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Original Research Paper

Applications of Pythagorean Fuzzy Superhypergraph Entropy in Uncertainty Optimization

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Abstract. This paper considers the Pythagorean fuzzy superhypergraphs (PFSh) as a motivation for generalizing of Pythagorean fuzzy graphs and the modeling of some complex real problems. We analyze the Pythagorean fuzzy quasi-superhypergraphs (PFQS), in the class of smooth, soft, rough, and coverable subcategories. It is investigated the impact of Pythagorean (α, β) -level subsets and considered the relation between Pythagorean fuzzy sets and quasi-superhypergraphs. To evaluate uncertainty, establish definitions for Pythagorean fuzzy entropy of supervertices, superedges, and links, proving the resulting superhypergraph entropy is a valid measure. This paper presents the fuzzy Kruskal's Algorithm to semi optimal superedge entropy via a spanning superhypertree. In final, afford applications in business superhypernetworks and social complex hypernetworks, via fuzzy entropy of Pythagorean fuzzy superedges.

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1 Introduction

The limitation of graphs in link of just two object as an edge in the same time, led to the development of hypergraphs, where an edge can connect any number of vertices [1]. Also, in a hypergraph, an edge cannot be defined as an exact relationship between objects; rather, it acts as a set of connections. To solve this limitation, superhypergraphs were introduced as an extended theoretical model that builds upon hypergraphs by enabling recursive, nested, or multi-scale interactions among vertices and hyperedges. Smarandache offered the n -superhypergraph and plithogenic n -superhypergraph, providing a framework for handling complex real-world scenarios with increased granularity [18]. Building on this, Hamidi et al. [4] formally proposed the quasi-superhypergraph model, exploring its properties and potential applications. Our work directly extends these structural innovations by fusing them with the uncertainty representation of Pythagorean fuzzy sets. Superhypergraphs allow hyperedges to connect not only individual vertices but also other hyperedges, fostering hierarchical and self-referential structures.

Fuzzy set theory introduced by Lotfi A. Zadeh further revolutionized mathematics by generalizing classical sets through partial membership functions mapping elements to the interval $[0, 1]$, [22]. Rosenfeld extended these principles to create fuzzy graphs, as a representation of weighted social or traffic networks [17]. Kaufmann introduced fuzzy hypergraphs in the 1970s to accommodate hyperedges with fuzzy membership levels, thereby broadening the scope of fuzzy graph theory [13]. Also, recent advancements have integrated rough set and soft set theories with hypergraphs for enhanced decision-making under uncertainty. Hamidi et al., applied these notions of fuzzy subsets and superhypergraphs to decision-making processes in real-world scenarios [5, 6]. Fuzzy entropy, akin to Shannon entropy in information theory, has emerged as a pivotal tool for measuring randomness by evaluating ambiguity or disorder within fuzzy systems. Early foundational work by De Luca and Termini in 1972 established an axiomatic approach to adapt probabilistic entropy for use with membership functions [3]. Subsequent contributions from researchers such as Pal and Pal [15], Yager [21], Kosko [10], and Bhandari [2] have refined entropy metrics to address increasingly sophis-

ticated uncertainty challenges. In relational modeling, traditional graph theory struggles to capture the complexity of nuanced associations due to its reliance on binary representations. Real-world interactions whether human relationships, biological dependencies, social dynamics, or infrastructure networks often encompass varying levels of strength, reliability, or ambiguity that defy simple binary categorization [8, 9, 12, 16, 19].

Building on these foundations, this research advances the field by considering the Pythagorean fuzzy entropy superhypergraphs that suited for modeling systems with multi uncertainties. The study emphasizes contributions across three dimensions: the formulation and analysis of Pythagorean fuzzy entropy superhypergraph structures, an extension of valued-cut set concepts and the development of generalized fuzzy entropy measures specific to fuzzy superhypergraphs for quantifying uncertainty.

Motivation and contribution: Motivation for this research is based the creating advanced mathematical tools of representing the real problem in real-world systems, and presenting methods for quantifying uncertainty within complex combinatorial structures. Indeed, we extend the fuzzy entropy in complex networks to applications of Pythagorean fuzzy entropy superhypergraphs in addressing these challenges.

2 Preliminaries

In this section, we revisit some definitions and results that will be utilized in the subsequent discussions. Throughout the paper, X denotes a non-empty set.

Theorem 2.1. [11] *Given a connected, undirected, and weighted graph $G = (V, E)$, Kruskal's algorithm constructs a minimum spanning tree.*

Definition 2.2. [20] *A pair $P = (\mu_P, \nu_P) = \{(x, \mu_P(x), \nu_P(x)) \mid x \in X\}$ is a Pythagorean fuzzy subset of X (PFS), if $\mu_P, \nu_P : X \rightarrow [0, 1]$ and for any $x \in X$, $\mu_P^2(x) + \nu_P^2(x) \leq 1$.*

From now on, will denote any $FS, A = \{(x, \mu(x)) \mid x \in X\}$ by A and any $PFS, P = (\mu_P, \nu_P)$ by $P \in PFS(X)$ for simplify.

Definition 2.3. [19] *Let $P, Q \in PFS(X)$. An entropy measure $E(P)$ is an $E : PFS(X) \rightarrow [0, 1]$ that satisfies the following axioms:*

(EPFs-1): $0 \leq E(P) \leq 1$,

(EPFs-2): $E(P) = 0$ iff P is a crisp set $((\mu_P, \nu_P) = (1, 0) \text{ or } (\mu_P, \nu_P) = (0, 1))$,

(EPFs-3): $E(P) = 1$ iff $\mu_P = \nu_P$,

(EPFs-4): $E(P) = E(P^c)$, that $P^c = (\nu_P, \mu_P)$,

(EPFs-5): $E(P) \leq E(Q)$, iff $(\mu_P \leq \mu_Q \text{ and } \nu_P \geq \nu_Q, \text{ for } \mu_Q \leq \nu_Q) \text{ or } (\mu_P \geq \mu_Q \text{ and } \nu_P \leq \nu_Q, \text{ for } \mu_Q \geq \nu_Q)$.

Definition 2.4. [14] A pair $G = (P, Q)$ is a Pythagorean fuzzy graph (PFG) of X , if $P, Q \in PFS(X)$ and for any $xy := (x, y) \in X \times X$, $\mu_Q(xy) \leq \min\{\mu_P(x), \mu_P(y)\}$ and $\nu_Q(xy) \geq \max\{\nu_P(x), \nu_P(y)\}$.

Definition 2.5. [4]

(i) $H = (X, S = \{S_i\}_{i=1}^k, \Phi = \{\varphi_{i,j}\}_{i,j})$ is called a quasi-superhypergraph (QSH) if, $\Phi \neq \emptyset$ and $X = \bigcup_{i=1}^n S_i$, where $k \geq 2$. Each set $S_i \in P^*(X)$ is referred to as a "supervertex," and for every pair $i \neq j$, the mapping $\varphi_{i,j} : S_i \rightarrow S_j$, which creates a link from S_i to S_j , is termed a "superedge."

(ii) The quasi-superhypergraph $H = (X, \{S_i\}_{i=1}^k, \{\varphi_{i,j}\}_{i,j})$ is a superhypergraph (SH) if for any supervertex $S_i \in P^*(X)$, there exists at least one $S_j \in P^*(X)$ such that S_i has a link(connection) to S_j . However, not all supervertices are required to be connected.

(iii) A superhypergraph $H = (X, \{S_i\}_{i=1}^k, \{\varphi_{i,j}\}_{i,j})$ is considered trivial when $k = 1$. In this case, the single supervertex S_1 cannot establish a link to itself.

Let $\mathcal{SH}(X) = \{H \mid H \text{ is a SH on } X\}$ and $\mathcal{SH}(n_1, n_2, \dots, n_k) = \{(X, \{S_i\}_{i=1}^k, \{\varphi_{i,j}\}) \in \mathcal{SH} \mid |S_i| = n_i\}$.

Theorem 2.6. [4] Let $|X| = n$. If $\alpha = \{(n_1, n_2, \dots, n_r) \mid \sum_{i=1}^r n_i = n, n_i, r \in \mathbb{N}\}$ and $m = |\{i \mid n_i = n_j\}|$, then

$$|\mathcal{SH}(X)| = \sum_{\alpha} \left(\frac{1}{m!} \prod_{i=1}^r \binom{n - \sum_{j=1}^{i-1} n_j}{n_i} (\sum_{1 \leq i \neq j \leq r} n_i^{n_j}) \right).$$

3 On Pythagorean Fuzzy (Quasi) Superhypergraph

This section considers the concepts of Pythagorean fuzzy supervertices, Pythagorean fuzzy superedges, Pythagorean fuzzy links, and Pythagorean fuzzy superhypergraphs and the relation of Pythagorean fuzzy graphs and Pythagorean fuzzy superhypergraphs is investigated.

Let $P \in PFS(X)$, then define support of P , by $P^* = \{x \in X \mid \mu_P(x) > 0, \nu_P(x) < 1\}$.

Hamidi et.al, have been introduced the notion of Pythagorean fuzzy quasi superhypergraph as follows.

Definition 3.1. [7] Assume that $\mathcal{V} = \{P_i = (\mu_{P_i}, \nu_{P_i})\}_{i=1}^v, \mathcal{E} = \{Q_{i,j} = (\mu_{Q_{i,j}}, \nu_{Q_{i,j}})\}_{i,j \geq 1}^e, \mathcal{D} = \{R_i = (\mu_{R_i}, \nu_{R_i})\}_{i=1}^d$ is a finite family of non-trivial PFS of X . Then $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$ is a Pythagorean fuzzy quasi superhypergraph (PFQS) on X , if $X = \bigcup_{i=1}^v P_i^*$ and $\forall xy \in R_i^*, \forall xy \in Q_{i,j}^*,$
 (i) $\mu_{R_i}(xy) \leq \min\{\mu_{P_i}(x), \mu_{P_i}(y)\}, \nu_{R_i}(xy) \geq \max\{\nu_{P_i}(x), \nu_{P_i}(y)\},$
 (ii) $\mu_{Q_{i,j}}(xy) \leq \min\{\frac{\mu_{P_i}(x)}{w(P_i)}, \frac{\mu_{P_j}(y)}{w(P_j)}\}, \nu_{Q_{i,j}}(xy) \geq \max\{\frac{\nu_{P_i}(x)}{w(P_i)}, \frac{\nu_{P_j}(y)}{w(P_j)}\},$
 where $w(P_i) = \sum_{x \in P_i^*} (\mu_{P_i} + \nu_{P_i})(x)$. We will call $\mathcal{V}$ by Pythagorean fuzzy supervertices, $\mathcal{E}$ by Pythagorean fuzzy superedges and $\mathcal{D}$ by Pythagorean fuzzy links. Will call a PFQS $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$ as a Pythagorean fuzzy superhypergraph (PFSh) on X , if $|\mathcal{V}| \leq |\mathcal{E}|$ and for any $1 \leq i \leq d, |R_i^*| \geq 1$. If for any $1 \leq i \leq v, w(P_i) < 1$, H is a Soft PFSh, if for any $1 \leq i \leq v, w(P_i) > 1$, H is a Rough PFSh and if for any $1 \leq i \leq v, w(P_i) = 1$, H is a Smooth PFSh. A PFQS $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$ is Coverable, if any member of $\mathcal{E}$ is an onto map and $\mu_{R_i}(xy) + \mu_{P_i}(x) \leq \nu_{R_i}(xy) + \nu_{P_i}(x), \mu_{R_i}(xy) + \mu_{P_i}(y) \leq \nu_{R_i}(xy) + \nu_{P_i}(y)$ and similar to for Q_{ij} .

Example 3.2. Let $X = \{x_i\}_{i=1}^7$. Then $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$ is a rough and coverable PFSh on X as Figure 1, where $\mathcal{V} = \{P_1, P_2, P_3\}, \mathcal{E} = \{Q_{12}, Q_{13}, Q_{23}\}$ and $\mathcal{D} = \{R_1, R_2, R_3\}$.

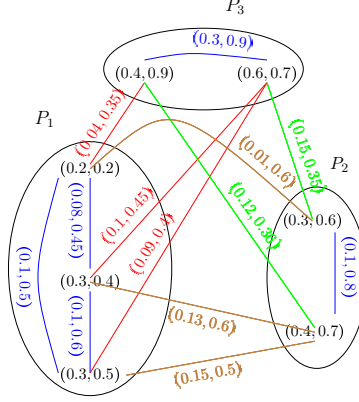


Figure 1: Rough and Coverable PFSh $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$

Indeed,

$$\begin{aligned}
 P_1 &= \{(x_1, 0.2, 0.2), (x_2, 0.3, 0.4), (x_3, 0.3, 0.5)\}, P_2 = \{(x_4, 0.3, 0.6), \\
 &(x_5, 0.4, 0.7)\}, P_3 = \{(x_6, 0.4, 0.9), (x_7, 0.6, 0.7)\}, \\
 Q_{12} &= \{(x_1x_4, 0.01, 0.6), (x_2x_5, 0.13, 0.6), (x_3x_5, 0.15, 0.5)\}, \\
 Q_{13} &= \{(x_1x_6, 0.04, 0.35), (x_2x_7, 0.1, 0.45), (x_3x_7, 0.09, 0.4)\}, \\
 Q_{23} &= \{(x_4x_7, 0.15, 0.35), (x_5x_6, 0.12, 0.36)\}, \\
 R_1 &= \{(x_1x_2, 0.08, 0.45), (x_2x_3, 0.1, 0.6), (x_1x_3, 0.1, 0.5)\}, \\
 R_2 &= \{(x_4x_5, 0.1, 0.8)\}, R_3 = \{(x_6x_7, 0.3, 0.9)\}.
 \end{aligned}$$

In what follow, we presented the relation of PFSh and PFG.

Proposition 3.3. *Let X be a non-empty set. Then any smooth PFSh on X , is an extension of PFG.*

Proof. Since $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$ is a PFSh on X , for any $xy \in P_i^* \times P_j^*$,

$$\mu_{Q_{i,j}}(xy) \leq \min\left\{\frac{\mu_{P_i}(x)}{w(P_i)}, \frac{\mu_{P_j}(y)}{w(P_j)}\right\}, \nu_{Q_{i,j}}(xy) \geq \max\left\{\frac{\nu_{P_i}(x)}{w(P_i)}, \frac{\nu_{P_j}(y)}{w(P_j)}\right\}.$$

On the other hand, for any $P_i \in \mathcal{V}$, get $E \subseteq P_i^* \times P_i^*$ and $E \subseteq P_i^* \times P_j^*$,

because of $X = \bigcup_{i=1}^v P_i^*$, $\mu_{R_i}(xy) \leq \min\{\mu_{P_i}(x), \mu_{P_i}(y)\}$ and $\nu_{R_i}(xy) \geq \max\{\nu_{P_i}(x), \nu_{P_i}(y)\}$.

Now, $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$ is a smooth PFS on X , then $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$ is a PF G. $\square$

In what follows, we investigated the impact of Pythagorean (α, β) -level subset on PFS and the relation of PFS and QSH . Let A and E be FS and PFS of X , respectively and $\alpha, \beta \in \mathbb{I}$. Recall that $A^{(\alpha)} := \mu^{(\alpha)} = \{x \in X \mid \mu(x) \geq \alpha\}$ and $E^{(\alpha, \beta)} = \{x \in X \mid \mu_E^2(x) \geq \alpha, \nu_E^2(x) \leq \beta\}$ as known α -level subset of A and Pythagorean (α, β) -level subset of E , respectively.

Proposition 3.4. Let $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$ be a PFS on X and $\alpha, \alpha', \beta, \beta' \in \mathbb{I}$.

$$(i) \quad P_i^{(\alpha, \beta)} = \mu_{P_i}^{2^{(\alpha)}} \cap (1 - \nu_{P_i}^2)^{(1-\beta)}.$$

$$(ii) \quad \text{If } \alpha' \leq \alpha \text{ and } \beta \leq \beta', \text{ then } P_i^{(\alpha, \beta)} \subseteq P_i^{(\alpha', \beta')}.$$

$$(iii) \quad \text{For all } 1 \leq i \leq n, P_i^{(\alpha, 0)} = \emptyset \text{ and } P_i^{(0, \beta)} = (1 - \nu_{P_i}^2)^{(1-\beta)}.$$

$$(iv) \quad \text{If } H \text{ is non-trivial, then } P_i^{(1, \beta)} = \emptyset \text{ and } P_i^{(\alpha, 1)} = \mu_{P_i}^{2^{(\alpha)}}.$$

Proof. (i) Let $x \in X$.

$$\begin{aligned} x \in P_i^{(\alpha, \beta)} &\Leftrightarrow \mu_{P_i}^2(x) \geq \alpha \text{ and } \nu_{P_i}^2(x) \leq \beta \Leftrightarrow 1 - \nu_{P_i}^2(x) \geq 1 - \beta \Leftrightarrow \\ x &\in \mu_{P_i}^{2^{(\alpha)}} \cap (1 - \nu_{P_i}^2)^{(1-\beta)}. \end{aligned}$$

(ii) Let $x \in P_i^{(\alpha, \beta)}$. Then $\mu_{P_i}^2(x) \geq \alpha \geq \alpha'$ and $\nu_{P_i}^2(x) \leq \beta \leq \beta'$. Hence $P_i^{(\alpha, \beta)} \subseteq P_i^{(\alpha', \beta')}$.

(iii) Since $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$ is a PFS on X , get $X = \bigcup_{i=1}^v P_i^*$ and so for any $x \in X, \nu_{P_i}(x) > 0$ and so $\mathcal{V}^{(\alpha, 0)} = \emptyset$. Using (i), $P_i^{(0, \beta)} = \mu_{P_i}^{2^{(0)}} \cap (1 - \nu_{P_i}^2)^{(1-\beta)} = X \cap (1 - \nu_{P_i}^2)^{(1-\beta)} = (1 - \nu_{P_i}^2)^{(1-\beta)}$.

(iv) Since H is non-trivial, for any $x \in X, \nu_{P_i}(x) < 1$, so $P_i^{(1, \beta)} = \emptyset$ and applying (i), $P_i^{(\alpha, 1)} = \mu_{P_i}^{2^{(\alpha)}} \cap (1 - \nu_{P_i}^2)^{(0)} = \mu_{P_i}^{2^{(\alpha)}} \cap X = \mu_{P_i}^{2^{(\alpha)}}$. $\square$

For any $\alpha, \beta \in \mathbb{I}$, will denote

$$\begin{aligned} \mathcal{V}^{(\alpha, \beta)} &= \{P_i^{(\alpha, \beta)}\}_{i=1}^v, \mathcal{E}^{(\alpha, \beta)} = \{Q_{i,j}^{(\alpha, \beta)}\}_{i,j \geq 1}^e, \mathcal{D}^{(\alpha, \beta)} = \{R_i^{(\alpha, \beta)}\}_{i=1}^d, \text{ that} \\ P_i^{(\alpha, \beta)} &= \{x \in X \mid \mu_{P_i}^2(x) \geq \alpha, \nu_{P_i}^2(x) \leq \beta\}, \\ Q_{i,j}^{(\alpha, \beta)} &= \{xy \in P_i^* \times P_j^* \in X \mid \mu_{Q_{i,j}}^2(xy) \geq \alpha, \nu_{Q_{i,j}}^2(xy) \leq \beta\} \text{ and} \\ R_i^{(\alpha, \beta)} &= \{xy \in P_i^* \times P_j^* \mid \mu_{R_i}^2(xy) \geq \alpha, \nu_{R_i}^2(xy) \leq \beta\}. \end{aligned}$$

Definition 3.5. Let X be a non-avoided set. Then $H = (X, S = \{S_i\}_{i=1}^k, \Phi = \{\varphi_{i,j}\}_{i,j}, \Psi = \{\psi_i\}_i)$ is a strong SH, if $H = (X, S = \{S_i\}_{i=1}^k, \Phi = \{\varphi_{i,j}\}_{i,j})$ is a superhypergraph and $\Psi \neq \emptyset$, that the map $\psi_i : S_i \rightarrow S_i$ is self-linked.

Example 3.6. Let $X = \{x_i\}_{i=1}^5$. Then $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$ is a soft PFQS on X as Figure 2, where $\mathcal{V} = \{P_1, P_2, P_3\}$, $\mathcal{E} = \{Q_{12}, Q_{13}, Q_{32}\}$ and $\mathcal{D} = \{R_1, R_2\}$.

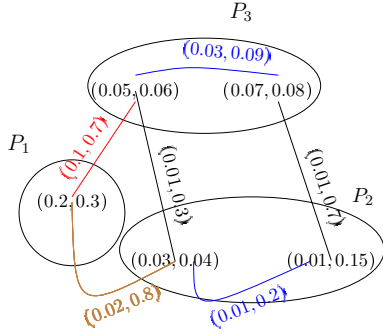


Figure 2: Soft PFSh $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$

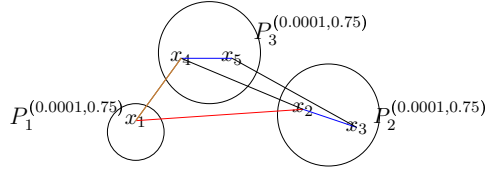


Figure 3: SH $H^{(0.0001, 0.75)}$

Indeed,

$$\begin{aligned} P_1 &= \{(x_1, 0.2, 0.3)\}, P_2 = \{(x_2, 0.03, 0.04), (x_3, 0.01, 0.15)\}, \\ P_3 &= \{(x_4, 0.05, 0.06), (x_5, 0.07, 0.08)\}, \\ Q_{12} &= \{(x_1x_2, 0.02, 0.8)\}, Q_{13} = \{(x_1x_4, 0.1, 0.7)\}, \\ Q_{32} &= \{(x_4x_2, 0.01, 0.3), (x_5x_3, 0.01, 0.7)\}, \\ R_1 &= \{(x_2x_3, 0.01, 0.2)\}, R_2 = \{(x_4x_5, 0.03, 0.09)\}. \end{aligned}$$

Now, for $(\alpha, \beta) = (0.0001, 0.75)$, the strong SH $H^{(0.0001, 0.75)}$ is computed as Figure 3.

From Proposition 3.4, have the following results.

Theorem 3.7. Let $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$ be a PFSh on X and $\alpha, \beta \in \mathbb{I}$.

- (i) If H is a rough PFSh, then $Q_{ij}^{(\alpha, \beta)} \subseteq \mu_{P_i}^{(\sqrt{\alpha})} \times \mu_{P_j}^{(\sqrt{\alpha})}$.
- (ii) If H is a soft PFSh, then $Q_{ij}^{(\alpha, \beta)} \subseteq (1 - \nu_{P_i})^{(\sqrt{1-\beta})} \times (1 - \nu_{P_j})^{(\sqrt{1-\beta})}$.
- (iii) If H is a smooth PFSh, then $Q_{ij}^{(\alpha, \beta)} \subseteq \mu_{P_i}^{(\sqrt{\alpha})} \times \nu_{P_j}^{(\sqrt{\beta})}$.

Proof. Let $xy \in Q_{ij}^{(\alpha, \beta)}$.

(i) Then $\mu_{Q_{i,j}}^2(xy) \geq \alpha$ and $\nu_{Q_{i,j}}^2(xy) \leq \beta$. Since for any $1 \leq i \leq v$, $w(P_i) \geq 1$ and $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$ is a PFSh on X , $\forall xy \in P_i^* \times P_j^*$,

$$\begin{aligned} \alpha \leq \mu_{Q_{i,j}}^2(xy) &\leq \min\left\{\frac{\mu_{P_i}^2(x)}{w^2(P_i)}, \frac{\mu_{P_j}^2(y)}{w^2(P_j)}\right\} \leq \frac{\mu_{P_i}^2(x)}{w^2(P_i)} \leq \mu_{P_i}^2(x), \\ \alpha \leq \mu_{Q_{i,j}}^2(xy) &\leq \min\left\{\frac{\mu_{P_i}^2(x)}{w^2(P_i)}, \frac{\mu_{P_j}^2(y)}{w^2(P_j)}\right\} \leq \frac{\mu_{P_j}^2(y)}{w^2(P_j)} \leq \mu_{P_j}^2(y). \end{aligned}$$

Hence $Q_{ij}^{(\alpha, \beta)} \subseteq \mu_{P_i}^{(\sqrt{\alpha})} \times \mu_{P_j}^{(\sqrt{\alpha})}$.

(ii) Then $\mu_{Q_{i,j}}^2(xy) \geq \alpha$ and $\nu_{Q_{i,j}}^2(xy) \leq \beta$. Since for any $1 \leq i \leq v$, $w(P_i) \leq 1$ and $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$ is a PFSh on X , $\forall xy \in P_i^* \times P_j^*$,

$$\begin{aligned} \beta \geq \nu_{Q_{i,j}}^2(xy) &\geq \max\left\{\frac{\nu_{P_i}^2(x)}{w^2(P_i)}, \frac{\nu_{P_j}^2(y)}{w^2(P_j)}\right\} \geq \frac{\nu_{P_i}^2(x)}{w^2(P_i)} \geq \nu_{P_i}^2(x) \\ &\Rightarrow 1 - \nu_{P_i}^2(x) \geq 1 - \beta, \\ \beta \geq \nu_{Q_{i,j}}^2(xy) &\geq \max\left\{\frac{\nu_{P_i}^2(x)}{w^2(P_i)}, \frac{\nu_{P_j}^2(y)}{w^2(P_j)}\right\} \geq \frac{\nu_{P_j}^2(y)}{w^2(P_j)} \geq \nu_{P_j}^2(y) \\ &\Rightarrow 1 - \nu_{P_j}^2(y) \geq 1 - \beta. \end{aligned}$$

Hence $Q_{ij}^{(\alpha, \beta)} \subseteq (1 - \nu_{P_i})^{(\sqrt{1-\beta})} \times (1 - \nu_{P_j})^{(\sqrt{1-\beta})}$.

(iii) Based (i), (ii) is clear. $\square$

Corollary 3.8. *Let $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$ be a $PFSH$ on X and $\alpha, \beta \in \mathbb{I}$. Then $R_i^{(\alpha, \beta)} \subseteq \mu_{P_i}^{(\sqrt{\alpha})} \times \nu_{P_i}^{(\sqrt{\beta})}$.*

Theorem 3.9. *Let $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$ be a rough $PFSH$ on X . If $\alpha \leq \bigwedge_{i=1}^d \mu_{R_i}^2$ and $\beta \geq (\bigvee_{i=1}^d \nu_{R_i}^2) \vee (\bigvee_{i,j \geq 1}^e \nu_{Q_{i,j}}^2)$, then $H^{(\alpha, \beta)} = (\mathcal{V}^{(\alpha, \beta)}, \mathcal{E}^{(\alpha, \beta)}, \mathcal{D}^{(\alpha, \beta)})$ is a strong SH .*

Proof. By definition, $X = \bigcup_{i=1}^v P_i^*$. Now, $\alpha \leq \bigwedge_{i=1}^e \mu_{R_i}^2$, and $\beta \geq (\bigvee_{i=1}^d \nu_{R_i}^2) \vee (\bigvee_{i,j \geq 1}^e \nu_{Q_{i,j}}^2)$ imply $\forall xy \in P_i^* \times P_j^*$,

$$\begin{aligned} \alpha &\leq \bigwedge_{i=1}^d \mu_{R_i}^2 \leq \bigwedge_{i=1}^d \{\mu_{P_i}^2(x), \mu_{P_i}^2(y)\} \leq \mu_{P_i}^2(x), \beta \geq \bigvee_{i=1}^d \nu_{R_i}^2 \\ &\geq \bigvee_{i=1}^d \{\nu_{P_i}^2(x), \nu_{P_i}^2(y)\} \geq \nu_{P_i}^2(y), \\ \beta &\geq \bigvee_{i,j \geq 1}^e \nu_{Q_{i,j}}^2 \geq \bigvee_{i=1}^e \{\nu_{P_i}^2(x), \nu_{P_j}^2(y)\} \geq \nu_{P_i}^2(y), \end{aligned}$$

so

$$\begin{aligned} \bigcup_{i=1}^v \mathcal{V}^{(\alpha, \beta)} &= \bigcup_{i=1}^v \{x \in X \mid \mu_{P_i}^2(x) \geq \alpha, \nu_{P_i}^2(x) \leq \beta\} \\ &\supseteq \bigcup_{i=1}^v \{x \in X \mid \mu_{P_i}(x) > 0, \nu_{P_i}(x) < 1\} = X. \end{aligned}$$

Hence, $\bigcup_{i=1}^v \mathcal{V}^{(\alpha, \beta)} = X$. Now, let $xy \in \mathcal{E}^{(\alpha, \beta)}$, then there are $1 \leq i, j \leq e$, that $xy \in Q_{i,j}^{(\alpha, \beta)}$. Since $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$ is a rough $PFSH$ on X ,

$$\begin{aligned} \alpha &\leq \mu_{Q_{i,j}}^2(xy) \leq \min\left\{\frac{\mu_{P_i}^2(x)}{w^2(P_i)}, \frac{\mu_{P_j}^2(y)}{w^2(P_j)}\right\} \leq \frac{\mu_{P_i}^2(x)}{w^2(P_i)} \leq \mu_{P_i}^2(x) \text{ and} \\ \beta &\geq \nu_{Q_{i,j}}^2(xy) \geq \max\left\{\frac{\nu_{P_i}^2(x)}{w^2(P_i)}, \frac{\nu_{P_j}^2(y)}{w^2(P_j)}\right\} \geq \frac{\nu_{P_i}^2(y)}{w^2(P_i)}. \end{aligned}$$

On the other hand, by hypotheses, $\beta \geq (\bigvee_{i=1}^d \nu_{R_i}^2) \vee (\bigvee_{i,j \geq 1}^e \nu_{Q_{i,j}}^2) \geq \nu_{P_i}^2(y)$,

so $\beta \geq \max\{\frac{\nu_{P_i}^2(y)}{w^2(P_i)}, \nu_{P_i}^2(y)\} = \nu_{P_i}^2(y)$, because of $w^2(P_i) > 1$. Thus $xy \in P_i^{(\alpha)} \times P_i^{(\beta)}$ and so $\mathcal{E}^{(\alpha,\beta)} \subseteq \mathcal{V}^{(\alpha,\beta)} \times \mathcal{V}^{(\alpha,\beta)}$. Assume $xy \in \mathcal{D}^{(\alpha,\beta)}$. Then there is $1 \leq i \leq d$, that $xy \in R_i^{(\alpha,\beta)}$, so $\alpha \leq \mu_{R_i}^2(xy)$ and $\nu_{R_i}^2(xy) \geq \beta$. Hence,

$$\begin{aligned} \alpha &\leq \mu_{R_i}^2(xy) \leq \min\{\mu_{P_i}^2(x), \mu_{P_i}^2(y)\}, \beta \geq \nu_{R_i}^2(xy) \geq \max\{\nu_{P_i}^2(x), \nu_{P_i}^2(y)\} \\ &\Rightarrow \alpha \leq \mu_{P_i}^2(x), \alpha \leq \mu_{P_i}^2(y), \beta \geq \nu_{P_i}^2(x), \beta \geq \nu_{P_i}^2(y) \\ &\Rightarrow xy \in P_i^{(\alpha)} \times P_i^{(\beta)} \Rightarrow R_i^{(\alpha,\beta)} \subseteq P_i^{(\alpha)} \times P_i^{(\beta)} \Rightarrow \mathcal{D}^{(\alpha,\beta)} \subseteq \mathcal{V}^{(\alpha,\beta)} \times \mathcal{V}^{(\alpha,\beta)}. \end{aligned}$$

Therefore, $H^{(\alpha,\beta)} = (\mathcal{V}^{(\alpha,\beta)}, \mathcal{E}^{(\alpha,\beta)}, \mathcal{D}^{(\alpha,\beta)})$ is a strong SH. $\square$

Theorem 3.10. *Let $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$ be a soft PFSH on X . If $\alpha \leq (\bigwedge_{i,j \geq 1}^e \mu_{Q_{i,j}}^2) \wedge (\bigwedge_{i=1}^d \mu_{R_i}^2)$ and $\beta \geq \bigvee_{i=1}^d \nu_{R_i}^2$, then $H^{(\alpha,\beta)} = (\mathcal{V}^{(\alpha,\beta)}, \mathcal{E}^{(\alpha,\beta)}, \mathcal{D}^{(\alpha,\beta)})$ is a strong SH.*

Proof. By definition, $X = \bigcup_{i=1}^v P_i^*$. Now, $\alpha \leq (\bigwedge_{i,j \geq 1}^e \mu_{Q_{i,j}}^2) \wedge (\bigwedge_{i=1}^d \mu_{R_i}^2)$ and

$$\beta \geq \bigvee_{i=1}^d \nu_{R_i}^2 \text{ imply } \forall xy \in P_i^* \times P_j^*,$$

$$\begin{aligned} \alpha &\leq \bigwedge_{i=1}^d \mu_{R_i}^2(xy) \leq \bigwedge_{i=1}^d \{\mu_{P_i}^2(x), \mu_{P_i}^2(y)\} \leq \mu_{P_i}^2(x), \\ \alpha &\leq \bigwedge_{i,j \geq 1}^e \mu_{Q_{i,j}}^2(xy) \leq \bigwedge_{i,j \geq 1}^e \{\mu_{P_i}^2(x), \mu_{P_i}^2(y)\} \leq \mu_{P_i}^2(x), \\ \beta &\geq \bigvee_{i=1}^d \nu_{R_i}^2 \geq \bigvee_{i=1}^d \{\nu_{P_i}^2(x), \nu_{P_i}^2(y)\} \geq \nu_{P_i}^2(y), \end{aligned}$$

so

$$\begin{aligned} \bigcup_{i=1}^v \mathcal{V}^{(\alpha,\beta)} &= \bigcup_{i=1}^v \{x \in X \mid \mu_{P_i}^2(x) \geq \alpha, \nu_{P_i}^2(x) \leq \beta\} \\ &\supseteq \bigcup_{i=1}^v \{x \in X \mid \mu_{P_i}(x) > 0, \nu_{P_i}(x) < 1\} = X. \end{aligned}$$

Hence, $\bigcup_{i=1}^v \mathcal{V}^{(\alpha, \beta)} = X$. Now, let $xy \in \mathcal{E}^{(\alpha, \beta)}$, then there are $1 \leq i, j \leq e$, that $xy \in Q_{i,j}^{(\alpha, \beta)}$. Since $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$ is a soft $PFSH$ on X ,

$$\begin{aligned} \alpha \leq \mu_{Q_{i,j}}^2(xy) &\leq \min\left\{\frac{\mu_{P_i}^2(x)}{w^2(P_i)}, \frac{\mu_{P_j}^2(y)}{w^2(P_j)}\right\} \leq \frac{\mu_{P_i}^2(x)}{w^2(P_i)} \text{ and} \\ \beta \geq \nu_{Q_{i,j}}^2(xy) &\geq \max\left\{\frac{\nu_{P_i}^2(x)}{w^2(P_i)}, \frac{\nu_{P_j}^2(y)}{w^2(P_j)}\right\} \geq \frac{\nu_{P_i}^2(y)}{w^2(P_i)} \geq \nu_{P_i}^2(y). \end{aligned}$$

On the other hand, by hypotheses, $\alpha \leq (\bigwedge_{i,j \geq 1}^e \mu_{Q_{i,j}}^2) \wedge (\bigwedge_{i=1}^d \mu_{R_i}^2)$, so $\alpha \leq \min\left\{\frac{\mu_{P_i}^2(x)}{w^2(P_i)}, \mu_{P_i}^2(x)\right\} = \mu_{P_i}^2(x)$, because of $w^2(P_i) < 1$. Thus $xy \in P_i^{(\alpha)} \times P_i^{(\beta)}$ and so $\mathcal{E}^{(\alpha, \beta)} \subseteq \mathcal{V}^{(\alpha, \beta)} \times \mathcal{V}^{(\alpha, \beta)}$.

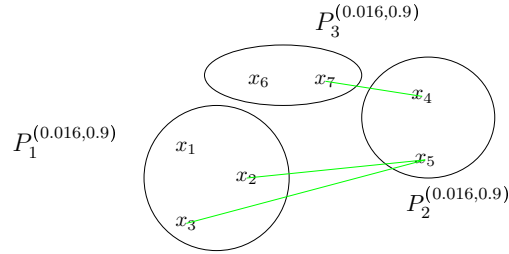
Assume $xy \in \mathcal{D}^{(\alpha, \beta)}$. Then there is $1 \leq i \leq d$, that $xy \in R_i^{(\alpha, \beta)}$, so $\alpha \leq \mu_{R_i}^2(xy)$ and $\nu_{R_i}^2(xy) \geq \beta$. Hence

$$\begin{aligned} \alpha \leq \mu_{R_i}^2(xy) &\leq \min\{\mu_{P_i}^2(x), \mu_{P_i}^2(y)\}, \beta \geq \nu_{R_i}^2(xy) \geq \max\{\nu_{P_i}^2(x), \nu_{P_i}^2(y)\} \\ \Rightarrow \alpha &\geq \mu_{P_i}^2(x), \alpha \geq \mu_{P_i}^2(y), \beta \leq \nu_{P_i}^2(x), \beta \leq \nu_{P_i}^2(y) \\ \Rightarrow xy &\in P_i^{(\alpha)} \times P_i^{(\beta)} \Rightarrow R_i^{(\alpha, \beta)} \subseteq P_i^{(\alpha)} \times P_i^{(\beta)} \Rightarrow \mathcal{D}^{(\alpha, \beta)} \subseteq \mathcal{V}^{(\alpha, \beta)} \times \mathcal{V}^{(\alpha, \beta)}. \end{aligned}$$

Therefore, $H^{(\alpha, \beta)} = (\mathcal{V}^{(\alpha, \beta)}, \mathcal{E}^{(\alpha, \beta)}, \mathcal{D}^{(\alpha, \beta)})$ is a strong SH. $\square$

Corollary 3.11. *Let $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$ be a smooth $PFSH$ on X . If $\alpha \leq (\bigwedge_{i,j \geq 1}^e \mu_{Q_{i,j}}^2) \wedge (\bigwedge_{i=1}^d \mu_{R_i}^2)$ and $\beta \geq (\bigvee_{i=1}^d \nu_{R_i}^2) \vee (\bigvee_{i,j \geq 1}^e \nu_{Q_{i,j}}^2)$, then $H^{(\alpha, \beta)} = (\mathcal{V}^{(\alpha, \beta)}, \mathcal{E}^{(\alpha, \beta)}, \mathcal{D}^{(\alpha, \beta)})$ is a strong SH.*

Example 3.12. *Consider the $PFQS$, $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$ in Example 3.2 as Figure 1. Then $H^{(0.016, 0.9)} = (\mathcal{V}^{(0.016, 0.9)}, \mathcal{E}^{(0.016, 0.9)}, \mathcal{D}^{(0.016, 0.9)})$ is a quasi superhypergraph as Figure 4, while it is not an SH.*

Figure 4: QSH $H^{(0.016,0.9)}$

4 Pythagorean Fuzzy Entropy of Superhypergraph

In this section, we consider an entropy measure on any fuzzy superhypergraphs and investigate its properties.

Let $|X| = n$. A real function $E^* : PFS(X) \rightarrow [0, 1]$ is a Pythagorean fuzzy entropy, by $E^*(P) = \frac{1}{n} \sum_{i=1}^n (1 - |\mu_P(x_i) - \nu_P(x_i)|)$ and we apply it in what follows. Hamidi et. al, introduced the notion of Pythagorean fuzzy entropy on superhypergraphs as follows.

Definition 4.1. [7] Let $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$ be a PFQS on X , $|\mathcal{V}| = v$, $|\mathcal{E}| = e$ and $|\mathcal{D}| = d$. Define

$$(i) \quad E^*(\mathcal{V}) = \frac{1}{v} \sum_{i=1}^v \sum_{j=1}^{|P_i^*|} \frac{1}{|P_i^*|} (1 - |\mu_{P_i}(x_j) - \nu_{P_i}(x_j)|),$$

$$(ii) \quad E^*(\mathcal{E}) = \frac{1}{e} \sum_{i,j \geq 1}^e \sum_{k \neq l \geq 1}^{|Q_{i,j}^*|} \frac{1}{|Q_{i,j}^*|} (1 - |\mu_{Q_{i,j}}((x_k, y_l)) - \nu_{Q_{i,j}}((x_k, y_l))|),$$

$$(iii) \quad E^*(\mathcal{D}) = \frac{1}{d} \sum_{i=1}^d \sum_{k \neq l \geq 1}^{|R_i^*|} \frac{1}{|R_i^*|} (1 - |\mu_{R_i}(x_k y_l) - \nu_{R_i}(x_k y_l)|),$$

as Pythagorean fuzzy entropy of supervertices, Pythagorean fuzzy entropy of superedges and Pythagorean fuzzy entropy of links, respectively.

Example 4.2. Consider the *PFS* as Example 3.2. Then $E^*(\mathcal{V}) \approx 0.7667$, $E^*(\mathcal{E}) \approx 0.6622$, $E^*(\mathcal{D}) \approx 0.4256$.

Let $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$ be a (*PFQS*) on X . Clearly, for any $1 \leq i, j \leq v$, $Q_{i,j}^* \subseteq P_i^* \times P_j^*$ and have the following result.

Theorem 4.3. Let $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$ be a smooth and coverable *PFS* on X . Then $E^*(\mathcal{E}) \leq E^*(\mathcal{V})$.

Proof. Let $|\mathcal{V}| = v$, $|\mathcal{E}| = e$ and $|\mathcal{D}| = d$. By definition, for any $xy \in P_i^* \times P_i^*$,

$$\mu_{Q_{i,j}}(xy) \leq \min\left\{\frac{\mu_{P_i}(x)}{w(P_i)}, \frac{\mu_{P_j}(y)}{w(P_j)}\right\} \text{ and } \nu_{Q_{i,j}}(xy) \geq \max\left\{\frac{\nu_{P_i}(x)}{w(P_i)}, \frac{\nu_{P_j}(y)}{w(P_j)}\right\}.$$

Since $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$ is a *PFS* on X , get $v \leq e$, and since is coverable, get $|Q_{i,j}^*| \geq \max\{|P_i^*|, |P_j^*|\}$. Now, because $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$ is a smooth *PFS*, have

$$\begin{aligned} E^*(\mathcal{E}) &= \frac{1}{e} \sum_{i,j \geq 1}^e \sum_{k \neq l \geq 1}^{|Q_{i,j}^*|} \frac{1}{|Q_{i,j}^*|} (1 - |\mu_{Q_{i,j}}((x_k, y_l)) - \nu_{Q_{i,j}}((x_k, y_l))|) \leq \\ &\frac{1}{e} \sum_{i,j \geq 1}^e \sum_{k \neq l \geq 1}^{|Q_{i,j}^*|} \frac{1}{|Q_{i,j}^*|} (1 - |\min\left\{\frac{\mu_{P_i}(x_k)}{w(P_i)}, \frac{\mu_{P_j}(y_l)}{w(P_j)}\right\} - \max\left\{\frac{\nu_{P_i}(x_k)}{w(P_i)}, \frac{\nu_{P_j}(y_l)}{w(P_j)}\right\}|) \\ &\leq \frac{1}{e} \sum_{i,j \geq 1}^e \sum_{k \neq l \geq 1}^{|Q_{i,j}^*|} \frac{1}{|Q_{i,j}^*|} (1 - |\mu_{P_i}(x_k) - \nu_{P_i}(y_l)|) \\ &\leq \frac{1}{v} \sum_{i,j \geq 1}^v \sum_{k \neq l \geq 1}^{|P_i^*|} \frac{1}{|P_i^*|} (1 - |\mu_{P_i}(x_k) - \nu_{P_i}(y_l)|) = E^*(\mathcal{V}). \end{aligned}$$

□

Theorem 4.4. Let $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$ be a coverable *PFS* on X . Then $E^*(\mathcal{D}) \leq E^*(\mathcal{V})$.

Proof. Since $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$ is a coverable *PFS* on X , for any $1 \leq i \leq d$, $|R_i^*| \geq 1$, so for any $xy \in P_i^* \times P_i^*$,

$\mu_{R_i}(xy) \leq \min\{\mu_{P_i}(x), \mu_{P_i}(y)\}$ and $\nu_{R_i}(xy) \geq \max\{\nu_{P_i}(x), \nu_{P_i}(y)\}$.
Hence, by coverable

$$\begin{aligned} E^*(\mathcal{D}) &= \frac{1}{d} \sum_{i=1}^d \sum_{k \neq l \geq 1} \frac{|R_i^*|}{|R_i^*|} (1 - |\mu_{R_i}((x_k, y_l)) - \nu_{R_i}((x_k, y_l))|) \\ &\leq \frac{1}{d} \sum_{i=1}^d \sum_{k \neq l \geq 1} \frac{|R_i^*|}{|R_i^*|} (1 - |\mu_{P_i}(x_k) - \nu_{P_i}(x_k)|). \end{aligned}$$

Let $S_i(x_k) = 1 - |\mu_{P_i}(x_k) - \nu_{P_i}(x_k)|$ be the entropy term for element x_k in supervertex P_i , then

$$E^*(\mathcal{D}) = \frac{1}{d} \sum_{i=1}^d \frac{1}{|R_i^*|} \sum_{(x_k, x_l) \in R_i^*} S_i(x_k) \text{ and } E^*(\mathcal{V}) = \frac{1}{v} \sum_{i=1}^v \frac{1}{|P_i^*|} \sum_{k=1}^{|P_i^*|} S_i(x_k).$$

On the other hand, For each supervertex P_i , number of possible links is $|R_i^*| \leq \binom{|P_i^*|}{2}$ and each x_k appears in at most $|P_i^*| - 1$ links. Hence,

$$\begin{aligned} \sum_{(x_k, x_l) \in R_i^*} S_i(x_k) &\leq (|P_i^*| - 1) \sum_{k=1}^{|P_i^*|} S_i(x_k) \Rightarrow E^*(\mathcal{D}) \leq \\ &\frac{1}{d} \sum_{i=1}^d \frac{|P_i^*| - 1}{|R_i^*|} \cdot \frac{1}{|P_i^*|} \sum_{k=1}^{|P_i^*|} S_i(x_k). \end{aligned}$$

Since $|R_i^*| \geq 1$ and $|P_i^*| \geq 2$, get $\frac{|P_i^*| - 1}{|R_i^*|} \leq |P_i^*| - 1 \leq |P_i^*|$. It concludes that

$$E^*(\mathcal{D}) \leq \frac{1}{d} \sum_{i=1}^d \sum_{k=1}^{|P_i^*|} S_i(x_k) \leq E^*(\mathcal{V}). \quad \square$$

Let $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$ be a $(PFQS)$ on X . Clearly, for any $1 \leq i, j \leq v$, $Q_{i,j}^* \subseteq P_i^* \times P_j^* = R_i^* \times R_j^*$ and have the following result.

Corollary 4.5. *Let $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$ be a smooth and coverable $PFSh$ on X . Then $E^*(\mathcal{D}) \vee E^*(\mathcal{E}) \leq E^*(\mathcal{V})$.*

Let $\mathcal{P} = \{(\mu_{P_i}, \nu_{P_i})\}_{i \in I}$ be a family of PFS of X . Then $\mathcal{P}$ is a crisp set, if any member of $\mathcal{P}$ is a crisp set and $\mathcal{P}^c = \{(\nu_{P_i}, \mu_{P_i})\}_{i \in I}$. In addition, for another family of PFS of X as $\mathcal{P}' = \{(\mu_{P'_i}, \nu_{P'_i})\}_{i \in I}$, get that $\mathcal{P} \subseteq \mathcal{P}'$ iff for any $x \in X, i \in I$, $\mu_{P_i}(x) \leq \mu_{P'_i}(x) \leq \nu_{P'_i}(x) \leq \nu_{P_i}(x)$ or $\nu_{P_i}(x) \leq \nu_{P'_i}(x) \leq \mu_{P'_i}(x) \leq \mu_{P_i}(x)$.

Theorem 4.6. *Let $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$ be a (PFQS) on X and $|\mathcal{V}| = v$. Then $E^*(\mathcal{V})$ satisfies in (EPFs-1), (EPFs-2), (EPFs-3), (EPFs-4).*

Proof. (EPFs-1): For any $1 \leq i \leq v$ and any $1 \leq j \leq |P_i^*|$, get $\frac{1}{|P_i^*|}(1 - |\mu_{P_i}(x_j) - \nu_{P_i}(x_j)|) \leq \frac{1}{|P_i^*|}$. Thus

$$\begin{aligned} \sum_{j=1}^{|P_i^*|} \frac{1}{|P_i^*|} (1 - |\mu_{P_i}(x_j) - \nu_{P_i}(x_j)|) &\leq \sum_{j=1}^{|P_i^*|} \frac{1}{|P_i^*|} \leq 1 \\ \Rightarrow \sum_{i=1}^v \sum_{j=1}^{|P_i^*|} \frac{1}{|P_i^*|} (1 - |\mu_{P_i}(x_j) - \nu_{P_i}(x_j)|) &\leq \sum_{i=1}^v \sum_{j=1}^{|P_i^*|} \frac{1}{|P_i^*|} \leq \sum_{i=1}^v (1) \Rightarrow \frac{1}{v} \sum_{i=1}^v \sum_{j=1}^{|P_i^*|} \frac{1}{|P_i^*|} (1 - |\mu_{P_i}(x_j) - \nu_{P_i}(x_j)|) \\ &\leq \left(\frac{1}{v}\right)(v) = 1. \end{aligned}$$

Hence, $0 \leq E^*(\mathcal{V}) \leq 1$.

$$\begin{aligned} EPFs - 2 : E^*(\mathcal{V}) = 0 &\Leftrightarrow \frac{1}{v} \sum_{i=1}^v \sum_{j=1}^{|P_i^*|} \frac{1}{|P_i^*|} (1 - |\mu_{P_i}(x_j) - \nu_{P_i}(x_j)|) = 0 \\ &\Leftrightarrow \sum_{i=1}^v \sum_{j=1}^{|P_i^*|} \frac{1}{|P_i^*|} (1 - |\mu_{P_i}(x_j) - \nu_{P_i}(x_j)|) = 0, \\ &\Leftrightarrow \sum_{j=1}^{|P_i^*|} \frac{1}{|P_i^*|} (1 - |\mu_{P_i}(x_j) - \nu_{P_i}(x_j)|) = 0, \\ &\Leftrightarrow \forall 1 \leq i \leq v, \forall 1 \leq j \leq |P_i^*|, 1 - |\mu_{P_i}(x_j) - \nu_{P_i}(x_j)| = 0, \\ &\Leftrightarrow \forall 1 \leq i \leq v, \forall 1 \leq j \leq |P_i^*|, 1 = |\mu_{P_i}(x_j) - \nu_{P_i}(x_j)|, \\ &\Leftrightarrow \forall 1 \leq i \leq v, \forall 1 \leq j \leq |P_i^*|, (1 = \mu_{P_i}(x_j), \nu_{P_i}(x_j) = 0) \text{ or } \\ &\quad (0 = \mu_{P_i}(x_j), \nu_{P_i}(x_j) = 1), \\ &\Leftrightarrow 1 \leq i \leq v, P_i \text{ is a crisp set} \Leftrightarrow \mathcal{V} \text{ is a crisp set.} \end{aligned}$$

$$\begin{aligned}
EPFs-3 : E^*(\mathcal{V}) = 1 &\Leftrightarrow \frac{1}{v} \sum_{i=1}^v \sum_{j=1}^{|P_i^*|} \frac{1}{|P_i^*|} (1 - |\mu_{P_i}(x_j) - \nu_{P_i}(x_j)|) \\
&\Leftrightarrow \sum_{i=1}^v \sum_{j=1}^{|P_i^*|} \frac{1}{|P_i^*|} (1 - |\mu_{P_i}(x_j) - \nu_{P_i}(x_j)|) = v \\
&\Leftrightarrow \forall 1 \leq i \leq v, \sum_{j=1}^{|P_i^*|} \frac{1}{|P_i^*|} (1 - |\mu_{P_i}(x_j) - \nu_{P_i}(x_j)|) = 1, \\
&\Leftrightarrow \forall 1 \leq i \leq v, \sum_{j=1}^{|P_i^*|} (1 - |\mu_{P_i}(x_j) - \nu_{P_i}(x_j)|) = |P_i^*|, \\
&\Leftrightarrow \forall 1 \leq i \leq v, \forall 1 \leq j \leq |P_i^*|, 1 - |\mu_{P_i}(x_j) - \nu_{P_i}(x_j)| = 1, \\
&\Leftrightarrow \forall 1 \leq i \leq v, \forall 1 \leq j \leq |P_i^*|, |\mu_{P_i}(x_j) - \nu_{P_i}(x_j)| = 0, \\
&\Leftrightarrow \forall 1 \leq i \leq v, \forall 1 \leq j \leq |P_i^*|, \mu_{P_i}(x_j) = \nu_{P_i}(x_j).
\end{aligned}$$

(EPFs-4) :

$$\begin{aligned}
E^*(\mathcal{V}^c) &= \frac{1}{v} \sum_{i=1}^v \sum_{j=1}^{|P_i^*|} \frac{1}{|P_i^*|} (1 - |\nu_{P_i}(x_j) - \mu_{P_i}(x_j)|) = \\
&\frac{1}{v} \sum_{i=1}^v \sum_{j=1}^{|P_i^*|} \frac{1}{|P_i^*|} (1 - |\mu_{P_i}(x_j) - \nu_{P_i}(x_j)|) = E^*(\mathcal{V}).
\end{aligned}$$

□

Theorem 4.7. Let $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$ be a (PFQS) on X , $|\mathcal{E}| = e$ and $|\mathcal{D}| = d$. Then

- (i) $E^*(\mathcal{D})$ satisfies in (EPFs-1), (EPFs-2), (EPFs-3), (EPFs-4).
- (ii) $E^*(\mathcal{E})$ satisfies in (EPFs-1), (EPFs-2), (EPFs-3), (EPFs-4).

Proof. It is similar to Theorem 4.6. □

4.1 Minimum Pythagorean fuzzy entropy of superedges

This subsection focuses on analyzing a PFQS and determining the minimum Pythagorean fuzzy entropy of superedges through the entropy of a

ring topology. The goal is to optimize the $PFQS$ while maintaining its structure, achieved by eliminating only redundant or costly Pythagorean fuzzy of superedges. A $PFQS$ is connected, if for any two supervertices P_i and P_j there exists a sequence of Pythagorean fuzzy superedges $Q_{i,k_1}, Q_{k_1,k_2}, \dots, Q_{m,j}$ such that for any $xy \in P_i^* \times P_i^*$, $\mu_{Q_{l,s}}(xy) > 0$ and $\nu_{Q_{l,s}}(xy) < 1$, that will call it by Pythagorean fuzzy path or path and, if $Q_{i,k_1} = Q_{m,j}$, will denote by a Pythagorean fuzzy cyclic or cyclic for simplicity.

Say a spanning Pythagorean fuzzy subhypergraph of a $PFQS$ is a connected, acyclic Pythagorean fuzzy subhypergraph that includes all supervertices while preserving the Pythagorean fuzzy constraints on superedges and links, indeed a $H' = (\mathcal{V}', \mathcal{E}', \mathcal{D}')$ satisfies in (1); $V' = V$ (all original supervertices are retained) and (2); $\forall xy \in P_i^* \times P_i^*$, $\mu_{R'_i}(xy) \leq \mu_{R_i}(xy)$, $\nu_{R'_i}(xy) \geq \nu_{R_i}(xy)$, (3); $\forall xy \in P_i^* \times P_j^*$, $\mu'_{Q_{i,j}}(xy) \leq \mu_{Q_{i,j}}(xy)$ and $\nu_{Q'_{i,j}}(xy) \geq \nu_{Q_{i,j}}(xy)$. Also a spanning Pythagorean fuzzy superhypertree is a special case where H' is a connected Pythagorean fuzzy superhypergraph without cycles, covering all vertices.

In what follows, we present an algorithm to reduce the Pythagorean fuzzy entropy of superedges of $PFQS$ via spanning Pythagorean fuzzy Superhypertree. Indeed, we apply the *Fuzzy Kruskal's Algorithm* in this minimizing Pythagorean fuzzy entropy of superedges and will denote minimum(semioptimal) total Pythagorean fuzzy entropy of superedges by $E_{min}(\mathcal{E})$.

Algorithm 4.8. *Get the $PFQS, H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$:*

1. Consider $\mathcal{E} = \{Q_{i,j} = (\mu_{Q_{i,j}}, \nu_{Q_{i,j}})\}_{i,j \geq 1}^e$,
2. For any $i, j \geq 1$, calculate $E^*(Q_{i,j})$,
3. Sort Pythagorean fuzzy superedges $Q_{i,j}$ in ascending order based on entropy $E^*(Q_{i,j})$,
4. Examine Pythagorean fuzzy superedges in ascending entropy order.
5. Add the Pythagorean fuzzy superedges to the Pythagorean fuzzy Superhypergraph if it doesn't form a cycle.
6. Repeat until $n - 1$ edges are added $n = |X|$.
7. $E_{min}^*(\mathcal{E}) = \sum_{Q_{i,j}} E^*(Q_{i,j})$, from step 5.

Applying the Theorem 2.1, have the following result.

Example 4.9. Consider a PFQS $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$ with the following components:

$$\begin{aligned} P_1 &= \{(x_1, 0.3, 0.4), (x_2, 0.5, 0.3), (x_3, 0.4, 0.5)\}, \\ P_2 &= \{(x_4, 0.6, 0.2), (x_5, 0.4, 0.4)\}, \\ Q_{12} &= \{(x_1x_4, 0.12, 0.50), (x_2x_5, 0.20, 0.40), (x_3x_4, 0.16, 0.45)\}, \\ R_1 &= \{(x_1x_2, 0.25, 0.45), (x_2x_3, 0.30, 0.50), (x_1x_3, 0.20, 0.55)\}, \\ R_2 &= \{(x_4x_5, 0.35, 0.40)\}. \end{aligned}$$

Applying Algorithm 4.8, $E_{min}(\mathcal{E}) = 0.4734$.

Theorem 4.10. Let $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$ be a PFQS. Then $E_{min}(\mathcal{E}) = \min\{\sum_{uv} En(Q_{i,j})(uv)\}$.

5 Application of Fuzzy Pythagorean Fuzzy Superhypergraph in Real World

In this section, we apply the concept of FPFSh in the real world.

Business superhypernetworks: In industrial park X , there are four factories represented as $X = \{F_1, F_2, F_3, F_4\}$. Each factory operates two plastic production machines. Within each factory, these machines utilize specialized sources of raw materials to process them into usable plastic in various forms.

Every machine comes with two critical considerations: first, the time and waste associated with receiving raw materials, and second, the quantity of goods it can produce. These options are detailed in Table 1, which outlines the production characteristics for this industrial zone.

Table 1: Machine-Level Data

Factory	Machine	Wastage(μ_{P_i})	Production(ν_{P_i})
F_1	M_1	0.25	0.85
	M_2	0.35	0.75
F_2	M'_1	0.15	0.92
	M'_2	0.45	0.65
F_3	M''_1	0.10	0.95
	M''_2	0.55	0.55
F_4	M'''_1	0.30	0.85
	M'''_2	0.40	0.80

Based on Table 1, and examining the flow of raw materials into and goods out of each device, factory management has determined that implementing a smart accelerator as a connector between the two devices would help minimize waste and boost production efficiency. The resulting improvements are highlighted in Table 2, based $\forall (M_i, M_j) \in F_i^* \times F_j^*, \mu_{R_i}(M_i M_j) \leq \min\{\mu_{P_i}(x), \mu_{P_i}(M_j)\}$ and $\nu_{R_i}(M_i M_j) \geq \max\{\nu_{P_i}(M_i), \nu_{P_i}(M_j)\}$. Indeed, by Table 2, internal links optimize intra-factory performance by bounding wastage and production values.

Table 2: Internal Links Within Factories

Factory	Link	Min	Max
F_1	M_1 - M_2	0.25	0.85
F_2	M'_1 - M'_2	0.15	0.92
F_3	M''_1 - M''_2	0.10	0.95
F_4	M'''_1 - M'''_2	0.30	0.85

We aim to model serve as intelligent, machine-specific connections that optimize cross-factory resource allocation and production coordination. In fact, our goal is to optimize the entire town in terms of reducing waste and increasing production, so i consider the outcome of waste and production of goods in each factory, and by creating links between the factories of the industrial town, in equality based the

$$\forall (M_i, M_j \in F_i^* \times F_j^*, \mu_{Q_{i,j}}(M_i M_j) \leq \min\left\{\frac{\mu_{P_i}(M_i)}{w(P_i)}, \frac{\mu_{P_j}(M_j)}{w(P_j)}\right\} \text{ and}$$

$\nu_{Q_{i,j}}(M_i M_j) \geq \max\{\frac{\nu_{P_i}(M_i)}{w(P_i)}, \frac{\nu_{P_j}(M_j)}{w(P_j)}\}$. Computations show that, $w(F_1) = 2.20$, $w(F_2) = 2.17$, $w(F_3) = 2.15$, $w(F_4) = 2.35$ and get the Table 3.

Table 3: Superedges Between Factories

Connection	Machine Pair	Min	Max
F_1-F_2	$M_1-M'_1$	0.1591	0.3409
	$M_2-M'_2$	0.1591	0.3409
F_2-F_3	$M'_1-M''_1$	0.0465	0.4419
	$M'_2-M''_2$	0.2074	0.2995
F_3-F_4	$M''_1-M'''_1$	0.0465	0.4419
	$M''_2-M'''_2$	0.1702	0.3404
F_4-F_1	M'''_1-M_1	0.1136	0.3864
	M'''_2-M_2	0.1591	0.3409

Now, $E^*(\mathcal{V}) = 0.52875$, $E^*(\mathcal{D}) = 0.3075$, and see that $E^*(\mathcal{V}) \geq E^*(\mathcal{D})$ and conclude that internal links within factories optimize the outcoming of industrial park. Apply $\mu_{Q_{i,j}}(M_i M_j) \leq \min\{\frac{\mu_{P_i}(M_i)}{w(P_i)}, \frac{\mu_{P_j}(M_j)}{w(P_j)}\}$ and $\nu_{Q_{i,j}}(M_i M_j) \geq \max\{\frac{\nu_{P_i}(M_i)}{w(P_i)}, \frac{\nu_{P_j}(M_j)}{w(P_j)}\}$ and modelled this problem by an optimized Pythagorean fuzzy quasi superhypergraph $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$, where $\mathcal{V} = \{F_1, F_2, F_3, F_4\}$, $\mathcal{E} = \{Q_{12}, Q_{23}, Q_{34}, Q_{41}\}$ and $\mathcal{D} = \{R_1, R_2, R_3, R_4\}$ as follows:

$$\begin{aligned}
F_1 &= \{(M_1, 0.25, 0.85), (M_2, 0.35, 0.75)\}, F_2 = \{(M'_1, 0.15, 0.92), \\
& (M'_2, 0.45, 0.65)\}, F_3 = \{(M''_1, 0.10, 0.95), (M''_2, 0.55, 0.55)\}, \\
F_4 &= \{(M'''_1, 0.30, 0.85), (M'''_2, 0.40, 0.80)\}, F_{12} = \{(M_1 M'_1, 0.05, 0.9), \\
& (M_2 M'_2, 0.01, 0.85)\}, F_{23} = \{(M'_1 M''_1, 0.03, 0.95), \\
& (M'_2 M''_2, 0.02, 0.85)\}, F_{34} = \{(M''_1 M'''_1, 0.02, 0.98), \\
& (M''_2 M'''_2, 0.01, 0.97)\}, F_{41} = \{(M'''_1 M_1, 0.03, 0.89), (M'''_2 M_2, 0.01, 0.88)\}, \\
R_1 &= \{(M_1 M_2, 0.25, 0.85)\}, R_2 = \{(M'_1 M'_2, 0.15, 0.92)\}, \\
R_3 &= \{(M''_1 M''_2, 0.10, 0.95)\}, R_4 = \{(M'''_1 M'''_2, 0.30, 0.85)\}.
\end{aligned}$$

Computations show that, $E^*(\mathcal{E}) = 0.11375$. Now, apply the algorithm

4.8, and compute the Pythagorean fuzzy entropy of superedges in Table 4 and so $H^{op} = (\mathcal{V}, \mathcal{E}, \mathcal{D})$ is an optimized Pythagorean fuzzy quasi superhypergraph with $E^*(\mathcal{E}) = \frac{1}{3}(0.04+0.125+0.135)$, where $\mathcal{V} = \{F_1, F_2, F_3, F_4\}$, $\mathcal{E} = \{Q_{23}, Q_{34}, Q_{41}\}$ and $\mathcal{D} = \{R_1, R_2, R_3, R_4\}$ as follows:

$$\begin{aligned} F_1 &= \{(M_1, 0.25, 0.85), (M_2, 0.35, 0.75)\}, F_2 = \{(M'_1, 0.15, 0.92), \\ & (M'_2, 0.45, 0.65)\}, F_3 = \{(M''_1, 0.10, 0.95), (M''_2, 0.55, 0.55)\}, \\ F_4 &= \{(M'''_1, 0.30, 0.85), (M'''_2, 0.40, 0.80)\}, F_{23} = \{(M'_1 M'_2, 0.03, 0.95), \\ & (M'_2 M'_3, 0.02, 0.85)\}, F_{34} = \{(M''_1 M''_2, 0.02, 0.98), \\ & (M''_2 M''_3, 0.01, 0.97)\}, F_{41} = \{(M'''_1 M'''_2, 0.03, 0.89), \\ & (M'''_2 M'''_3, 0.01, 0.88)\}, R_1 = \{(M_1 M_2, 0.25, 0.85)\}, \\ R_2 &= \{(M'_1 M'_2, 0.15, 0.92)\}, R_3 = \{(M''_1 M''_2, 0.10, 0.95)\}, \\ R_4 &= \{(M'''_1 M'''_2, 0.30, 0.85)\}. \end{aligned}$$

Table 4: Algorithm's process 4.8

Pythagorean Fuzzy Superedge	Entropy	Check
F_{34}	0.04	✓
F_{23}	0.125	✓
F_{41}	0.135	✓
F_{12}	0.155	✗

Social complex hypernetwork: Within a community, a group of 12 individuals decides to organize themselves into three smaller groups, each consisting of four members with diverse temperaments and dynamics. Following this, members within each group will engage in thorough, one-on-one interactions, ensuring all exchanges are unique and non-repetitive. These communications aim to support a comprehensive field study about their community, as outlined in the tables referenced as Tables 5, 6, and 7.

Table 5: Groups and Member Attributes

Group	Member	Isolation	Intelligence
A	A_1	0.3	0.8
	A_2	0.4	0.7
	A_3	0.2	0.9
	A_4	0.5	0.6
B	B_1	0.25	0.75
	B_2	0.15	0.85
	B_3	0.35	0.65
	B_4	0.3	0.7
C	C_1	0.1	0.9
	C_2	0.4	0.6
	C_3	0.2	0.8
	C_4	0.5	0.5

Table 6: Groups and Member Attributes

Group	Member 1	Member 2	Shyness	Motivation
A	A_1	A_2	0.3	0.8
	A_3	A_4	0.2	0.9
B	B_1	B_2	0.15	0.85
	B_3	B_4	0.3	0.7
C	C_1	C_2	0.1	0.9
	C_3	C_4	0.2	0.8

Table 7: Group Links

Link Pair of Distinct Groups	challenge	Empathy
A_1B_1	≤ 0.06	≥ 0.2
A_2B_2	≤ 0.03	≥ 0.25
A_3B_3	≤ 0.04	≥ 0.21
A_4B_4	≤ 0.07	≥ 0.22
B_1C_1	≤ 0.02	≥ 0.24
B_2C_2	≤ 0.03	≥ 0.23
B_3C_3	≤ 0.05	≥ 0.2
B_4C_4	≤ 0.07	≥ 0.27

Computations show that $H = (\mathcal{V}, \mathcal{E}, \mathcal{D})$ is a Pythagorean fuzzy superhypergraph, that

$$\begin{aligned}
P_1 &= \{(A_1, 0.3, 0.8), (A_2, 0.4, 0.7), (A_3, 0.2, 0.9), (A_4, 0.5, 0.6)\}, \\
P_2 &= \{(B_1, 0.25, 0.75), (B_2, 0.15, 0.85), (B_3, 0.35, 0.65), (B_4, 0.3, 0.7)\}, \\
P_3 &= \{(C_1, 0.1, 0.9), (C_2, 0.4, 0.6), (C_3, 0.2, 0.8), (C_4, 0.5, 0.5)\}, \\
R_1 &= \{(A_1A_2, 0.3, 0.8), (A_3A_4, 0.2, 0.9)\}, R_2 = \{(B_1B_2, 0.15, 0.85), \\
&\quad (B_3B_4, 0.3, 0.7)\}, R_3 = \{(C_1C_2, 0.1, 0.9), (C_3C_4, 0.2, 0.8)\}, \\
Q_{12} &= \{(A_1B_1, 0.06, 0.2), (A_2B_2, 0.03, 0.25), (A_3B_3, 0.04, 0.21), \\
&\quad (A_4B_4, 0.07, 0.22)\}, Q_{23} = \{(B_1C_1, 0.02, 0.24), (B_2C_2, 0.03, 0.23), \\
&\quad (B_3C_3, 0.05, 0.2), (B_4C_4, 0.07, 0.27)\}.
\end{aligned}$$

Now, apply the fuzzy entropy, so $E^*(\mathcal{V}) = 0.575$, $E^*(\mathcal{D}) = 0.383$, $E^*(\mathcal{E}) = 0.82$. The entropy values highlight a network characterized by stable intra-group relationships but highly uncertain inter-group connections. Mitigating superedge ambiguity plays a crucial role in improving overall network resilience.

Proceed by applying Theorem 4.3, semi optimizing Table 5 into Table 8, and calculating Tables 9 and 10 accordingly, following the guidelines provided by Algorithm 5.1.

Algorithm 5.1. *Step 1: Apply the Theorem 4.3, reduce the Pythagorean fuzzy supervertices to construct the soft PFSH and arrange by Algorithm 4.8 for semi optimizing.*

Table 8: Groups and Member Attributes

Group	Member	Isolation	Intelligence
A	A_1	0.03	0.08
	A_2	0.04	0.07
	A_3	0.02	0.09
	A_4	0.05	0.06
B	B_1	0.025	0.075
	B_2	0.015	0.085
	B_3	0.035	0.065
	B_4	0.03	0.07
C	C_1	0.01	0.09
	C_2	0.04	0.06
	C_3	0.02	0.08
	C_4	0.05	0.05

Step 2: Compute the Pythagorean fuzzy links as Table 9.

Table 9: Groups and Member Attributes

Group	Member 1	Member 2	Shyness	Motivation
A	A_1	A_2	0.03	0.08
	A_3	A_4	0.02	0.09
B	B_1	B_2	0.015	0.085
	B_3	B_4	0.03	0.07
C	C_1	C_2	0.01	0.09
	C_3	C_4	0.02	0.08

Step 3: Compute the Pythagorean fuzzy links as Table 10 and Algorithm 4.8 for optimizing.

Table 10: Group Links

Link Pair of Distinct Groups	challenge	Empathy
A_1B_1	0.0625	0.1875
A_2B_2	0.0375	0.2125
A_3B_3	0.0455	0.2045
A_4B_4	0.075	0.175
B_1C_1	0.025	0.225
B_2C_2	0.0375	0.2125
B_3C_3	0.05	0.2
B_4C_4	0.075	0.175

Step 3: Compute the Pythagorean fuzzy links as Tables 8, 9, and 10. Now, apply the fuzzy entropy as follows:

$$E^*(\mathcal{E}) = 0.85175 < E^*(\mathcal{D}) = 0.9383 < E^*(\mathcal{V}) = 0.9575.$$

High entropy within supervertices indicates a lack of clarity in individual traits, while lower entropy in superedges suggests more deterministic relationships between groups, but is not optimal.

5.1 Comparative analysis to other models:

This subsection provides a comparison of Pythagorean fuzzy Superhypergraphs (PFSh) with other fuzzy subsets.

Step 1: Model Comparison Framework: We evaluate PFSh in the following Tables:

Table 11: Key Characteristics Comparison

Model	Uncertainty Type	Entropy
PFSh	Pythagorean	Yes
Fuzzy Graphs	linear	Basic

Step 2: Performance Evaluation:

Table 12: Performance Metrics (Business Case Study)

Model	Uncertainty Capture	Optimization	Time
PFSh	92%	95%	8.2s
Fuzzy Graphs	65%	70%	1.2s

Thus have the better uncertainty modeling and comprehensive entropy.

6 Conclusion

This research shows that Pythagorean fuzzy superhypergraphs (PFSh) provide a mathematical way to model complex real-world systems. Indeed, PFSh can represent multi relationships where entire groups link to other groups. By using values-cut sets, analysts can focus on the most reliable core connections or examine more uncertain peripheral relationships, providing flexibility in understanding complex networks. We investigated an important principle as entropy fuzzy on Pythagorean fuzzy superedges that cannot be greater than the entropy fuzzy on Pythagorean fuzzy supervertices. This means that to improve reliability between the Pythagorean fuzzy supervertices. The method can become computationally demanding for very large networks, and it requires careful setup of the uncertainty parameters that is a limitation. We hope that this research be helpful in the modeling of complex hypernetworks and wish to extended it in the another optimal algorithms in the computations of entropy fuzzy of Pythagorean fuzzy superedges and Pythagorean fuzzy supervertices in the next works.

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