Journal of Mathematical Extension
Vol. 9, No. 4, (2015), 81-97
ISSN: 1735-8299
URL: http://www.ijmex.com

# Characterization of Generalized Derivations Associate with Hochschild 2-Cocycles on Triangular Banach Algebras

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**Abstract.** Let $\mathcal{A}$ and $\mathcal{B}$ be unital Banach algebras and $\mathcal{M}$ be a left $\mathcal{A}$-module and right $B$-module. We consider generalized derivations associate with Hochschild 2-cocycles on triangular Banach algebra $T$ (related to $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{M}$). We characterize this new version of generalized derivations on triangular Banach algebras and we obtain some results for $l^1-direct$ summands of Banach algebras.

**AMS Subject Classification:** 47B47; 46H25
**Keywords and Phrases:** Banach algebra, cocycle, generalized derivation, triangular Banach algebra

## 1. Introduction

Let $\mathcal{A}$ be a Banach algebra, and let $X$ be a Banach $\mathcal{A}$-bimodule. A derivation is a linear map $D : \mathcal{A} \longrightarrow X$ such that

$$D(ab) = a \cdot D(b) + D(a) \cdot b \qquad (a, b \in \mathcal{A}).$$

For $x \in X$, set $ad_x : a \mapsto a \cdot x - x \cdot a$, $\mathcal{A} \longrightarrow X$. Then $ad_x$ is the inner derivation induced by $x$.

Received:April 2015; Accepted: June 2015
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The linear space of bounded derivations from $\mathcal{A}$ into $X$ denoted by $Z^1(\mathcal{A}, X)$ and the linear subspace of inner derivations denoted by $N^1(\mathcal{A}, X)$. We consider the quotient space $H^1(\mathcal{A}, X) = Z^1(\mathcal{A}, X)/N^1(\mathcal{A}, X)$, called the *first Hochschild cohomology group* of $\mathcal{A}$ with coefficients in $X$.
Let $\mathcal{A}$ be a Banach algebra, and let $X$ be a Banach $\mathcal{A}$-bimodule. By $B^n(\mathcal{A}, X)$, we mean that the space of bounded $n$-linear maps form $\mathcal{A}^n$ into $X$. A 2-linear map $\gamma \in B^2(\mathcal{A}, X)$ is called Hochschild *2-cocycle* if it satisfies in the following equation

$$a \cdot \gamma(b, c) - \gamma(ab, c) + \gamma(a, bc) - \gamma(a, b) \cdot c = 0,$$

for every $a, b, c \in \mathcal{A}$. The space of Hochschild 2-cocycles is a linear subspace of $B^2(\mathcal{A}, X)$, which denoted by $Z^2(\mathcal{A}, X)$. Here in after we used the word 2-cocycle instead Hochschild 2-cocycle. Let $\varphi \in \Delta\ \mathcal{A}$, where $\Delta(\mathcal{A})$ is the carrier space of $\mathcal{A}$, then A 2-linear map $\gamma \in B^2(\mathcal{A}, X)$ is called *point 2-cocycle* at $\varphi$ if it satisfies in the following equation

$$\varphi(a)\gamma(b, c) - \gamma(ab, c) + \gamma(a, bc) - \gamma(a, b)\varphi(c) = 0,$$

for every $a, b, c \in \mathcal{A}$. For given $T \in B(\mathcal{A}, X)$, we let

$$(\delta^1 T)(a, b) = a \cdot T(b) - T(ab) + T(a) \cdot b,$$

for every $a, b \in \mathcal{A}$ and $\delta^1 : B(\mathcal{A}, X) \longrightarrow B^2(\mathcal{A}, X)$. Then the maps $\{\delta^1 T : \ T \in B(\mathcal{A}, X)\}$ is a linear subspace of $Z^2(\mathcal{A}, X)$. These maps are called *2-coboundaries*. The collection of all 2-coboundaries is denoted by $N^2(\mathcal{A}, X)$.
An additive map $D : \mathcal{A} \longrightarrow X$ called generalized 2-cocycle derivation if there exists a 2-cocycle $\gamma$ such that

$$D(xy) = x \cdot D(y) + D(x) \cdot y + \gamma(x, y) \tag{1}$$

for every $x, y \in \mathcal{A}$. Similarly, an additive map $D : \mathcal{A} \longrightarrow X$ called generalized 2-cocycle Jordan derivation if there exists a 2-cocycle $\gamma$ such that

$$D(x^2) = x \cdot D(x) + D(x) \cdot y + \gamma(x, x) \tag{2}$$

for every $x \in \mathcal{A}$. This definitions introduced by Nakajima in [5] and he gave some examples for this new definition. In [3] authors considered this new notion for some algebras such as von-Neumann and they showed that generalized Jordan derivation of this type from von-Neumann algebras into themselves is a generalized derivation (under some conditions). Similar result obtained by authors in [4] for triangular algebras. An additive map $D : \mathcal{A} \longrightarrow X$ called generalized 2-coboundry derivation if there exists a 2-coboundry $\delta^1$ such that

$$D(xy) = x \cdot D(y) + D(x) \cdot y + (\delta^1 F)(x, y), \tag{3}$$

for every $x, y \in \mathcal{A}$ and $F \in B(\mathcal{A}, X)$. Let $\mathcal{A}$ and $\mathcal{B}$ be unital Banach algebras with units $e_{\mathcal{A}}$ and $e_{\mathcal{B}}$, respectively. Suppose that $\mathcal{M}$ is a unital Banach $\mathcal{A}, \mathcal{B}$-bimodule. We define the triangular Banach algebra

$$\mathcal{T} = \begin{bmatrix} \mathcal{A} & \mathcal{M} \\ & \mathcal{B} \end{bmatrix},$$

with the sum and product being giving by the usual $2 \times 2$ matrix operations and internal module actions. The norm on $\mathcal{T}$ is

$$\| \begin{bmatrix} a & m \\ & b \end{bmatrix} \| = \|a\|_{\mathcal{A}} + \|m\|_{\mathcal{M}} + \|b\|_{\mathcal{B}}.$$

The Banach algebra $\mathcal{T}$ as a Banach space is isomorphic to the $\ell^1$-direct sum of $\mathcal{A}, \mathcal{B}$ and $\mathcal{M}$. Forrest and Marcoux introduced and studied derivation of triangular Banach algebras in [1].

Let $\mathcal{T}$ be a triangular Banach algebras defined as above, and let $\gamma \in B^2(\mathcal{T}, \mathcal{T})$. Let $\gamma_1 : \mathcal{T} \times \mathcal{T} \longrightarrow \mathcal{A}, \gamma_2 : \mathcal{T} \times \mathcal{T} \longrightarrow \mathcal{M}$ and $\gamma_3 : \mathcal{T} \times \mathcal{T} \longrightarrow \mathcal{B}$ denote the coordinate functions associated to $\gamma$. That is

$$\gamma(T_1, T_2) = \begin{bmatrix} \gamma_1(T_1, T_2) & \gamma_2(T_1, T_2) \\ & \gamma_3(T_1, T_2) \end{bmatrix},$$

for $T_1, T_2 \in \mathcal{T}$. Let $\gamma : \mathcal{T} \times \mathcal{T} \longrightarrow \mathcal{T}$ be a 2-cocycle. The coordinate function $\gamma_1$ is said to correspond to a 2-cocycle (2-coboundary) on $\mathcal{A} \times \mathcal{A}$ if there exists a 2-cocycle (2-coboundary) $\tau_{\mathcal{A}}$ on $\mathcal{A} \times \mathcal{A}$ such that $\gamma_1(T_1, T_2) = \tau_{\mathcal{A}}(a_1, a_2)$, where $T_i = \begin{bmatrix} a_i & m_i \\ & b_i \end{bmatrix}$, for $i = 1, 2$.

Similarly, $\gamma_3$ is said to correspond to a 2-cocycle (2-coboundary) on $\mathcal{B} \times \mathcal{B}$ if there exists a 2-cocycle (2-coboundary) $\tau_{\mathcal{B}}$ on $\mathcal{B} \times \mathcal{B}$ such that $\gamma_3(T_1, T_2) = \tau_{\mathcal{B}}(b_1, b_2)$.

**Definition 1.1.** *Let $\gamma \in B^2(\mathcal{T}, \mathcal{T})$, $\gamma_1 : \mathcal{T} \times \mathcal{T} \longrightarrow \mathcal{A}, \gamma_2 : \mathcal{T} \times \mathcal{T} \longrightarrow \mathcal{M}$ and $\gamma_3 : \mathcal{T} \times \mathcal{T} \longrightarrow \mathcal{B}$ denote the coordinate functions associated to $\gamma$. That is*

$$\gamma(T_1, T_2) = \begin{bmatrix} \gamma_1(T_1, T_2) & \gamma_2(T_1, T_2) \\ & \gamma_3(T_1, T_2) \end{bmatrix},$$

*for $T_1, T_2 \in \mathcal{T}$. Let $\gamma \in B^2(\mathcal{T}, \mathcal{T})$ be a 2-cocycle (2-coboundries). We say that $\gamma_1$ corresponds to a 2-cocycle (2-coboundries) on $\mathcal{A} \times \mathcal{A}$ if there exists a 2-cocycle (2-coboundries) $\tau_{\mathcal{A}}$ on $\mathcal{A} \times \mathcal{A}$ such that $\gamma_1(T_1, T_2) = \tau_{\mathcal{A}}(a_1, a_2)$, where $T_i = \begin{bmatrix} a_i & m_i \\ & b_i \end{bmatrix}$, for $i = 1, 2$.*
*Similarly, we say that $\gamma_3$ corresponds to a 2-cocycle (2-coboundries) on $\mathcal{B} \times \mathcal{B}$ if there exists a 2-cocycle (2-coboundries) $\tau_{\mathcal{B}}$ on $\mathcal{B} \times \mathcal{B}$ such that $\gamma_3(T_1, T_2) = \tau_{\mathcal{B}}(b_1, b_2)$. Second order cohomology of triangular Banach algebras studied by Forrest and Marcoux in [2].*

**Lemma 1.2.** *Let $\gamma \in B^2(\mathcal{T}, \mathcal{T})$ be a 2-cocycle. Then there are continuous corresponding 2-cocycles $\tau_{\mathcal{A}}$ and $\tau_{\mathcal{B}}$ on $\mathcal{A} \times \mathcal{A}$ and $\mathcal{B} \times \mathcal{B}$, respectively. Furthermore*

1. $\tau_{\mathcal{A}}(a, 0) = \tau_{\mathcal{B}}(0, b) = \tau_{\mathcal{A}}(e_{\mathcal{A}}, 0) = \tau_{\mathcal{B}}0, e_{\mathcal{B}}) = \tau_{\mathcal{A}}(0, 0) = \tau_{\mathcal{B}}(0, 0) = 0.$
2. $\gamma_2(e_{11}, e_{11}) = 0, \gamma_2(e_{11}, 0) = 0.$
3. $\gamma_2(e_{22}, e_{22}) = 0.$
4. $\tau_{\mathcal{A}}(a, e_{\mathcal{A}}) = 0$ and $\tau_{\mathcal{A}}(e_{\mathcal{A}}, e_{\mathcal{A}}) = 0.$
5. $\tau_{\mathcal{B}}(b, e_{\mathcal{B}}) = 0$ and $\tau_{\mathcal{B}}(e_{\mathcal{B}}, e_{\mathcal{B}}) = 0.$
6. $\gamma_2(b_{22}, e_{22}) = 0$ and $\gamma_2(e_{11}, a_{11}) = 0.$

*where $e_{11} = \begin{bmatrix} e_{\mathcal{A}} & 0 \\ & 0 \end{bmatrix}$, $e_{22} = \begin{bmatrix} 0 & 0 \\ & e_{\mathcal{B}} \end{bmatrix}$, $b_{22} = \begin{bmatrix} 0 & 0 \\ & b \end{bmatrix}$, $a_{11} = \begin{bmatrix} a & 0 \\ & 0 \end{bmatrix}$ and for $a \in \mathcal{A}, b \in \mathcal{B}$.*

**Proof.** Define $\tau_{\mathcal{A}} : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$ and $\tau_{\mathcal{B}} : \mathcal{B} \times \mathcal{B} \longrightarrow \mathcal{B}$ as follows

$$\tau_{\mathcal{A}}(a_1, a_2) = e_{\mathcal{A}}\gamma\left(\begin{bmatrix} a_1 & 0 \\ & 0 \end{bmatrix}, \begin{bmatrix} a_2 & 0 \\ & 0 \end{bmatrix}\right)e_{\mathcal{A}},$$

and

$$\tau_{\mathcal{B}}(b_1, b_2) = e_{\mathcal{B}}\gamma\left(\begin{bmatrix} 0 & 0 \\ & b_1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ & b_2 \end{bmatrix}\right)e_{\mathcal{B}}.$$

It is easy to check that $\tau_{\mathcal{A}}$ and $\tau_{\mathcal{B}}$ are 2-cocycle. Continuity of $\tau_{\mathcal{A}}$ and $\tau_{\mathcal{B}}$ is clear. Since $\tau_{\mathcal{A}}$ and $\tau_{\mathcal{B}}$ are 2-linear so (1) is clear. For the rest consider the following

$$\gamma(\begin{bmatrix} e_{\mathcal{A}} & 0 \\ & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ & e_{\mathcal{B}} \end{bmatrix}) = \begin{bmatrix} \tau_{\mathcal{A}}(e_{\mathcal{A}}, 0) & \gamma_2(e_{11}, e_{22}) \\ & \tau_{\mathcal{B}}(0, e_{\mathcal{B}}) \end{bmatrix},$$

and

$$e_{11}\gamma(e_{11}, e_{22}) - \gamma(e_{11}, e_{22}) + \gamma(e_{11}, 0) - \gamma(e_{11}, e_{11})e_{22} = 0.$$

Then,

$$\begin{bmatrix} \tau_{\mathcal{A}}(e_{\mathcal{A}}, 0) & \gamma_2(e_{11}, e_{22}) \\ & 0 \end{bmatrix} - \begin{bmatrix} \tau_{\mathcal{A}}(e_{\mathcal{A}}, 0) & \gamma_2(e_{11}, e_{22}) \\ & \tau_{\mathcal{B}}(0, e_{\mathcal{B}}) \end{bmatrix}$$

$$+ \begin{bmatrix} \tau_{\mathcal{A}}(e_{\mathcal{A}}, 0) & \gamma_2(e_{11}, 0) \\ & 0 \end{bmatrix} + \begin{bmatrix} 0 & \gamma_2(e_{11}, e_{11}) \\ & 0 \end{bmatrix} = 0.$$

This follows that $\gamma_2(e_{11}, e_{11}) + \gamma_2(e_{11}, 0) = 0$. From

$$e_{11}\gamma(e_{11}, e_{11}) - \gamma(e_{11}, e_{11}) + \gamma(e_{11}, e_{11}) - \gamma(e_{11}, e_{11})e_{11} = 0,$$

we conclude that $\gamma_2(e_{11}, e_{11}) = 0$. This implies that $\gamma_2(e_{11}, 0) = 0$. Similarly, one can show that $\gamma_2(e_{22}, e_{22}) = 0$. From

$$a_{11}\gamma(e_{11}, e_{11}) - \gamma(a_{11}, e_{11}) + \gamma(a_{11}, e_{11}) - \gamma(a_{11}, e_{11})e_{11} = 0,$$

we conclude that $\tau_A(a, e_{\mathcal{A}}) = 0$. Since $a$ was arbitrary so $\tau_A(e_{\mathcal{A}}, e_{\mathcal{A}}) = 0$. By the similar methods, we obtain the other cases. $\square$

Let $\mathcal{T}$ be a triangular Banach algebra and let $\mathcal{X}$ be a unital Banach $\mathcal{T}$-bimodule, then we use these notations in this paper: $\mathcal{X}_{\mathcal{AA}} = e_{\mathcal{A}} \cdot \mathcal{X} \cdot e_{\mathcal{A}}, \mathcal{X}_{\mathcal{BB}} = e_{\mathcal{B}} \cdot \mathcal{X} \cdot e_{\mathcal{B}}, \mathcal{X}_{\mathcal{AB}} = e_{\mathcal{A}} \cdot \mathcal{X} \cdot e_{\mathcal{B}}$, and $\mathcal{X}_{\mathcal{BA}} = e_{\mathcal{B}} \cdot \mathcal{X} \cdot e_{\mathcal{A}}$. If $\mathcal{X}$ replaced by $\mathcal{T}$, we have $\mathcal{X}_{\mathcal{AA}} = \mathcal{A}, \mathcal{X}_{\mathcal{BB}} = \mathcal{B}, \mathcal{X}_{\mathcal{AB}} = \mathcal{M}$, and $\mathcal{X}_{\mathcal{BA}} = 0$.

**Lemma 1.3.** *Let $\mathcal{X}$ be a $\mathbb{T}$-bimodule, $\delta_{\mathcal{A}} : \mathcal{A}\times\mathcal{A} \longrightarrow \mathcal{X}_{\mathcal{AA}}$, $\delta_{\mathcal{B}} : \mathcal{B}\times\mathcal{B} \longrightarrow \mathcal{X}_{\mathcal{BB}}$ be 2-cocycles, and $\mathcal{X}_{\mathcal{AB}} = 0$. Then there exists a 2-cocycle mapping from $\mathbb{T} \times \mathbb{T}$ into $\mathcal{X}$.*

**Proof.** For every $\begin{bmatrix} a_1 & m_1 \\ & b_1 \end{bmatrix}, \begin{bmatrix} a_2 & m_2 \\ & b_2 \end{bmatrix} \in \mathbb{T}$, define $D : \mathbb{T} \times \mathbb{T} \longrightarrow \mathcal{X}$

by

$$D\left(\begin{bmatrix} a_1 & m_1 \\ & b_1 \end{bmatrix}, \begin{bmatrix} a_2 & m_2 \\ & b_2 \end{bmatrix}\right) = \delta_{\mathcal{A}}(a_1, a_2) + \delta_{\mathcal{B}}(b_1, b_2).$$

We claim that $D$ is a 2-cocycle. Because for every
$T_i = \begin{bmatrix} a_i & m_i \\ & b_i \end{bmatrix}, (i = 1, 2, 3)$, we have

$$\begin{aligned}
&T_1 \cdot D(T_2, T_3) - D(T_1T_2, T_3) + D(T_1, T_2T_3) - D(T_1, T_2) \cdot T_3 \\
&= T_1 \cdot (\delta_{\mathcal{A}}(a_2, a_3) + \delta_{\mathcal{B}}(b_2, b_3)) - \delta_{\mathcal{A}}(a_1a_2, a_3) - \delta_{\mathcal{B}}(b_1b_2, b_3) \\
&+ \delta_{\mathcal{A}}(a_1, a_2a_3) + \delta_{\mathcal{B}}(b_1, b_2b_3) - (\delta_{\mathcal{A}}(a_1, a_2) + \delta_{\mathcal{B}}(b_1, b_2)) \cdot T_3 \\
&= T_1 \cdot \tau(e_{\mathcal{A}})\delta_{\mathcal{A}}(a_2, a_3) - \delta_{\mathcal{A}}(a_1a_2, a_3) + \delta_{\mathcal{A}}(a_1, a_2a_3) - \delta_{\mathcal{A}}(a_1, a_2) \cdot e_{\mathcal{A}} \cdot T_3 \\
&+ T_1 \cdot \tau(e_{\mathcal{B}})\delta_{\mathcal{B}}(b_2, b_3) - \delta_{\mathcal{B}}(b_1b_2, b_3) + \delta_{\mathcal{B}}(b_1, b_2b_3) - \delta_{\mathcal{B}}(b_1, b_2) \cdot e_{\mathcal{B}} \cdot T_3 \\
&= a_1 \cdot \delta_{\mathcal{A}}(a_2, a_3) - \delta_{\mathcal{A}}(a_1a_2, a_3) + \delta_{\mathcal{A}}(a_1, a_2a_3) - \delta_{\mathcal{A}}(a_1, a_2) \cdot a_3 \\
&+ b_1 \cdot \delta_{\mathcal{B}}(b_2, b_3) - \delta_{\mathcal{B}}(b_1b_2, b_3) + \delta_{\mathcal{B}}(b_1, b_2b_3) - \delta_{\mathcal{B}}(b_1, b_2) \cdot b_3 \\
&= 0.
\end{aligned}$$

This proves our claim. $\square$

**Lemma 1.4.** [2, Lemma 3.1] *Let $\delta_{\mathcal{A}}$ and $\delta_{\mathcal{B}}$ be 2-coboundaries on $\mathcal{A} \times \mathcal{A}$ and $\mathcal{B} \times \mathcal{B}$, respectively. Then there exists a 2-coboundaries $\delta$ on $\mathbb{T} \times \mathbb{T}$ such that $\delta_1$ corresponds to $\delta_{\mathcal{A}}$ and $\delta_2$ corresponds to $\delta_{\mathcal{B}}$, where $\delta_1$ and $\delta_2$ are coordinate functions associated to $\delta$.*

## 2. Characterization of Generalized 2-Cocycle Derivations

In this section we prove main results of paper. We characterize generalized 2-cocycle derivations on triangular Banach algebras and by taking $\mathcal{M} = 0$ we consider generalized 2-cocycle derivations on $\mathcal{A} \oplus_1 \mathcal{B}$, where $\mathcal{A}$ and $\mathcal{B}$ are Banach algebras.

**Theorem 2.1.** *Let $\mathbb{T}$ be a triangular Banach algebra and $\mathcal{A}, \mathcal{B}$ be unital Banach algebras with units $e_{\mathcal{A}}$ and $e_{\mathcal{B}}$, respectively, and $M$ be a unitary Banach $\mathcal{A}, \mathcal{B}$-bimodule. Let $D : \mathbb{T} \longrightarrow \mathbb{T}$ be a generalized 2-cocycle derivation associate with $\gamma$, then there exist element $m_D \in M$, corresponding 2-cocycles $\tau_{\mathcal{A}}$ and $\tau_{\mathcal{B}}$ on $\mathcal{A} \times \mathcal{A}$ and $\mathcal{B} \times \mathcal{B}$, respectively, and mappings $D_{\mathcal{A}} : \mathcal{A} \longrightarrow \mathcal{A}$, $D_{\mathcal{B}} : \mathcal{B} \longrightarrow \mathcal{B}$ and $\tau_{\mathcal{M}} : \mathcal{M} \longrightarrow \mathcal{M}$ such that*

1. $D\Big(\begin{bmatrix} e_{\mathcal{A}} & 0 \\ & 0 \end{bmatrix}\Big) = \begin{bmatrix} 0 & m_D \\ & 0 \end{bmatrix}.$

2. $D\Big(\begin{bmatrix} a & 0 \\ & 0 \end{bmatrix}\Big) = \begin{bmatrix} D_{\mathcal{A}}(a) & am_D + \gamma_2(a_{11}, e_{11}) \\ & 0 \end{bmatrix}.$

3. $D\Big(\begin{bmatrix} 0 & 0 \\ & e_{\mathcal{B}} \end{bmatrix}\Big) = \begin{bmatrix} 0 & -m_D - \gamma_2(e_{11}, e_{22}) \\ & 0 \end{bmatrix}.$

4. $D\Big(\begin{bmatrix} 0 & 0 \\ & b \end{bmatrix}\Big) = \begin{bmatrix} 0 & -m_D b - \gamma_2(e_{11}, b_{22}) \\ & D_{\mathcal{B}}(b) \end{bmatrix}$

5. $D\Big(\begin{bmatrix} 0 & m \\ & 0 \end{bmatrix}\Big) = \begin{bmatrix} 0 & \tau_{\mathcal{M}}(m) \\ & 0 \end{bmatrix}.$

6. $\gamma_1(\begin{bmatrix} a & 0 \\ & 0 \end{bmatrix}, \begin{bmatrix} 0 & m \\ & 0 \end{bmatrix}) = 0$, $\gamma_3(\begin{bmatrix} a & 0 \\ & 0 \end{bmatrix}, \begin{bmatrix} 0 & m \\ & 0 \end{bmatrix}) = 0$ *and*

$$\tau_{\mathcal{M}}(a \cdot m) = a \cdot \tau_{\mathcal{M}}(m) + D_{\mathcal{A}}(a) \cdot m + \gamma_2(\begin{bmatrix} a & 0 \\ & 0 \end{bmatrix}, \begin{bmatrix} 0 & m \\ & 0 \end{bmatrix}).$$

*7.* $\gamma_1(\begin{bmatrix} 0 & m \\ & \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ & b \end{bmatrix}) = 0$, $\gamma_3(\begin{bmatrix} 0 & m \\ & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ & b \end{bmatrix}) = 0$ *and*

$$\tau_{\mathcal{M}}(m \cdot b) = \tau_{\mathcal{M}}(m) \cdot b + m \cdot D_{\mathcal{B}}(b) + \gamma_2(\begin{bmatrix} 0 & m \\ & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ & b \end{bmatrix}).$$

*Furthermore, $D_{\mathcal{A}} : \mathcal{A} \longrightarrow \mathcal{A}$ and $D_{\mathcal{B}} : \mathcal{B} \longrightarrow \mathcal{B}$ are generalized 2-cocycle derivations associate with 2-cocycles $\tau_{\mathcal{A}}$ and $\tau_{\mathcal{B}}$, respectively.*

**Proof.** Let $\gamma$ be the 2-cocycle related to $D$. Let $\gamma_1, \gamma_2$ and $\gamma_3$ be the coordinate functions associated to $\gamma$. By Lemma 1.2, there exists corresponding 2-cocycles $\tau_A$ and $\tau_{\mathcal{B}}$, respectively on $\mathcal{A} \times \mathcal{A}$ and $\mathcal{B} \times \mathcal{B}$ (continuity of these maps do not need).

**(1)** Put $e_{11} = \begin{bmatrix} e_{\mathcal{A}} & 0 \\ & 0 \end{bmatrix}$ and let $D(e_{11}) = \begin{bmatrix} p & q \\ & r \end{bmatrix}$. Then by Lemma 1.2

$$\begin{aligned} D(e_{11}) &= D(e_{11}e_{11}) = e_{11}D(e_{11}) + D(e_{11})e_{11} + \gamma(e_{11}, e_{11}) \\ &= \begin{bmatrix} e_{\mathcal{A}} & 0 \\ & 0 \end{bmatrix} \begin{bmatrix} p & q \\ & r \end{bmatrix} + \begin{bmatrix} p & q \\ & r \end{bmatrix} \begin{bmatrix} e_{\mathcal{A}} & 0 \\ & 0 \end{bmatrix} \\ &+ \begin{bmatrix} \gamma_1(e_{11}, e_{11}) & 0 \\ & \gamma_3(e_{11}, e_{11}) \end{bmatrix} \\ &= \begin{bmatrix} 2p & q \\ & 0 \end{bmatrix} + \begin{bmatrix} \tau_{\mathcal{A}}(e_{\mathcal{A}}, e_{\mathcal{A}}) & 0 \\ & \tau_B(0,0) \end{bmatrix} \\ &= \begin{bmatrix} 2p & q \\ & 0 \end{bmatrix}. \end{aligned}$$

Above relations follow that

$$D(e_{11}) = \begin{bmatrix} 0 & q \\ & 0 \end{bmatrix}.$$

Set $q = m_D$, then

$$D(e_{11}) = \begin{bmatrix} 0 & m_D \\ & 0 \end{bmatrix}.$$

**(2)** Put $a_{11} = \begin{bmatrix} a & 0 \\ & 0 \end{bmatrix}$ and $e_{11} = \begin{bmatrix} e_{\mathcal{A}} & 0 \\ & 0 \end{bmatrix}$. Let $D(a_{11}) = \begin{bmatrix} p & q \\ & r \end{bmatrix}$, then

$$\begin{aligned}
D(a_{11}) &= D(a_{11}e_{11}) = a_{11}D(e_{11}) + D(a_{11})e_{11} + \gamma(a_{11}, e_{11}) \\
&= \begin{bmatrix} a & 0 \\ & 0 \end{bmatrix}\begin{bmatrix} 0 & m_D \\ & 0 \end{bmatrix} + \begin{bmatrix} p & q \\ & r \end{bmatrix}\begin{bmatrix} e_A & 0 \\ & 0 \end{bmatrix} \\
&\quad + \begin{bmatrix} \gamma_1(a_{11}, e_{11}) & \gamma_2(a_{11}, e_{11}) \\ & \gamma_3(a_{11}, e_{11}) \end{bmatrix} \\
&= \begin{bmatrix} p & am_D \\ & 0 \end{bmatrix} + \begin{bmatrix} \tau_{\mathcal{A}}(a, e_{\mathcal{A}}) & \gamma_2(a_{11}, e_{11}) \\ & \tau_B(0,0) \end{bmatrix} \\
&= \begin{bmatrix} p & am_D + \gamma_2(a_{11}, e_{11}) \\ & 0 \end{bmatrix}.
\end{aligned}$$

Therefore

$$D(a_{11}) = \begin{bmatrix} D_{\mathcal{A}}(a) & am_D + \gamma_2(a_{11}, e_{11}) \\ & 0 \end{bmatrix},$$

where $D_{\mathcal{A}}(a) = p$.

**(3)** Set $e_{22} = \begin{bmatrix} 0 & 0 \\ & e_{\mathcal{B}} \end{bmatrix}$ and let $D(e_{22}) = \begin{bmatrix} p & q' \\ & r \end{bmatrix}$. Then

$$\begin{aligned}
D(e_{22}) &= D(e_{22}e_{22}) = e_{22}D(e_{22}) + D(e_{22})e_{22} + \gamma(e_{22}, e_{22}) \\
&= \begin{bmatrix} 0 & 0 \\ & e_{\mathcal{B}} \end{bmatrix}\begin{bmatrix} p & q' \\ & r \end{bmatrix} + \begin{bmatrix} p & q' \\ & r \end{bmatrix}\begin{bmatrix} 0 & 0 \\ & e_{\mathcal{B}} \end{bmatrix} \\
&+ \begin{bmatrix} \gamma_1(e_{22}, e_{22}) & \gamma_2(e_{22}, e_{22}) \\ & \gamma_3(e_{22}, e_{22}) \end{bmatrix} \\
&= \begin{bmatrix} 0 & q' \\ & 2r \end{bmatrix} + \begin{bmatrix} \tau_{\mathcal{A}}(0,0) & \gamma_2(e_{22}, e_{22}) \\ & \tau_{\mathcal{B}}(e_{\mathcal{B}}, e_{\mathcal{B}}) \end{bmatrix} \\
&= \begin{bmatrix} 0 & q' \\ & 2r \end{bmatrix}.
\end{aligned}$$

Above relations follow that

$$D(e_{22}) = \begin{bmatrix} 0 & q' \\ & 0 \end{bmatrix}.$$

Then

$$\begin{aligned}
0 &= D(0) = D(e_{11}e_{22}) = e_{11}D(e_{22}) + D(e_{11})e_{22} + \gamma(e_{11}, e_{22}) \\
&= \begin{bmatrix} 0 & q' \\ & 0 \end{bmatrix} + \begin{bmatrix} 0 & m_D \\ & 0 \end{bmatrix} + \begin{bmatrix} 0 & \gamma_2(e_{11}, e_{22}) \\ & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & m_D + q' + \gamma_2(e_{11}, e_{22}) \\ & 0 \end{bmatrix}, \qquad (1)
\end{aligned}$$

and

$$\begin{aligned}
0 &= D(0) = D(e_{22}e_{11}) = e_{22}D(e_{11}) + D(e_{22})e_{11} + \gamma(e_{22}, e_{11}) \\
&= \begin{bmatrix} 0 & \gamma_2(e_{22}, e_{11}) \\ & 0 \end{bmatrix}.
\end{aligned}$$

This follows that $\gamma_2(e_{22}, e_{11}) = 0$.Then we have $q' = -m_D - \gamma_2(e_{11}, e_{22})$.

**(4)** Set $e_{22} = \begin{bmatrix} 0 & 0 \\ & e_{\mathcal{B}} \end{bmatrix}$, $b_{22} = \begin{bmatrix} 0 & 0 \\ & b \end{bmatrix}$ and let $D(b_{22}) = \begin{bmatrix} p & q \\ & r \end{bmatrix}$. Then by Lemma 1.2 we have

$$\begin{aligned}
D(b_{22}) &= D(b_{22}e_{22}) = b_{22}D(e_{22}) + D(b_{22})e_{22} + \gamma(b_{22}, e_{22}) \\
&= \begin{bmatrix} 0 & 0 \\ & b_{22} \end{bmatrix} \begin{bmatrix} 0 & -m_D - \gamma_2(e_{11}, e_{22}) \\ & 0 \end{bmatrix} + \begin{bmatrix} p & q \\ & r \end{bmatrix} \begin{bmatrix} 0 & 0 \\ & e_{\mathcal{B}} \end{bmatrix} \\
&+ \begin{bmatrix} \gamma_1(b_{22}, e_{22}) & \gamma_2(b_{22}, e_{22}) \\ & \gamma_3(b_{22}, e_{22}) \end{bmatrix} \\
&= \begin{bmatrix} 0 & q \\ & r \end{bmatrix} + \begin{bmatrix} \tau_{\mathcal{A}}(0, 0) & \gamma_2(b_{22}, e_{22}) \\ & \tau_{\mathcal{B}}(b, e_{\mathcal{B}}) \end{bmatrix} \\
&= \begin{bmatrix} 0 & q \\ & r \end{bmatrix}.
\end{aligned}$$

By $0 = D(0) = D(e_{11}b_{22})$ we conclude that $q = -m_D b - \gamma_2(e_{11}, b_{22})$. If we set $D_{\mathcal{B}}(b) = r$ we obtain the desire.

**(5)** Set $e_{11} = \begin{bmatrix} e_{\mathcal{A}} & 0 \\ & 0 \end{bmatrix}$, $e_{22} = \begin{bmatrix} 0 & 0 \\ & e_B \end{bmatrix}$, $m_{12} = \begin{bmatrix} 0 & m \\ & 0 \end{bmatrix}$ and let

$D(m_{12}) = \begin{bmatrix} p & q \\ & r \end{bmatrix}$. Then by Lemma 1.2 we have

$$\begin{aligned}
D(m_{12}) &= D(m_{12}e_{22}) = m_{12}D(e_{22}) + D(m_{12})e_{22} + \gamma(m_{12}, e_{22}) \\
&= \begin{bmatrix} 0 & m \\ & 0 \end{bmatrix} \begin{bmatrix} 0 & -m_D - \gamma_2(e_{11}, e_{22}) \\ & 0 \end{bmatrix} + \begin{bmatrix} p & q \\ & r \end{bmatrix} \begin{bmatrix} 0 & 0 \\ & e_{\mathcal{B}} \end{bmatrix} \\
&+ \begin{bmatrix} \gamma_1(m_{12}, e_{22}) & \gamma_2(m_{12}, e_{22}) \\ & \gamma_3(m_{12}, e_{22}) \end{bmatrix} \\
&= \begin{bmatrix} 0 & q \\ & r \end{bmatrix} + \begin{bmatrix} \tau_{\mathcal{A}}(0,0) & \gamma_2(m_{12}, e_{22}) \\ & \tau_B(0, e_{\mathcal{B}}) \end{bmatrix} \\
&= \begin{bmatrix} 0 & q \\ & r \end{bmatrix}.
\end{aligned}$$

This shows that $\gamma_2(m_{12}, e_{22}) = 0$. On the other hand

$$\begin{aligned}
D(m_{12}) &= D(e_{11}m_{12}) = e_{11}D(m_{12}) + D(e_{11})m_{12} + \gamma(e_{11}, m_{12}) \\
&= \begin{bmatrix} e_{\mathcal{A}} & 0 \\ & 0 \end{bmatrix} \begin{bmatrix} 0 & q \\ & r \end{bmatrix} + \begin{bmatrix} 0 & m_D \\ & 0 \end{bmatrix} \begin{bmatrix} 0 & m \\ & 0 \end{bmatrix} \\
&+ \begin{bmatrix} \gamma_1(e_{11}, m_{12}) & \gamma_2(e_{11}, m_{12}) \\ & \gamma_3(e_{11}, m_{12}) \end{bmatrix} \\
&= \begin{bmatrix} 0 & q \\ & 0 \end{bmatrix} + \begin{bmatrix} \tau_{\mathcal{A}}(e_{\mathcal{A}}, 0) & \gamma_2(e_{11}, m_{12}) \\ & \tau_{\mathcal{B}}(0,0) \end{bmatrix} \\
&= \begin{bmatrix} 0 & q + \gamma_2(e_{11}, m_{12}) \\ & 0 \end{bmatrix}.
\end{aligned}$$

This implies that $\gamma_2(e_{11}, m_{12}) = 0$. Thus,

$$D(m_{12}) = \begin{bmatrix} 0 & q \\ & 0 \end{bmatrix}.$$

Now; set $q = \tau_M(m)$. Therefore the proof is complete.

**(6)** Let $a_{11} = \begin{bmatrix} a & 0 \\ & 0 \end{bmatrix}$ and $m_{12} = \begin{bmatrix} 0 & m \\ & 0 \end{bmatrix}$. Then by (2) and (5) we have

$$\begin{aligned}
\begin{bmatrix} 0 & \tau_{\mathcal{M}}(a\cdot m) \\ & 0 \end{bmatrix} &= D(a_{11}m_{12}) = a_{11}D(m_{12}) + D(a_{11})m_{12} + \gamma(a_{11}, m_{12}) \\
&= \begin{bmatrix} 0 & a\cdot\tau_{\mathcal{M}}(m) \\ & 0 \end{bmatrix} + \begin{bmatrix} 0 & D_{\mathcal{A}}(a)\cdot m \\ & 0 \end{bmatrix} \\
&+ \begin{bmatrix} \gamma_1(a_{11}, m_{12}) & \gamma_2(a_{11}, m_{12}) \\ & \gamma_3(a_{11}, m_{12}) \end{bmatrix} \\
&= \begin{bmatrix} \gamma_1(a_{11}, m_{12}) & a\cdot\tau_{\mathcal{M}}(m) + D_{\mathcal{A}}(a)\cdot m + \gamma_2(a_{11}, m_{12}) \\ & \gamma_3(a_{11}, m_{12}) \end{bmatrix}.
\end{aligned}$$

Thus, $\gamma_1(a_{11}, m_{12}) = 0$, $\gamma_3(a_{11}, m_{12}) = 0$ and $\tau_{\mathcal{M}}(a\cdot m) = a\cdot\tau_{\mathcal{M}}(m) + D_{\mathcal{A}}(a)\cdot m + \gamma_2(a_{11}, m_{12})$. By the similar method we can prove the case (7).

**(7)** Finally we show that $D_{\mathcal{A}}$ is a generalized 2-cocycle derivation associative with 2-cocycle $\tau_{\mathcal{A}}$. Put $a'_{11} = \begin{bmatrix} a' & 0 \\ & 0 \end{bmatrix}$ and $(aa')_{11} = \begin{bmatrix} aa' & 0 \\ & 0 \end{bmatrix}$, then

$$\begin{aligned}
D(\begin{bmatrix} a & 0 \\ & 0 \end{bmatrix}\begin{bmatrix} a' & 0 \\ & 0 \end{bmatrix}) &= D(a_{11}a'_{11}) = a_{11}D(a'_{11}) + D(a_{11})a'_{11} + \gamma(a_{11}, a'_{11}) \\
&= \begin{bmatrix} a & 0 \\ & 0 \end{bmatrix}\begin{bmatrix} D_{\mathcal{A}}(a') & a'm_D + \gamma_2(a'_{11}, e_{11}) \\ & 0 \end{bmatrix} \\
&+ \begin{bmatrix} D_{\mathcal{A}}(a) & am_D + \gamma_2(a_{11}, e_{11}) \\ & 0 \end{bmatrix}\begin{bmatrix} a' & 0 \\ & 0 \end{bmatrix} \\
&+ \begin{bmatrix} \tau_A(a, a') & \gamma_2(a_{11}, a'_{11}) \\ & \tau_B(0,0) \end{bmatrix} \\
&= \begin{bmatrix} aD_{\mathcal{A}}(a') + a'D_{\mathcal{A}}(a) + \tau_{\mathcal{A}}(a, a') & aa'm_D + a\gamma_2(a'_{11}, e_{11}) + \gamma_2(a_{11}, a'_{11}) \\ & 0 \end{bmatrix}
\end{aligned}$$

On the other hand we have

$$\begin{aligned}
D(\begin{bmatrix} a & 0 \\ & 0 \end{bmatrix}\begin{bmatrix} a' & 0 \\ & 0 \end{bmatrix}) &= D(\begin{bmatrix} aa' & 0 \\ & 0 \end{bmatrix}) \\
&= \begin{bmatrix} D_{\mathcal{A}}(aa') & aa'm_D + \gamma_2((aa')_{11}, e_{11}) \\ & 0 \end{bmatrix}.
\end{aligned}$$

Above relations follow that

$$D_{\mathcal{A}}(aa') = aD_{\mathcal{A}}(a') + a'D_{\mathcal{A}}(a) + \tau_{\mathcal{A}}(a, a'),$$

and

$$a\gamma_2(a'_{11}, e_{11}) + \gamma_2(a_{11}, a'_{11}) = \gamma_2(a_{11}a'_{11}, e_{11}).$$

Thus $D_{\mathcal{A}}$ is generalized 2-cocycle derivation. The same way of $D_{\mathcal{A}}$ proves that $D_{\mathcal{B}}$ is a generalized 2-cocycle derivation. $\square$

**Note 2.2.** In the Theorem 2.1, if $D : \mathbb{T} \longrightarrow \mathbb{T}$ is continuous and $\gamma \in Z^2(\mathbb{T}, \mathbb{T})$ then all obtained generalized derivations and 2-cocycles will be continuous. From now on, we suppose that all maps (generalized derivations, module mappings and 2-cocycles) are continuous.

**Proposition 2.3.** *Let $D_{\mathcal{A}} : \mathcal{A} \longrightarrow A$ and $D_{\mathcal{B}} : \mathcal{B} \longrightarrow \mathcal{B}$ be generalized 2-coclyes derivation associate with 2-cocycles $\tau_{\mathcal{A}}$ and $\tau_{\mathcal{B}}$, respectively, and let $\tau_{\mathcal{M}} : \mathcal{M} \longrightarrow \mathcal{M}$ be a linear map that satisfies in conditions (6) and (7) of Theorem 2.1 such that*

$$\gamma_2(T_1, T_2) = \gamma_2\Big(\begin{bmatrix} a_1 & 0 \\ & 0 \end{bmatrix}, \begin{bmatrix} 0 & m_2 \\ & 0 \end{bmatrix}\Big) + \gamma_2\Big(\begin{bmatrix} 0 & m_1 \\ & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ & b_2 \end{bmatrix}\Big), \tag{2}$$

*and*

$$T'\gamma_2(T_1T_2) - \gamma_2(T'T_1, T_2) + \gamma_2(T', T_1T_2) - \gamma_2(T', T_1)T_2 = \tau_{\mathcal{A}}(a', a_1)\cdot m_2 - m'\cdot\tau_{\mathcal{B}}(b_1, b_2), \tag{3}$$

*for all* $T_1 = \begin{bmatrix} a_1 & m_1 \\ & b_1 \end{bmatrix}, T_2 = \begin{bmatrix} a_2 & m_2 \\ & b_2 \end{bmatrix}, T' = \begin{bmatrix} a' & m' \\ & b' \end{bmatrix} \in \mathbb{T}$.
*Then the map*

$$D\Big(\begin{bmatrix} a & m \\ & b \end{bmatrix}\Big) = \begin{bmatrix} D_{\mathcal{A}}(a) & \tau_{\mathcal{M}}(m) \\ & D_{\mathcal{B}}(b) \end{bmatrix},$$

*is a generalized 2-cocycle derivation with associate 2-cocycle* $\gamma = \begin{bmatrix} \tau_{\mathcal{A}} & \gamma_2 \\ & \tau_{\mathcal{B}} \end{bmatrix}$.

**Proof.** Clearly by (2.3) $\gamma$ is a 2-cocycle on $\mathbb{T} \times \mathbb{T}$. Then for every $T_1 =$

$\begin{bmatrix} a_1 & m_1 \\ & b_1 \end{bmatrix}, T_2 = \begin{bmatrix} a_2 & m_2 \\ & b_2 \end{bmatrix} \in \mathbb{T}$, we have

$$\begin{aligned}
& T_1D(T_2) + D(T_1)T_2 + \gamma(T_1,T_2) \\
= & \begin{bmatrix} a_1 & m_1 \\ & b_1 \end{bmatrix}\begin{bmatrix} D_{\mathcal{A}}(a_2) & \tau_{\mathcal{M}}(m_2) \\ & D_{\mathcal{B}}(b_2) \end{bmatrix} + \begin{bmatrix} D_{\mathcal{A}}(a_1) & \tau_{\mathcal{M}}(m_1) \\ & D_{\mathcal{B}}(b_1) \end{bmatrix}\begin{bmatrix} a_2 & m_2 \\ & b_2 \end{bmatrix} \\
+ & \begin{bmatrix} \tau_{\mathcal{A}}(a_1,a_2) & \gamma_2(T_1,T_2) \\ & \tau_{\mathcal{B}}(b_1,b_2) \end{bmatrix} \\
= & \begin{bmatrix} a_1D_{\mathcal{A}}(a_2) + D_{\mathcal{A}}(a_1)a_2 + \tau_{\mathcal{A}}(a_1,a_2) & a_1 \cdot \tau_{\mathcal{M}}(m_2) + m_1 \cdot D_{\mathcal{B}}(b_2) + D_{\mathcal{A}}(a_1) \cdot m_2 \\ & +\tau_{\mathcal{M}}(m_1) \cdot b_2 + \gamma_2(T_1,T_2) \\ & b_1D_{\mathcal{B}}(b_2) + D_{\mathcal{B}}(b_1)b_2 + \tau_{\mathcal{B}}(b_1,b_2) \end{bmatrix}.
\end{aligned}$$

As well as, together with cases (6) and (7) of Theorem 2.1 and (2.2) we have

$$\begin{aligned}
D(T_1T_2) &= D\Big(\begin{bmatrix} a_1a_2 & a_1 \cdot m_2 + m_1 \cdot b_2 \\ & b_1b_2 \end{bmatrix}\Big) \\
&= \begin{bmatrix} D_{\mathcal{A}}(a_1a_2) & \tau_{\mathcal{M}}(a_1 \cdot m_2 + m_1 \cdot b_2) \\ & D_{\mathcal{B}}(b_1b_2) \end{bmatrix} \\
&= \begin{bmatrix} a_1D_{\mathcal{A}}(a_2) + D_{\mathcal{A}}(a_1)a_2 + \tau_{\mathcal{A}}(a_1,a_2) & a_1 \cdot \tau_{\mathcal{M}}(m_2) + m_1 \cdot D_{\mathcal{B}}(b_2) + D_{\mathcal{A}}(a_1) \cdot m_2 \\ & +\tau_{\mathcal{M}}(m_1) \cdot b_2 + \gamma_2(T_1,T_2) \\ & b_1D_{\mathcal{B}}(b_2) + D_{\mathcal{B}}(b_1)b_2 + \tau_{\mathcal{B}}(b_1,b_2) \end{bmatrix}
\end{aligned}$$

Thus $D$ is a generalized derivation associate with 2-cocycle $\gamma$. $\square$

If $\mathcal{M} = 0$, then triangular Banach algebra reformed to $\mathcal{A} \oplus_1 \mathcal{B}$ with the following sum and product

$$(a,b) + (a',b') = (a + a, b + b'), \text{ and } (a,b)(a',b') = (aa, bb'),$$

for every $(a,b),(a',b') \in \mathcal{A} \oplus_1 \mathcal{B}$. It becomes a Banach algebra with defined norm as $\|(a,b)\| = \|a\|_{\mathcal{A}} + \|b\|_{\mathcal{B}}$. We set $\mathfrak{A} = \mathcal{A} \oplus_1 \mathcal{B}$. Let $\gamma \in Z^2(\mathfrak{A},\mathfrak{A})$, $\gamma_1 : \mathfrak{A} \times \mathfrak{A} \longrightarrow \mathcal{A}$ and $\gamma_2 : \mathfrak{A} \times \mathfrak{A} \longrightarrow \mathcal{B}$ be the coordinate functions associated to $\gamma$ that is

$$\gamma((a_1,b_1),(a_2,b_2)) = (\gamma_1((a_1,b_2),(a_2,b_2)), \gamma_2((a_1,b_2),(a_2,b_2))),$$

for all $(a_1,b_1),(a_2,b_2) \in \mathfrak{A}$. If $\gamma_1$ corresponds to a 2-cocycle on $\mathcal{A} \times \mathcal{A}$ then there exists a 2-cocycle $\tau_{\mathcal{A}}$ on $\mathcal{A} \times \mathcal{A}$ such that $\gamma_1((a_1,b_1),(a_2,b_2)) =$

$\tau_{\mathcal{A}}(a_1, a_2)$. Similarly, if $\gamma_2$ corresponds to a 2-cocycle on $\mathcal{B}\times\mathcal{B}$ then there exists a 2-cocycle $\tau_{\mathcal{B}}$ on $\mathcal{B}\times\mathcal{B}$ such that $\gamma_1((a_1, b_1), (a_2, b_2)) = \tau_{\mathcal{B}}(b_1, b_2)$. We reduce the Lemma 1.2. into this Banach algebra as follows.

**Lemma 2.4.** *Let $\gamma \in B^2(\mathfrak{A}, \mathfrak{A})$. Then $\gamma$ is a 2-cocycle if and only if there are 2-cocycles $\tau_{\mathcal{A}}$ and $\tau_{\mathcal{B}}$ on $\mathcal{A}\times\mathcal{A}$ and $\mathcal{B}\times\mathcal{B}$, respectively, such that $\gamma = (\tau_A, \tau_{\mathcal{B}})$.*

**Proof.** The case where $\gamma$ is a 2-cocycle existence of $\tau_{\mathcal{A}}$ and $\tau_{\mathcal{B}}$ are clear by the same reasoning in Lemma 1.2. For converse, let $\tau_{\mathcal{A}} \in Z^2(\mathcal{A}, \mathcal{A})$ and $\tau_{\mathcal{B}} \in Z^2(\mathcal{B}, \mathcal{B})$. We shall show that $\gamma = (\tau_A, \tau_{\mathcal{B}})$ belongs to $Z^2(\mathfrak{A}, \mathfrak{A})$. For every $(a_1, b_1), (a_2, b_2), (a_3, b_3) \in \mathfrak{A}$,

$$\begin{aligned}
&(a_1, b_1)\gamma((a_2, b_2), (a_3, b_3)) - \gamma((a_1, b_1)(a_2, b_2), (a_3, b_3))\\
&\quad + \quad \gamma((a_1, b_1), (a_2, b_2)(a_3, b_3))\\
&\quad - \quad \gamma((a_1, b_1), (a_2, b_2))(a_3, b_3)\\
&\quad = \quad (a_1, b_1)\gamma((a_2, b_2), (a_3, b_3)) - \gamma((a_1a_2, b_1b_2), (a_3, b_3))\\
&\quad + \quad \gamma((a_1, b_1), (a_2a_3, b_2b_3))\\
&\quad - \quad \gamma((a_1, b_1), (a_2, b_2))(a_3, b_3)\\
&\quad = \quad (a_1, b_1)(\gamma_1((a_2, b_2), (a_3, b_3)), \gamma_2((a_2, b_2), (a_3, b_3)))\\
&\quad - \quad (\gamma_1((a_1a_2, b_2b_2), (a_3, b_3)), \gamma_2((a_1a_2, b_2b_2), (a_3, b_3)))\\
&\quad + \quad (\gamma_1((a_1, b_1), (a_2a_3, b_2b_3)), \gamma_2((a_1, b_1), (a_2a_3, b_2b_3)))\\
&\quad - \quad (\gamma_1((a_1, b_1), (a_2, b_2)), \gamma((a_1, b_1), (a_2, b_2)))(a_3, b_3)\\
&\quad = \quad (a_1, b_1)(\tau_{\mathcal{A}}((a_2, a_3)), \tau_{\mathcal{B}}((b_2, b_3))) - (\tau_{\mathcal{A}}((a_1a_2, a_3)), \tau_{\mathcal{B}}((b_1b_2, b_3)))\\
&\quad + \quad (\tau_{\mathcal{A}}((a_1, a_2a_3)), \tau_{\mathcal{B}}((b_1, b_2b_3))) - (\tau_{\mathcal{A}}((a_1, a_2)), \tau_{\mathcal{B}}((b_1, b_2)))(a_3, b_3)\\
&\quad = \quad (a_1\tau_{\mathcal{A}}((a_2, a_3)) - \tau_{\mathcal{A}}((a_1a_2, a_3)), b_1\tau_{\mathcal{B}}((b_2, b_3)) - \tau_{\mathcal{B}}((b_1b_2, b_3)))\\
&\quad + \quad (\tau_{\mathcal{A}}((a_1, a_2a_3)) - \tau_{\mathcal{A}}((a_1, a_2))a_3, \tau_{\mathcal{B}}((b_1, b_2b_3)) - \tau_{\mathcal{B}}((b_1, b_2))b_3)\\
&\quad = \quad 0.
\end{aligned}$$

Thus $\gamma$ is a 2-cocycle. $\square$

**Proposition 2.5.** *Let $\mathfrak{A} = \mathcal{A}\oplus_1\mathcal{B}$, where $\mathcal{A}$ and $\mathcal{B}$ are Banach algebras. If $D : \mathfrak{A} \longrightarrow \mathfrak{A}$ is a generalized derivation associate with 2-cocycle $\gamma$, then there are generalized derivations $D_{\mathcal{A}}$ and $D_{\mathcal{B}}$ associate with 2-cocycles*

*$\tau_{\mathcal{A}}$ and $\tau_{\mathcal{B}}$, respectively, such that $\gamma = (\tau_{\mathcal{A}}, \tau_{\mathcal{B}})$.*

**Proof.** Define $D_{\mathcal{A}} : \mathcal{A} \longrightarrow \mathcal{A}$ by $D_{\mathcal{A}}(a) = e_{\mathcal{A}} D((a, 0)) e_{\mathcal{A}}$ for all $a \in \mathcal{A}$. By Lemma 2.4, there are 2-cocycles $\tau_{\mathcal{A}}$ and $\tau_{\mathcal{B}}$ such that $\gamma((a_1, b_1), (a_2, b_2)) = (\tau_{\mathcal{A}}(a_1, a_2), \tau_{\mathcal{B}}(b_1, b_2))$ for all $(a_1, b_1), (a_2, b_2) \in \mathfrak{A}$. Then

$$\begin{aligned} D_{\mathcal{A}}(a_1 a_2) &= e_{\mathcal{A}} D((a_1 a_2, 0)) e_{\mathcal{A}} = e_{\mathcal{A}} D((a_1, 0)(a_2, 0)) e_{\mathcal{A}} \\ &= e_{\mathcal{A}}(a_1, 0) D((a_2, 0)) e_{\mathcal{A}} + e_{\mathcal{A}} D((a_1, 0))(a_2, 0) e_{\mathcal{A}} \\ &+ e_{\mathcal{A}} \gamma((a_1, 0), (a_2, 0)) e_{\mathcal{A}} \\ &= e_{\mathcal{A}}(a_1, 0) D((a_2, 0)) e_{\mathcal{A}} + e_{\mathcal{A}} D((a_1, 0))(a_2, 0) e_{\mathcal{A}} \\ &+ e_{\mathcal{A}}(\tau_{\mathcal{A}}(a_1, a_2), 0) e_{\mathcal{A}} \\ &= a_1 D_{\mathcal{A}}(a_2) + D_{\mathcal{A}}(a_1) a_2 + \tau_{\mathcal{A}}(a_1, a_2), \end{aligned}$$

for every $a_1, a_2 \in \mathcal{A}$. Similarly, if we define $D_{\mathcal{B}} : \mathcal{B} \longrightarrow \mathcal{B}$ by $D_{\mathcal{B}}(b) = e_{\mathcal{B}} D((0, b)) e_{\mathcal{B}}$ for all $b \in \mathcal{B}$, then by the same reasoning for proof of $D_{\mathcal{A}}$, $D_{\mathcal{B}}$ become a generalized derivation associate with 2-cocycle $\tau_{\mathcal{B}}$. □

**Proposition 2.6.** *Let $\mathfrak{A} = \mathcal{A} \oplus_1 \mathcal{B}$, where $\mathcal{A}$ and $\mathcal{B}$ are Banach algebras. If $D_{\mathcal{A}}$ and $D_{\mathcal{B}}$ are generalized derivation associate with 2-cocycles $\tau_{\mathcal{A}}$ and $\tau_{\mathcal{B}}$, respectively, then $D : \mathfrak{A} \longrightarrow \mathfrak{A}$ defined by $D = (D_{\mathcal{A}}, D_{\mathcal{B}})$ is a generalized derivation associate with 2-cocycle $\gamma = (\tau_{\mathcal{A}}, \tau_{\mathcal{B}})$.*

**Proof.** By Lemma 2.4, $\gamma = (\tau_{\mathcal{A}}, \tau_{\mathcal{B}})$ is a 2-cocycle. Then

$$\begin{aligned} D((a_1, b_1)(a_2, b_2)) &= D((a_1 a_2, b_1 b_2)) = (D_{\mathcal{A}}(a_1 a_2), D_{\mathcal{B}}(b_1 b_2)) \\ &= (a_1 D_{\mathcal{A}}(a_2) + D_{\mathcal{A}}(a_1) a_2 + \tau_{\mathcal{A}}(a_1, a_2), b_1 D_{\mathcal{B}}(b a_2) \\ &+ D_{\mathcal{B}}(b_1) b_2 + \tau_{\mathcal{B}}(b_1, b_2)), \end{aligned}$$

and

$$\begin{aligned} &(a_1, b_1) D((a_2, b_2)) + D((a_1, b_1))(a_2, b_2) + \gamma((a_1, b_1), (a_2, b_2)) \\ &= (a_1 D_{\mathcal{A}}(a_2), b_1 D_{\mathcal{B}}(b a_2)) + (D_{\mathcal{A}}(a_1) a_2, D_{\mathcal{B}}(b_1) b_2) \\ &+ (\tau_{\mathcal{A}}(a_1, a_2), \tau_{\mathcal{B}}(b_1, b_2)), \end{aligned}$$

for all $(a_1, b_1), (a_2, b_2) \in \mathfrak{A}$. Thus, $D$ is a generalized derivation associate with 2-cocycle $\gamma$. □

## References

[1] B. E. Forrest and L. W. Marcoux, Derivations of triangular Banach algebras, *Indiana Univ. Math. J.,* 45 (1996), 441-462.

[2] B. E. Forrest and L. W. Marcoux, Second order cohomology of triangular Banach algebras, *Houston J. Math.,* 30 (4) (2004), 1157-1176.

[3] J. Li and J. Zhou, Generalized Jordan derivation associate with Hochschild 2-cocycles on some algebras, *Czechoslovak Mathematical Journal,* 60 (135) (2010), 909-932.

[4] A. Majieed and J. Zhou, Generalized Jordan derivation associate with Hochschild 2-cocycles of triangular algebras, *Czechoslovak Mathematical Journal,* 60 (135) (2010), 211-219.

[5] A. Nakajima, Note on generalized Jordan derivation associate with Hochschild 2-cocycles of rings, *Turk. J. Math.*, 30 (2006), 403-411.

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