if there exists a 2-cocycle $\gamma$ such that

$$D(xy) = x \cdot D(y) + D(x) \cdot y + \gamma(x, y)$$

(1)

for every $x, y \in \mathcal{A}$. Similarly, an additive map $D : \mathcal{A} \rightarrow X$ called generalized 2-cocycle Jordan derivation if there exists a 2-cocycle $\gamma$ such that

$$D(x^2) = x \cdot D(x) + D(x) \cdot y + \gamma(x, x)$$

(2)

for every $x \in \mathcal{A}$. This definitions introduced by Nakajima in [5] and he gave some examples for this new definition. In [3] authors considered
this new notion for some algebras such as von-Neumann and they showed that generalized Jordan derivation of this type from von-Neumann algebras into themselves is a generalized derivation (under some conditions). Similar result obtained by authors in [4] for triangular algebras.

An additive map $D : \mathcal{A} \rightarrow X$ called generalized 2-coboundry derivation if there exists a 2-coboundary $\delta^1$ such that

$$D(xy) = x \cdot D(y) + D(x) \cdot y + (\delta^1 F)(x, y),$$

for every $x, y \in \mathcal{A}$ and $F \in B(\mathcal{A}, X)$. Let $\mathcal{A}$ and $\mathcal{B}$ be unital Banach algebras with units $e_\mathcal{A}$ and $e_\mathcal{B}$, respectively. Suppose that $\mathcal{M}$ is a unital Banach $\mathcal{A}, \mathcal{B}$-bimodule. We define the triangular Banach algebra

$$\mathcal{T} = \begin{bmatrix} \mathcal{A} & \mathcal{M} \\ \mathcal{B} \end{bmatrix},$$

with the sum and product being given by the usual $2 \times 2$ matrix operations and internal module actions. The norm on $\mathcal{T}$ is

$$\| \begin{bmatrix} a & m \\ b \end{bmatrix} \| = \|a\|_\mathcal{A} + \|m\|_\mathcal{M} + \|b\|_\mathcal{B}. $$

The Banach algebra $\mathcal{T}$ as a Banach space is isomorphic to the $\ell^1$-direct sum of $\mathcal{A}, \mathcal{B}$ and $\mathcal{M}$. Forrest and Marcoux introduced and studied derivation of triangular Banach algebras in [1].

Let $\mathcal{T}$ be a triangular Banach algebras defined as above, and let $\gamma \in B^2(\mathcal{T}, \mathcal{T})$. Let $\gamma_1 : \mathcal{T} \times \mathcal{T} \rightarrow \mathcal{A}, \gamma_2 : \mathcal{T} \times \mathcal{T} \rightarrow \mathcal{M}$ and $\gamma_3 : \mathcal{T} \times \mathcal{T} \rightarrow \mathcal{B}$ denote the coordinate functions associated to $\gamma$. That is

$$\gamma(T_1, T_2) = \begin{bmatrix} \gamma_1(T_1, T_2) & \gamma_2(T_1, T_2) \\ \gamma_3(T_1, T_2) \end{bmatrix},$$

for $T_1, T_2 \in \mathcal{T}$. Let $\gamma : \mathcal{T} \times \mathcal{T} \rightarrow \mathcal{T}$ be a 2-cocycle. The coordinate function $\gamma_1$ is said to correspond to a 2-cocycle (2-coboundary) on $\mathcal{A} \times \mathcal{A}$ if there exists a 2-cocycle (2-coboundary) $\tau_\mathcal{A}$ on $\mathcal{A} \times \mathcal{A}$ such that $\gamma_1(T_1, T_2) = \tau_\mathcal{A}(a_1, a_2)$, where $T_i = \begin{bmatrix} a_i & m_i \\ b_i \end{bmatrix}$, for $i = 1, 2$. 
Similarly, $\gamma_3$ is said to correspond to a 2-cocycle (2-coboundary) on $B \times B$ if there exists a 2-cocycle (2-coboundary) $\tau_B$ on $B \times B$ such that $\gamma_3(T_1,T_2) = \tau_B(b_1,b_2)$.

**Definition 1.1.** Let $\gamma \in B^2(T,T)$, $\gamma_1 : T \times T \to A$, $\gamma_2 : T \times T \to M$ and $\gamma_3 : T \times T \to B$ denote the coordinate functions associated to $\gamma$. That is $\gamma(T_1,T_2) = \left[ \begin{array}{c} \gamma_1(T_1,T_2) \\ \gamma_2(T_1,T_2) \\ \gamma_3(T_1,T_2) \end{array} \right]$, for $T_1, T_2 \in T$. Let $\gamma \in B^2(T,T)$ be a 2-cocycle (2-coboundries). We say that $\gamma_1$ corresponds to a 2-cocycle (2-coboundries) on $A \times A$ if there exists a 2-cocycle (2-coboundries) $\tau_A$ on $A \times A$ such that $\gamma_1(T_1,T_2) = \tau_A(a_1,a_2)$, where $T_i = \left[ \begin{array}{c} a_i \\ m_i \\ b_i \end{array} \right]$, for $i = 1,2$.

Similarly, we say that $\gamma_3$ corresponds to a 2-cocycle (2-coboundries) on $B \times B$ if there exists a 2-cocycle (2-coboundries) $\tau_B$ on $B \times B$ such that $\gamma_3(T_1,T_2) = \tau_B(b_1,b_2)$. Second order cohomology of triangular Banach algebras studied by Forrest and Marcoux in [2].

**Lemma 1.2.** Let $\gamma \in B^2(T,T)$ be a 2-cocycle. Then there are continuous corresponding 2-cocycles $\tau_A$ and $\tau_B$ on $A \times A$ and $B \times B$, respectively. Furthermore

1. $\tau_A(a,0) = \tau_B(0,b) = \tau_A(e_A,0) = \tau_B(e_A,0) = \tau_A(0,0) = \tau_B(0,0) = 0$.
2. $\gamma_2(e_{11},e_{11}) = 0, \gamma_2(e_{11},0) = 0$.
3. $\gamma_2(e_{22},e_{22}) = 0$.
4. $\tau_A(a,e_A) = 0$ and $\tau_A(e_A,e_A) = 0$.
5. $\tau_B(b,e_B) = 0$ and $\tau_B(e_B,e_B) = 0$.
6. $\gamma_2(b_{22},e_{22}) = 0$ and $\gamma_2(e_{11},a_{11}) = 0$.

where $e_{11} = \left[ \begin{array}{c} e_A \\ 0 \\ 0 \end{array} \right]$, $e_{22} = \left[ \begin{array}{c} 0 \\ 0 \\ e_B \end{array} \right]$, $b_{22} = \left[ \begin{array}{c} 0 \\ 0 \\ b \end{array} \right]$, $a_{11} = \left[ \begin{array}{c} a \\ 0 \\ 0 \end{array} \right]$ and for $a \in A, b \in B$. 

Proof. Define $\tau_A : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$ and $\tau_B : \mathcal{B} \times \mathcal{B} \rightarrow \mathcal{B}$ as follows

$$
\tau_A(a_1, a_2) = e_A \gamma \left( \begin{bmatrix} a_1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} a_2 & 0 \\ 0 & 0 \end{bmatrix} \right) e_A,
$$

and

$$
\tau_B(b_1, b_2) = e_B \gamma \left( \begin{bmatrix} 0 & 0 \\ b_1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & b_2 \end{bmatrix} \right) e_B.
$$

It is easy to check that $\tau_A$ and $\tau_B$ are 2-cocycle. Continuity of $\tau_A$ and $\tau_B$ is clear. Since $\tau_A$ and $\tau_B$ are 2-linear so (1) is clear. For the rest consider the following

$$
\gamma \left( \begin{bmatrix} e_A & 0 \\ 0 & e_B \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ e_B & 0 \end{bmatrix} \right) = \begin{bmatrix} \tau_A(e_A, 0) & \gamma_2(e_{11}, e_{22}) \\ \tau_B(0, e_B) & 0 \end{bmatrix},
$$

and

$$
e_{11} \gamma(e_{11}, e_{22}) - \gamma(e_{11}, e_{22}) + \gamma(e_{11}, 0) - \gamma(e_{11}, e_{11})e_{22} = 0.
$$

Then,

$$
\begin{bmatrix} \tau_A(e_A, 0) & \gamma_2(e_{11}, e_{22}) \\ \tau_B(0, e_B) & 0 \end{bmatrix} - \begin{bmatrix} \tau_A(e_A, 0) & \gamma_2(e_{11}, e_{22}) \\ \tau_B(0, e_B) & 0 \end{bmatrix}
$$

$$
+ \begin{bmatrix} \tau_A(e_A, 0) & \gamma_2(e_{11}, 0) \\ 0 & \gamma_2(e_{11}, e_{11}) \end{bmatrix} = 0.
$$

This follows that $\gamma_2(e_{11}, e_{11}) + \gamma_2(e_{11}, 0) = 0$. From

$$
e_{11} \gamma(e_{11}, e_{11}) - \gamma(e_{11}, e_{11}) + \gamma(e_{11}, e_{11}) - \gamma(e_{11}, e_{11})e_{11} = 0,
$$

we conclude that $\gamma_2(e_{11}, e_{11}) = 0$. This implies that $\gamma_2(e_{11}, 0) = 0$. Similarly, one can show that $\gamma_2(e_{22}, e_{22}) = 0$. From

$$
a_{11} \gamma(e_{11}, e_{11}) - \gamma(a_{11}, e_{11}) + \gamma(a_{11}, e_{11}) - \gamma(a_{11}, e_{11})e_{11} = 0,
$$

we conclude that $\tau_A(a, e_A) = 0$. Since $a$ was arbitrary so $\tau_A(e_A, e_A) = 0$. By the similar methods, we obtain the other cases. □
Let $T$ be a triangular Banach algebra and let $\mathcal{X}$ be a unital Banach $T$-bimodule, then we use these notations in this paper: $\mathcal{X}_{AA} = e_A \cdot \mathcal{X} \cdot e_A$, $\mathcal{X}_{BB} = e_B \cdot \mathcal{X} \cdot e_B$, $\mathcal{X}_{AB} = e_A \cdot \mathcal{X} \cdot e_B$, and $\mathcal{X}_{BA} = e_B \cdot \mathcal{X} \cdot e_A$. If $\mathcal{X}$ replaced by $T$, we have $\mathcal{X}_{AA} = A, \mathcal{X}_{BB} = B, \mathcal{X}_{AB} = M,$ and $\mathcal{X}_{BA} = 0$.

**Lemma 1.3.** Let $\mathcal{X}$ be a $\mathbb{T}$-bimodule, $\delta_A : A \times A \to \mathcal{X}_{AA}$, $\delta_B : B \times B \to \mathcal{X}_{BB}$ be 2-cocycles, and $\mathcal{X}_{AB} = 0$. Then there exists a 2-cocycle mapping from $\mathbb{T} \times \mathbb{T}$ into $\mathcal{X}$.

**Proof.** For every $\begin{bmatrix} a_1 & m_1 \\ b_1 & 1 \end{bmatrix}, \begin{bmatrix} a_2 & m_2 \\ b_2 & 1 \end{bmatrix} \in \mathbb{T}$, define $D : \mathbb{T} \times \mathbb{T} \to \mathcal{X}$ by

\[
D \left( \begin{bmatrix} a_1 & m_1 \\ b_1 & 1 \end{bmatrix}, \begin{bmatrix} a_2 & m_2 \\ b_2 & 1 \end{bmatrix} \right) = \delta_A(a_1,a_2) + \delta_B(b_1,b_2).
\]

We claim that $D$ is a 2-cocycle. Because for every $T_i = \begin{bmatrix} a_i & m_i \\ b_i & 1 \end{bmatrix}, (i = 1, 2, 3)$, we have

\[
T_1 \cdot D(T_2,T_3) - D(T_1T_2,T_3) + D(T_1,T_2T_3) - D(T_1,T_2) \cdot T_3
\]

\[
= T_1 \cdot (\delta_A(a_2,a_3) + \delta_B(b_2,b_3)) - \delta_A(a_1a_2,a_3) - \delta_B(b_1b_2,b_3)
\]

\[
+ \delta_A(a_1,a_2a_3) + \delta_B(b_1,b_2b_3) - (\delta_A(a_1,a_2) + \delta_B(b_1,b_2)) \cdot T_3
\]

\[
= T_1 \cdot \tau(e_A)\delta_A(a_2,a_3) - \delta_A(a_1a_2,a_3) - \delta_A(a_1,a_2a_3) - \delta_A(a_1,a_2) \cdot e_A \cdot T_3
\]

\[
+ T_1 \cdot \tau(e_B)\delta_B(b_2,b_3) - \delta_B(b_1b_2,b_3) + \delta_B(b_1,b_2b_3) - \delta_B(b_1,b_2) \cdot e_B \cdot T_3
\]

\[
= a_1 \cdot \delta_A(a_2,a_3) - \delta_A(a_1a_2,a_3) + \delta_A(a_1,a_2a_3) - \delta_A(a_1,a_2) \cdot a_3
\]

\[
+ b_1 \cdot \delta_B(b_2,b_3) - \delta_B(b_1b_2,b_3) + \delta_B(b_1,b_2b_3) - \delta_B(b_1,b_2) \cdot b_3
\]

\[
= 0.
\]

This proves our claim. \qed

**Lemma 1.4.** [2, Lemma 3.1] Let $\delta_A$ and $\delta_B$ be 2-coboundaries on $A \times A$ and $B \times B$, respectively. Then there exists a 2-coboundary $\delta$ on $\mathbb{T} \times \mathbb{T}$ such that $\delta_1$ corresponds to $\delta_A$ and $\delta_2$ corresponds to $\delta_B$, where $\delta_1$ and $\delta_2$ are coordinate functions associated to $\delta$. 
2. Characterization of Generalized 2-Cocycle Derivations

In this section we prove main results of paper. We characterize generalized 2-cocycle derivations on triangular Banach algebras and by taking $\mathcal{M} = 0$ we consider generalized 2-cocycle derivations on $A \oplus_1 B$, where $A$ and $B$ are Banach algebras.

**Theorem 2.1.** Let $T$ be a triangular Banach algebra and $A, B$ be unital Banach algebras with units $e_A$ and $e_B$, respectively, and $M$ be a unital Banach $A, B$-bimodule. Let $D : T \to T$ be a generalized 2-cocycle derivation associate with $\gamma$, then there exist element $m_D \in M$, corresponding 2-cocycles $\tau_A$ and $\tau_B$ on $A \times A$ and $B \times B$, respectively, and mappings $D_A : A \to A$, $D_B : B \to B$ and $\tau_M : M \to M$ such that

1. $D\left(\begin{bmatrix} e_A & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & m_D \\ 0 & 0 \end{bmatrix}$.

2. $D\left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} D_A(a) & am_D + \gamma_2(a_{11}, e_{11}) \\ 0 & 0 \end{bmatrix}$.

3. $D\left(\begin{bmatrix} 0 & 0 \\ e_B & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & -m_D - \gamma_2(e_{11}, e_{22}) \\ 0 & 0 \end{bmatrix}$.

4. $D\left(\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}\right) = \begin{bmatrix} 0 & -m_Db - \gamma_2(e_{11}, b_{22}) \\ D_B(b) & 0 \end{bmatrix}$

5. $D\left(\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & \tau_M(m) \\ 0 & 0 \end{bmatrix}$.

6. $\gamma_1\left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}\right) = 0$, $\gamma_3\left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}\right) = 0$ and

$$\tau_M(a \cdot m) = a \cdot \tau_M(m) + D_A(a) \cdot m + \gamma_2\left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}\right).$$
7. \( \gamma_1\left(\begin{bmatrix} 0 & m \\ 0 & b \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right) = 0 \), \( \gamma_3\left(\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}\right) = 0 \) and \( \tau_M(m \cdot b) = \tau_M(m) \cdot b + m \cdot D_B(b) + \gamma_2\left(\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}\right)\).

Furthermore, \( D_A : A \rightarrow A \) and \( D_B : B \rightarrow B \) are generalized 2-cocycle derivations associate with 2-cocycles \( \tau_A \) and \( \tau_B \), respectively.

**Proof.** Let \( \gamma \) be the 2-cocycle related to \( D \). Let \( \gamma_1, \gamma_2 \) and \( \gamma_3 \) be the coordinate functions associated to \( \gamma \). By Lemma 1.2, there exists corresponding 2-cocycles \( \tau_A \) and \( \tau_B \), respectively on \( A \times A \) and \( B \times B \) (continuity of these maps do not need).

(1) Put \( e_{11} = \begin{bmatrix} e_A & 0 \\ 0 & 0 \end{bmatrix} \) and let \( D(e_{11}) = \begin{bmatrix} p & q \\ r & 0 \end{bmatrix} \). Then by Lemma 1.2

\[
D(e_{11}) = D(e_{11}e_{11}) = e_{11}D(e_{11}) + D(e_{11})e_{11} + \gamma(e_{11}, e_{11}) = \begin{bmatrix} e_A & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} p & q \\ r & 0 \end{bmatrix} + \begin{bmatrix} p & q \\ r & 0 \end{bmatrix}\begin{bmatrix} e_A & 0 \\ 0 & 0 \end{bmatrix} + \gamma_1(e_{11}, e_{11}) + \gamma_3(e_{11}, e_{11}) = \begin{bmatrix} 2p & q \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \tau_A(e_A, e_A) & 0 \\ \tau_B(0, 0) & \end{bmatrix}.
\]

Above relations follow that

\[
D(e_{11}) = \begin{bmatrix} 0 & q \\ 0 & 0 \end{bmatrix}.
\]

Set \( q = m_D \), then

\[
D(e_{11}) = \begin{bmatrix} 0 & m_D \\ 0 & 0 \end{bmatrix}.
\]
(2) Put $a_{11} = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$ and $e_{11} = \begin{bmatrix} e_A & 0 \\ 0 & 0 \end{bmatrix}$. Let $D(a_{11}) = \begin{bmatrix} p & q \\ r & 0 \end{bmatrix}$, then

$$D(a_{11}) = D(a_{11}e_{11}) = a_{11}D(e_{11}) + D(a_{11})e_{11} + \gamma(a_{11}, e_{11})$$

$$= \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & m_D \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} p & q \\ r & 0 \end{bmatrix} \begin{bmatrix} e_A & 0 \\ 0 & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} \gamma_1(a_{11}, e_{11}) & \gamma_2(a_{11}, e_{11}) \\ \gamma_3(a_{11}, e_{11}) & \gamma_4(a_{11}, e_{11}) \end{bmatrix}$$

$$= \begin{bmatrix} p & am_D \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \tau_A(a, e_A) & \gamma_2(a_{11}, e_{11}) \\ \tau_B(0, 0) & 0 \end{bmatrix}$$

$$= \begin{bmatrix} p & am_D + \gamma_2(a_{11}, e_{11}) \\ 0 & 0 \end{bmatrix}.$$ 

Therefore

$$D(a_{11}) = \begin{bmatrix} D_A(a) & am_D + \gamma_2(a_{11}, e_{11}) \\ 0 & 0 \end{bmatrix},$$

where $D_A(a) = p$.

(3) Set $e_{22} = \begin{bmatrix} 0 & 0 \\ e_B & 0 \end{bmatrix}$ and let $D(e_{22}) = \begin{bmatrix} p & q' \\ r & 0 \end{bmatrix}$. Then

$$D(e_{22}) = D(e_{22}e_{22}) = e_{22}D(e_{22}) + D(e_{22})e_{22} + \gamma(e_{22}, e_{22})$$

$$= \begin{bmatrix} 0 & 0 \\ e_B & 0 \end{bmatrix} \begin{bmatrix} p & q' \\ r & 0 \end{bmatrix} + \begin{bmatrix} p & q' \\ r & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ e_B & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} \gamma_1(e_{22}, e_{22}) & \gamma_2(e_{22}, e_{22}) \\ \gamma_3(e_{22}, e_{22}) & \gamma_4(e_{22}, e_{22}) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & q' \\ 2r & 0 \end{bmatrix} + \begin{bmatrix} \tau_A(0, 0) & \gamma_2(e_{22}, e_{22}) \\ \tau_B(e_B, e_B) & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & q' \\ 2r & 0 \end{bmatrix}.$$ 

Above relations follow that

$$D(e_{22}) = \begin{bmatrix} 0 & q' \\ 0 & 0 \end{bmatrix}. $$
Then

\[
0 = D(0) = D(e_{11}e_{22}) = e_{11}D(e_{22}) + D(e_{11})e_{22} + \gamma(e_{11}, e_{22})
\]

\[
= \begin{bmatrix}
0 & q' \\
0 & 0
\end{bmatrix} + \begin{bmatrix}
0 & m_D \\
0 & 0
\end{bmatrix} + \begin{bmatrix}
0 & \gamma_2(e_{11}, e_{22}) \\
0 & 0
\end{bmatrix}
\]

\[
= \begin{bmatrix}
0 & m_D + q' + \gamma_2(e_{11}, e_{22}) \\
0 & 0
\end{bmatrix}.
\]

(1)

and

\[
0 = D(0) = D(e_{22}e_{11}) = e_{22}D(e_{11}) + D(e_{22})e_{11} + \gamma(e_{22}, e_{11})
\]

\[
= \begin{bmatrix}
0 & \gamma_2(e_{22}, e_{11}) \\
0 & 0
\end{bmatrix}.
\]

This follows that \(\gamma_2(e_{22}, e_{11}) = 0\). Then we have \(q' = -m_D - \gamma_2(e_{11}, e_{22})\).

(4) Set \(e_{22} = \begin{bmatrix} 0 & 0 \\ e_B & 0 \end{bmatrix}\), \(b_{22} = \begin{bmatrix} 0 & 0 \\ b & 0 \end{bmatrix}\) and let \(D(b_{22}) = \begin{bmatrix} p & q \\ r & 0 \end{bmatrix}\).

Then by Lemma 1.2 we have

\[
D(b_{22}) = D(b_{22}e_{22}) = b_{22}D(e_{22}) + D(b_{22})e_{22} + \gamma(b_{22}, e_{22})
\]

\[
= \begin{bmatrix}
0 & 0 \\
0 & b_{22}
\end{bmatrix} \begin{bmatrix}
0 & -m_D - \gamma_2(e_{11}, e_{22}) \\
0 & 0
\end{bmatrix} + \begin{bmatrix}
p & q \\
r & 0
\end{bmatrix} \begin{bmatrix}
0 & 0 \\
e_B & 0
\end{bmatrix}
\]

\[
+ \begin{bmatrix}
\gamma_1(b_{22}, e_{22}) & \gamma_2(b_{22}, e_{22}) \\
\gamma_3(b_{22}, e_{22}) & 0
\end{bmatrix}
\]

\[
= \begin{bmatrix}
0 & q \\
r & 0
\end{bmatrix} + \begin{bmatrix}
\tau_A(0, 0) & \gamma_2(b_{22}, e_{22}) \\
\tau_B(b, e_B) & 0
\end{bmatrix}
\]

\[
= \begin{bmatrix}
0 & q \\
r & 0
\end{bmatrix}.
\]

By \(0 = D(0) = D(e_{11}b_{22})\) we conclude that \(q = -m_Db - \gamma_2(e_{11}, b_{22})\). If we set \(D_B(b) = r\) we obtain the desire.

(5) Set \(e_{11} = \begin{bmatrix} e_A & 0 \\ 0 & 0 \end{bmatrix}\), \(e_{22} = \begin{bmatrix} 0 & 0 \\ e_B & 0 \end{bmatrix}\), \(m_{12} = \begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}\) and let
$D(m_{12}) = \begin{bmatrix} p & q \\ r & 0 \end{bmatrix}$. Then by Lemma 1.2 we have

\[
D(m_{12}) = D(m_{12}e_{22}) = m_{12}D(e_{22}) + D(m_{12})e_{22} + \gamma(m_{12}, e_{22})
\]
\[
= \begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -m_D - \gamma_2(e_{11}, e_{22}) \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} p & q \\ r & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & e_B \end{bmatrix}
\]
\[
+ \begin{bmatrix} \gamma_1(m_{12}, e_{22}) & \gamma_2(m_{12}, e_{22}) \\ \gamma_3(m_{12}, e_{22}) & \gamma_4(m_{12}, e_{22}) \end{bmatrix}
\]
\[
= \begin{bmatrix} 0 & q \\ r & 0 \end{bmatrix} + \begin{bmatrix} \tau_A(0, 0) & \gamma_2(m_{12}, e_{22}) \\ \gamma_3(0, 0) & \gamma_4(0, 0) \end{bmatrix}
\]
\[
= \begin{bmatrix} 0 & q \\ r & 0 \end{bmatrix}.
\]

This shows that $\gamma_2(m_{12}, e_{22}) = 0$. On the other hand

\[
D(m_{12}) = D(e_{11}m_{12}) = e_{11}D(m_{12}) + D(e_{11})m_{12} + \gamma(e_{11}, m_{12})
\]
\[
= \begin{bmatrix} e_A & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & q \\ r & 0 \end{bmatrix} + \begin{bmatrix} 0 & m_D \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}
\]
\[
+ \begin{bmatrix} \gamma_1(e_{11}, m_{12}) & \gamma_2(e_{11}, m_{12}) \\ \gamma_3(e_{11}, m_{12}) & \gamma_4(e_{11}, m_{12}) \end{bmatrix}
\]
\[
= \begin{bmatrix} 0 & q \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \tau_A(e_A, 0) & \gamma_2(e_{11}, m_{12}) \\ \gamma_3(0, 0) & \gamma_4(0, 0) \end{bmatrix}
\]
\[
= \begin{bmatrix} 0 & q + \gamma_2(e_{11}, m_{12}) \\ 0 & 0 \end{bmatrix}.
\]

This implies that $\gamma_2(e_{11}, m_{12}) = 0$. Thus,

\[
D(m_{12}) = \begin{bmatrix} 0 & q \\ 0 & 0 \end{bmatrix}.
\]

Now; set $q = \tau_M(m)$. Therefore the proof is complete.

(6) Let $a_{11} = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$ and $m_{12} = \begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}$. Then by (2) and (5) we
have
\[
\begin{bmatrix}
0 & \tau_M(a \cdot m) \\
\gamma_3(a_{11}, m_{12})
\end{bmatrix} = D(a_{11} m_{12}) = a_{11} D(m_{12}) + D(a_{11}) m_{12} + \gamma(a_{11}, m_{12})
\]
\[
= \begin{bmatrix}
0 & a \cdot \tau_M(m) \\
\gamma_1(a_{11}, m_{12}) & \gamma_2(a_{11}, m_{12})
\end{bmatrix} + \begin{bmatrix}
0 & D_A(a) \cdot m \\
\gamma_1(a_{11}, m_{12}) & \gamma_2(a_{11}, m_{12})
\end{bmatrix}
\]
\[
= \begin{bmatrix}
\gamma_1(a_{11}, m_{12}) \cdot a \cdot \tau_M(m) + D_A(a) \cdot m + \gamma_2(a_{11}, m_{12}) \\
\gamma_3(a_{11}, m_{12})
\end{bmatrix}.
\]

Thus, \(\gamma_1(a_{11}, m_{12}) = 0, \gamma_3(a_{11}, m_{12}) = 0\) and \(\tau_M(a \cdot m) = a \cdot \tau_M(m) + D_A(a) \cdot m + \gamma_2(a_{11}, m_{12})\). By the similar method we can prove the case (7).

(7) Finally we show that \(D_A\) is a generalized 2-cocycle derivation associative with 2-cocycle \(\tau_A\). Put \(a'_{11} = \begin{bmatrix} a' & 0 \\ 0 & 0 \end{bmatrix}\) and \((aa')_{11} = \begin{bmatrix} aa' & 0 \\ 0 & 0 \end{bmatrix}\), then
\[
D(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a' & 0 \\ 0 & 0 \end{bmatrix}) = D(a_{11} a'_{11}) = a_{11} D(a'_{11}) + D(a_{11}) a'_{11} + \gamma(a_{11}, a'_{11})
\]
\[
= \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} D_A(a') & a'm_D + \gamma_2(a'_{11}, e_{11}) \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} D_A(a) & a m_D + \gamma_2(a_{11}, e_{11}) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a' & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \tau_A(a, a') & \gamma_2(a_{11}, a'_{11}) \\ \tau_B(0, 0) & 0 \end{bmatrix}
\]
\[
= \begin{bmatrix} a D_A(a') + a' D_A(a) + \tau_A(a, a') & a a' m_D + a \gamma_2(a'_{11}, e_{11}) + \gamma_2(a_{11}, a'_{11}) \\ 0 & 0 \end{bmatrix}
\]

On the other hand we have
\[
D(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a' & 0 \\ 0 & 0 \end{bmatrix}) = D(\begin{bmatrix} aa' & 0 \\ 0 & 0 \end{bmatrix})
\]
\[
= D(\begin{bmatrix} aa' & 0 \\ 0 & 0 \end{bmatrix})
\]
\[
= \begin{bmatrix} D_A(aa') & aa' m_D + \gamma_2((aa')_{11}, e_{11}) \\ 0 & 0 \end{bmatrix}.
\]

Above relations follow that
\[
D_A(aa') = a D_A(a') + a' D_A(a) + \tau_A(a, a'),
\]
and
\[ a\gamma_2(a_{11}', e_{11}) + \gamma_2(a_{11}, a_{11}') = \gamma_2(a_{11}a_{11}', e_{11}). \]

Thus \( D_A \) is generalized 2-cocycle derivation. The same way of \( D_A \) proves that \( D_B \) is a generalized 2-cocycle derivation. \( \square \)

**Note 2.2.** In the Theorem 2.1, if \( D : \mathbb{T} \rightarrow \mathbb{T} \) is continuous and \( \gamma \in Z^2(\mathbb{T}, \mathbb{T}) \) then all obtained generalized derivations and 2-cocycles will be continuous. From now on, we suppose that all maps (generalized derivations, module mappings and 2-cocycles) are continuous.

**Proposition 2.3.** Let \( D_A : \mathcal{A} \rightarrow \mathcal{A} \) and \( D_B : \mathcal{B} \rightarrow \mathcal{B} \) be generalized 2-cocyles derivation associate with 2-cocycles \( \tau_A \) and \( \tau_B \), respectively, and let \( \tau_M : \mathcal{M} \rightarrow \mathcal{M} \) be a linear map that satisfies in conditions (6) and (7) of Theorem 2.1 such that

\[ \gamma_2(T_1, T_2) = \gamma_2\left( \begin{bmatrix} a_1 & 0 \\ 0 & m_2 \end{bmatrix}, \begin{bmatrix} 0 & m_1 \\ 0 & b_2 \end{bmatrix} \right), \]  

(2)

and

\[ T'\gamma_2(T_1T_2) - \gamma_2(T'T_1, T_2) + \gamma_2(T', T_1T_2) - \gamma_2(T', T_1)T_2 = \tau_A(a', a_1)m_2 - m'\tau_B(b_1, b_2), \]  

(3)

for all \( T_1 = \begin{bmatrix} a_1 & m_1 \\ b_1 & \end{bmatrix}, T_2 = \begin{bmatrix} a_2 & m_2 \\ b_2 & \end{bmatrix}, T' = \begin{bmatrix} a' & m' \\ b' & \end{bmatrix} \in \mathbb{T}. \)

Then the map

\[ D\left( \begin{bmatrix} a & m \\ b & \end{bmatrix} \right) = \begin{bmatrix} D_A(a) & \tau_M(m) \\ \tau_M(m) & D_B(b) \end{bmatrix}, \]

is a generalized 2-cocycle derivation with associate 2-cocycle \( \gamma = \begin{bmatrix} \tau_A & \gamma_2 \\ \gamma_2 & \tau_B \end{bmatrix} \).

**Proof.** Clearly by (2.3) \( \gamma \) is a 2-cocycle on \( \mathbb{T} \times \mathbb{T} \). Then for every \( T_1 = \)}
\[
\begin{bmatrix}
  a_1 & m_1 \\
  b_1 &
\end{bmatrix}, T_2 = \begin{bmatrix}
  a_2 & m_2 \\
  b_2 &
\end{bmatrix} \in \mathbb{T}, \text{ we have}
\]

\[
T_1 D(T_2) + D(T_1)T_2 + \gamma(T_1, T_2)
\]

\[
= \begin{bmatrix}
  a_1 & m_1 \\
  b_1 &
\end{bmatrix} \begin{bmatrix}
  D_A(a_2) & \tau_M(m_2) \\
  D_B(b_2) &
\end{bmatrix} + \begin{bmatrix}
  D_A(a_1) & \tau_M(m_1) \\
  D_B(b_1) &
\end{bmatrix} \begin{bmatrix}
  a_2 & m_2 \\
  b_2 &
\end{bmatrix}
\]

\[
+ \begin{bmatrix}
  \tau_A(a_1, a_2) & \gamma_2(T_1, T_2) \\
  \tau_B(b_1, b_2) &
\end{bmatrix}
\]

\[
= \begin{bmatrix}
  a_1 D_A(a_2) + D_A(a_1) a_2 + \tau_A(a_1, a_2) & a_1 \cdot \tau_M(m_2) + m_1 \cdot D_B(b_2) + D_A(a_1) \cdot m_2 \\
  b_1 D_B(b_2) + D_B(b_1) b_2 + \tau_B(b_1, b_2) &
\end{bmatrix}
\]

As well as, together with cases (6) and (7) of Theorem 2.1 and (2.2) we have

\[
D(T_1 T_2) = D\left(\begin{bmatrix}
  a_1 a_2 & a_1 \cdot m_2 + m_1 \cdot b_2 \\
  b_1 b_2 &
\end{bmatrix}\right)
\]

\[
= \begin{bmatrix}
  D_A(a_1 a_2) & \tau_M(a_1 \cdot m_2 + m_1 \cdot b_2) \\
  D_B(b_1 b_2) &
\end{bmatrix}
\]

\[
= \begin{bmatrix}
  a_1 D_A(a_2) + D_A(a_1) a_2 + \tau_A(a_1, a_2) & a_1 \cdot \tau_M(m_2) + m_1 \cdot D_B(b_2) + D_A(a_1) \cdot m_2 \\
  b_1 D_B(b_2) + D_B(b_1) b_2 + \tau_B(b_1, b_2) &
\end{bmatrix}
\]

Thus \(D\) is a generalized derivation associate with 2-cocycle \(\gamma\). \(\Box\)

If \(\mathcal{M} = 0\), then triangular Banach algebra reformed to \(\mathcal{A} \oplus_1 \mathcal{B}\) with the following sum and product

\[(a, b) + (a', b') = (a + a, b + b'), \text{ and } (a, b)(a', b') = (aa, bb'),\]

for every \((a, b), (a', b') \in \mathcal{A} \oplus_1 \mathcal{B}\). It becomes a Banach algebra with defined norm as \(\|(a, b)\| = \|a\|_\mathcal{A} + \|b\|_\mathcal{B}\). We set \(\mathfrak{A} = \mathcal{A} \oplus_1 \mathcal{B}\). Let \(\gamma \in Z^2(\mathfrak{A}, \mathfrak{A})\), \(\gamma_1 : \mathfrak{A} \times \mathfrak{A} \longrightarrow \mathcal{A}\) and \(\gamma_2 : \mathfrak{A} \times \mathfrak{A} \longrightarrow \mathcal{B}\) be the coordinate functions associated to \(\gamma\) that is

\[\gamma((a_1, b_1), (a_2, b_2)) = (\gamma_1((a_1, b_2), (a_2, b_2)), \gamma_2((a_1, b_2), (a_2, b_2))),\]

for all \((a_1, b_1), (a_2, b_2) \in \mathfrak{A}\). If \(\gamma_1\) corresponds to a 2-cocycle on \(\mathcal{A} \times \mathcal{A}\) then there exists a 2-cocycle \(\tau_\mathcal{A}\) on \(\mathcal{A} \times \mathcal{A}\) such that \(\gamma_1((a_1, b_1), (a_2, b_2)) = \tau_\mathcal{A}(a_1, a_2) b_1 b_2 + \tau_\mathcal{A}(b_1, b_2).\)
Proposition 2.5. Let \( \gamma \in B^2(\mathfrak{A}, \mathfrak{A}) \). Then \( \gamma \) is a 2-cocycle if and only if there are 2-cocycles \( \tau_A \) and \( \tau_B \) on \( A \times A \) and \( B \times B \), respectively, such that \( \gamma = (\tau_A, \tau_B) \).

Proof. The case where \( \gamma \) is a 2-cocycle existence of \( \tau_A \) and \( \tau_B \) are clear by the same reasoning in Lemma 1.2. For converse, let \( \tau_A \in Z^2(A, A) \) and \( \tau_B \in Z^2(B, B) \). We shall show that \( \gamma = (\tau_A, \tau_B) \) belongs to \( Z^2(\mathfrak{A}, \mathfrak{A}) \).

For every \( (a_1, b_1), (a_2, b_2), (a_3, b_3) \in \mathfrak{A}, \)
\[
(a_1, b_1)\gamma((a_2, b_2), (a_3, b_3)) - \gamma((a_1, b_1)(a_2, b_2), (a_3, b_3))
+ \gamma((a_1, b_1), (a_2, b_2)(a_3, b_3))
- \gamma((a_1, b_1), (a_2, b_2))(a_3, b_3)
= (a_1, b_1)\gamma((a_2, b_2), (a_3, b_3)) - \gamma((a_1a_2, b_1b_2), (a_3, b_3))
+ \gamma((a_1, b_1), (a_2a_3, b_2b_3))
- \gamma((a_1, b_1), (a_2, b_2))(a_3, b_3)
= (a_1, b_1)\gamma_1((a_2, b_2), (a_3, b_3)), \gamma_2((a_2, b_2), (a_3, b_3))
- (\gamma_1((a_1a_2, b_1b_2), (a_3, b_3)), \gamma_2((a_1a_2, b_2b_3), (a_3, b_3)))
+ (\gamma_1((a_1, b_1), (a_2a_3, b_2b_3)), \gamma_2((a_1, b_1), (a_2a_3, b_2b_3)))
- (\gamma_1((a_1, b_1), (a_2, b_2)), \gamma((a_1, b_1), (a_2, b_2))(a_3, b_3)
= (a_1, b_1)(\tau_A((a_2, a_3)), \tau_B((b_2, b_3))) - (\tau_A((a_1a_2, a_3)), \tau_B((b_1b_2, b_3)))
+ (\tau_A((a_1, a_2a_3)), \tau_B((b_1, b_2b_3))) - (\tau_A((a_1, a_2)), \tau_B((b_1, b_2)))(a_3, b_3)
= (a_1, b_1)(\tau_A((a_2, a_3)) - \tau_A((a_1a_2, a_3)), b_1 \tau_B((b_2, b_3)) - \tau_B((b_1b_2, b_3)))
+ (\tau_A((a_1, a_2a_3)) - \tau_A((a_1, a_2)a_3), \tau_B((b_1, b_2b_3)) - \tau_B((b_1, b_2))b_3)
= 0.

Thus \( \gamma \) is a 2-cocycle. \( \square \)

Proposition 2.5. Let \( \mathfrak{A} = A \oplus_1 B \), where \( A \) and \( B \) are Banach algebras. If \( D : \mathfrak{A} \longrightarrow \mathfrak{A} \) is a generalized derivation associate with 2-cocycle \( \gamma \), then there are generalized derivations \( D_A \) and \( D_B \) associate with 2-cocycles
Proof. Define $D_A : A \rightarrow A$ by $D_A(a) = e_A D((a,0)) e_A$ for all $a \in A$. By Lemma 2.4, there are 2-cocycles $\tau_A$ and $\tau_B$ such that $\gamma((a_1,b_1),(a_2,b_2)) = (\tau_A(a_1,a_2), \tau_B(b_1,b_2))$ for all $(a_1,b_1), (a_2,b_2) \in \mathfrak{A}$. Then

$$D_A(a_1a_2) = e_A D((a_1a_2,0)) e_A = e_A D((a_1,0)(a_2,0)) e_A$$

$$= e_A(a_1,0) D((a_2,0)) e_A + e_A D((a_1,0))(a_2,0) e_A$$

$$+ e_A \gamma((a_1,0),(a_2,0)) e_A$$

$$= e_A(a_1,0) D((a_2,0)) e_A + e_A D((a_1,0))(a_2,0) e_A$$

$$+ e_A \tau_A(a_1,a_2), 0) e_A$$

$$= a_1 D_A(a_2) + D_A(a_1)a_2 + \tau_A(a_1,a_2),$$

for every $a_1, a_2 \in A$. Similarly, if we define $D_B : B \rightarrow B$ by $D_B(b) = e_B D((0,b)) e_B$ for all $b \in B$, then by the same reasoning for proof of $D_A$, $D_B$ become a generalized derivation associate with 2-cocycle $\tau_B$. □

Proposition 2.6. Let $\mathfrak{A} = A \oplus_1 B$, where $A$ and $B$ are Banach algebras. If $D_A$ and $D_B$ are generalized derivation associate with 2-cocycles $\tau_A$ and $\tau_B$, respectively, then $D : \mathfrak{A} \rightarrow \mathfrak{A}$ defined by $D = (D_A, D_B)$ is a generalized derivation associate with 2-cocycle $\gamma = (\tau_A, \tau_B)$.

Proof. By Lemma 2.4, $\gamma = (\tau_A, \tau_B)$ is a 2-cocycle. Then

$$D((a_1,b_1)(a_2,b_2)) = D((a_1a_2,b_1b_2)) = (D_A(a_1a_2), D_B(b_1b_2))$$

$$= (a_1 D_A(a_2) + D_A(a_1)a_2 + \tau_A(a_1,a_2), b_1 D_B(b_2))$$

$$+ D_B(b_1)b_2 + \tau_B(b_1,b_2),$$

and

$$(a_1,b_1) D((a_2,b_2)) + D((a_1,b_1))(a_2,b_2) + \gamma((a_1,b_1),(a_2,b_2))$$

$$= (a_1 D_A(a_2), b_1 D_B(b_2)) + (D_A(a_1)a_2, D_B(b_1)b_2)$$

$$+ (\tau_A(a_1,a_2), \tau_B(b_1,b_2)),$$

for all $(a_1,b_1), (a_2,b_2) \in \mathfrak{A}$. Thus, $D$ is a generalized derivation associate with 2-cocycle $\gamma$. □
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