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A New Improved Method for Comparing Fuzzy Numbers by Centroid Point

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Abstract. Ranking fuzzy numbers is a very important decision-making procedure in decision analysis and applications. The last few decades have seen a large number of methods investigated for ranking fuzzy numbers, yet some of these approaches are non-intuitive and inconsistent. The most commonly used approaches for ranking fuzzy numbers are ranking indices based on centroid of fuzzy numbers. Despite their merits, there are some weakness associated with these indices. This paper review several recent fuzzy numbers ranking methods based on centroid points, then proposes a new centroid index ranking method that is capable of effectively ranking various types of fuzzy numbers. The presented method is compared with the given attitude by the way of centroid point. The contents herein present several comparative examples demonstrating the usage and advantages of the proposed centroid index ranking method for fuzzy numbers. Meanwhile, it can overcome the drawback of other methods.

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Keywords and Phrases: Centroid points, center of gravity, comparing, defuzzification, trapezoidal fuzzy numbers, triangular fuzzy numbers, ordering, ranking

1. Introduction

Ranking fuzzy numbers is an important tool in decision process. In fuzzy decision analysis, fuzzy quantities are used to describe the performance

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of alternative in modeling a real-world problem. Most of the ranking procedures proposed so far in literature cannot discriminate fuzzy quantities and some are counterintuitive. As fuzzy numbers are represented by possibility distributions, they may overlap with each other, and hence it is not possible to order them. It is true that fuzzy numbers are frequently partial order and cannot be compared like real numbers which can be linearly ordered. So far, more than 100 fuzzy ranking indices have been proposed since 1976 while the theory of fuzzy sets were first introduced by Zadeh [63]. In 1976 and 1977, Jain [37,38] proposed a method using the concept of maximizing set to order the fuzzy numbers. Jain's method is that the decision maker considers only the right side membership function. A canonical way to extend the natural ordering of real numbers to fuzzy numbers was suggested by Bass and Kwakernaak [10] as early as 1977. In 1979, Baldwin and Guild [9] indicated that these two methods have some disturbing disadvantages. Also, in 1980, Adamo [5] used the concept of α-level set in order to introduce α-preference rule. In 1981 Chang [14] introduced the concept of the preference function of an alternative. Yager [59-61] proposed four indices and which may be employed for the purpose of ordering fuzzy quantities in $[0, 1]$ and also in 1983 Murakami [45] developed the proposed ranking methods at that time to apply for control system. Bortolan and Degani have been compared and reviewed some of these ranking methods [12]. Chen [15] presented ranking fuzzy numbers with maximizing set and minimizing set. In 1987, Dubois and Prade [30] presented the mean value of a fuzzy number. Lee and Li [40] presented a comparison of fuzzy numbers based on the probability measure of fuzzy events. Delgado et al. [25] presented a procedure for ranking fuzzy numbers. Campos and Munz [13] presented a subjective approach for ranking fuzzy numbers. Kim and Park [39] presented a method of ranking fuzzy numbers with index of optimism. Yaun [62] presented a criterion for evaluating fuzzy ranking method. Heilpern [36] presented the expected value of a fuzzy number. Saad and Schwarzlander [50] presented order fuzzy sets over real line. Liou and Wang [41] presented ranking fuzzy numbers with integral value. Chen and Hwang [16] thoroughly reviewed the existing the approaches and pointed out some illogical conditions that arise among them. Choobineh [23], Cheng

[22] have presented some methods. Since then several methods have been proposed by various researchers [44, 34], Wang and Kerre [53] classified all the above ranking procedures into three classes. The first classes consist of ranking procedure based on mean and spread, and second class consist ranking procedures based on fuzzy scoring, whereas the third class consist of methods based on preference relations.The development in ranking fuzzy numbers can also be found in [32-34, 4, 48, 43, 6, 49, 51, 52, 31]. Most of the methods presented above are counter-intuitive and cannot discriminate fuzzy numbers, and some methods do not agree with human intuition, whereas some methods cannot rank crisp numbers, which are special case of fuzzy numbers.

Among the ranking approaches, the centroid methods are commonly used approaches to rank fuzzy numbers was suggested for ranking fuzzy numbers. Ever since Yager [60] presented the centroid concept in the ranking techniques using the centroid concept have been proposed and investigated. In a paper by Cheng [22], a centroid-based distance method presented. The method utilized the Euclidean distances from the origin to the centroid point of each fuzzy numbers to compare and rank the fuzzy numbers. Chu and Tsao [34] found that the distance method could not rank fuzzy numbers correctly if they are negative and therefore, suggested using the area between centroid point and the origin to rank fuzzy numbers. Deng et al. [27] utilized the centroid point of a fuzzy number and presented a new area method to rank fuzzy numbers with the radius of gyration (ROG) points to overcome the drawback of the Cheng's distance method and Tsao's area method when some fuzzy numbers have the same centroid point. However, ROG method cannot rank negative fuzzy numbers. Abbasbandy and Asady [2] found that Tsao's area method could sometimes lead to counterintuitive ranking and hence suggested a sign distance. In 2006, Wang et al. [55] pointed out that the centroid point formulas for fuzzy numbers provided by Cheng [22] are incorrect and have led to some misapplication such as by Chu and Tsao [24], Pan and Yeh [47] and Deng et al. [27]. They presented the correct centroid formulae for fuzzy numbers and justified them from the viewpoint of analytical geometry. Nevertheless, the main problem, about ranking fuzzy numbers methods, which used the centroid point, was reminded. In 2008 Wang and Lee [54] revised Chu and Tsao's method and suggested a new approach for ranking fuzzy numbers based on Chu and Tsao's method in away to similar original point. However, there is a shortcoming in some situations.

In 2011, Abbasbandy and Hajjari [4] improved cheng's distance method. Afterward, Pani Bushan Rao et al. [48] presented a new method for ranking fuzzy numbers based on the circumcenter of centroid and used an index of optimism to reflect the decision maker's optimistic attitude and also an index of modality that represented the neutrality of the decision maker. Luu Quoc Dat et al. [43] presented an improved ranking method for fuzzy numbers based on the centroid-index.

In 2012 Allahviranloo and Sanei [6] presented a deffuzification method for ranking fuzzy numbers based on centre of gravity. Following Pani Bushan Rao's method [48], in 2013, Rezvani [49] represented a method on the incenter of centroid and used Euclidean distance to rank fuzzy number. Recently, Hajjari [35] reviewed "Mag-method" and improved it in order to overcome the shortcoming of the "Mag-method". In the present paper, we discuss the problem of some methods then we give a new idea to overcome the shortcoming in those methods.

The rest of the paper is organized as follows. Section 2 contains the basic definitions and notations use in the remaining parts of the paper. In Section 3, we review some of recent ranking fuzzy numbers methods, which are based on centroid points and present an idea to remove the weaknesses and improve them. An improvement in centroid-index methods will be given in Section 4. Section 5 demonstrates by several numerical examples the advantages of proposed idea. The paper is concluded in Section 6.

2. Preliminaries

In this section, we briefly review some basic concepts of generalized fuzzy numbers and some existing methods for ranking fuzzy numbers. we will identify the name of the number with that of its membership function for simplicity. Throughout this paper, $\mathbb{R}$ stands for the set of all real numbers, E stands the set of fuzzy numbers, "A" expresses a fuzzy number and $A(x)$ for its membership function, $\forall x \in \mathbb{R}$.

2.1 Basic notations and definitions

A generalized fuzzy number "A" is a subset of the real line $\mathbb{R}$ with membership function $A(x) : \mathbb{R} \to [0, w]$ such that [61]:

$$A(x) = \begin{cases} L_A(x), & a \leqslant x \leqslant b, \\ \omega, & b \leqslant x \leqslant c, \\ U_A(x), & c \leqslant x \leqslant d, \\ 0, & otherwise, \end{cases} \tag{1}$$

where $0 < \omega \leqslant 1$ is a constant, $L_A(x) : [a, b] \to [0, \omega]$ and $U_A(x) : [c, d] \to [0, \omega]$ are two strictly monotonically and continuous mapping. If $\omega = 1$, then A is a normal fuzzy number. If $L_A(x) = \omega(x - a)/(b - a)$, and $U_A(x) = \omega(d - x)/(d - c)$ then it is a trapezoidal fuzzy number and is usually denoted by $A = (a, b, c, d; \omega)$ or $A = (a, b, c, d)$ if $\omega = 1$. In particular, when $b = c$, the trapezoidal fuzzy number is reduced to a triangular fuzzy number denoted by $A = (a, b, d; \omega)$ or $A = (a, b, d)$ if $\omega = 1$. Therefore, triangular fuzzy numbers are special cases of trapezoidal fuzzy numbers. We show the set of generalized fuzzy numbers by $F_w(\mathbb{R})$ or for simplicity by $F(\mathbb{R})$.

Since $L_A(x)$ and $U_A(x)$ are both strictly monotonically and continuous functions, their inverse functions exist and should also be continuous and strictly monotonically. Let $A_L : [0, \omega] \to [a, b]$ and $A_U : [0, \omega] \to [c, d]$ be the inverse functions of $L_A(x)$ and $U_A(x)$, respectively. Then A_L and A_U should be integrable on the close interval $[0, \omega]$. In other words, both $\int_0^\omega A_L(y)dy$ and $\int_0^\omega A_U(y)dy$ should exist. In the case of trapezoidal fuzzy number, the inverse functions A_L and A_U can be analytically expressed as

$$A_L(y) = a + (b - c)y/\omega, \qquad 0 \leqslant y \leqslant \omega, \tag{2}$$

$$A_U(y) = d - (d - c)y/\omega, \qquad 0 \leqslant y \leqslant \omega. \tag{3}$$

The functions $L_A(x)$ and $R_A(x)$ are also called the left and right side of the fuzzy number A, respectively [29].

In this paper, we assume that

$$\int_{-\infty}^{+\infty} A(x)dx < +\infty.$$

A useful tool for dealing with fuzzy numbers are their $\alpha-$cuts. The $\alpha-$cut of a fuzzy number A is non-fuzzy set defined as

$$A_\alpha = \{x \in \mathbb{R} : A(x) \geqslant \alpha\},$$

for $\alpha \in (0, 1]$ and $A_0 = cl(\cup_{\alpha \in (0,1]} A_\alpha)$. According to the definition of a fuzzy number, it is seen at once that every $\alpha-$cut of a fuzzy number is closed interval. Hence, for a fuzzy number A, we have $A(\alpha) = [A_L(\alpha), A_U(\alpha)]$ where

$$A_L(\alpha) = \inf\{x \in \mathbb{R} : A(x) \geqslant \alpha\},$$
$$A_U(\alpha) = \sup\{x \in \mathbb{R} : A(x) \geqslant \alpha\}.$$

If the left and right sides of the fuzzy number A are strictly monotone, as it is described, A_L and A_U are inverse functions of $L_A(x)$ and $U_A(x)$, respectively.

The set of all elements that have a nonzero degree of membership in a is called the *support* of A, i.e.

$$supp(A) = \{x \in X \mid A(x) > 0\}. \tag{4}$$

The set of elements having the largest degree of membership in $\tilde{A}$ is called the *core* of A, i.e.

$$core(A) = \{x \in X \mid A(x) = \sup_{x \in X} A(x)\}. \tag{5}$$

In the following, we will always assume that A is continuous and bounded support $supp(A) = (a, d)$. The strong support of A should be $\overline{supp}\,(A) = [a, d]$.

Definition 2.1.1 A function $s : [0, 1] \longrightarrow [0, 1]$ is a reducing function if s is increasing and $s(0) = 0$ and $s(1) = 1$. We say that s is a regular function if $\int_0^1 s(\alpha)\mathrm{d}\alpha = \frac{1}{2}$.

Definition 2.1.2 If A is a fuzzy number with $\alpha-$cut representation, $\left[A_L(\alpha), A_U(\alpha)\right]$, and s is a reducing function then the value of A (with respect to s) is defined by

$$Val(A) = \int_0^1 s(\alpha)\left[A_U(\alpha) + A_L(\alpha)\right]\mathrm{d}\alpha. \tag{6}$$

Definition 2.1.3 [62] If A is a fuzzy number with $\alpha-$cut representation, $\left[A_L(\alpha), A_U(\alpha)\right]$, and s is a reducing function then the ambiguity of $\tilde{A}$ (with respect to s) is defined by

$$Amb(A) = \int_0^1 s(\alpha)\left[A_U(\alpha) - A_L(\alpha)\right]\mathrm{d}\alpha. \tag{7}$$

Let also recall that the expected interval $EI(A)$ of a fuzzy number A is given by

$$EI(A) = \Big[\int_0^1 A_L(\alpha)\mathrm{d}\alpha, \int_0^1 A_U(\alpha)\mathrm{d}\alpha\Big]. \tag{8}$$

Another parameter is utilized for representing the typical value of the fuzzy number is the middle of the expected interval of a fuzzy number and it is called the expected value of a fuzzy number "A" i.e. number A is given by [11]

$$EV(A) = \frac{1}{2}\Big(\int_0^1 A_L(\alpha)\mathrm{d}\alpha + \int_0^1 A_U(\alpha)\mathrm{d}\alpha\Big). \tag{9}$$

3. Review on Some Centroid-Index for Ranking Fuzzy Numbers

Yager [60] was the first researcher to proposed a centroid-index ranking method to calculate the value x_0 for a fuzzy number A as

$$x_0 = \frac{\int_0^1 w(x)A(x)\mathrm{d}x}{\int_0^1 A(x)\mathrm{d}x} \tag{10}$$

where $w(x)$ is a weighting function measuring the importance of the value x and $A(x)$ denotes the membership function of the fuzzy number A. When $w(x) = x$, the value x_0 becomes the geometric Center of

Gravity (COG) with

$$x_0 = \frac{\int_0^1 xA(x)\mathrm{d}x}{\int_0^1 A(x)\mathrm{d}x}. \tag{11}$$

The larger the value is of x_0 the better ranking of A.

Cheng [22] used a centroid-based distance approach to rank fuzzy numbers. For trapezoidal fuzzy number $A = (a, b, c, d; \omega)$, the distance index can be defined as

$$R(A) = \sqrt{x_0^2 + y_0^2}, \tag{12}$$

where

$$x_0 = \frac{\int_a^b xL_A(x)\mathrm{d}x + \int_b^c x\mathrm{d}x + \int_c^d xU_A(x)\mathrm{d}x}{\int_a^b L_A(x)\mathrm{d}x + \int_b^c \mathrm{d}x + \int_c^d U_A(x)\mathrm{d}x}, \tag{13}$$

$$y_0 = \omega\frac{\int_0^1 yA_L(y)\mathrm{d}y + \int_0^1 yA_U(y)\mathrm{d}y}{\int_0^1 A_L(y)\mathrm{d}y + \int_0^1 A_U(y)\mathrm{d}y}. \tag{14}$$

U_A and L_A are the respective right and left membership function of A, and A_U and A_L, are the inverse of U_A and L_A respectively. The larger the value is of $R(A)$ the better ranking will be of A.

Chau and Tsao [24] found that the distance approach by Cheng [22] had shortcomings. Hence to overcome the problems, Cha and Tsao [24] proposed a new ranking index function $S(A) = x_0.y_0$, where x_0 is defined in Cheng [22] and

$$y_0 = \frac{\int_0^\omega yA_L(y)\mathrm{d}y + \int_0^\omega yA_U(y)\mathrm{d}y}{\int_0^\omega A_L(y)\mathrm{d}y + \int_0^\omega A_U(y)\mathrm{d}y}. \tag{15}$$

The larger the value is of $S(A)$ the better ranking will be of A.

In some special cases, Cha and Tsao's [24] approach also has the same shortcoming of Cheng's [22] and Cha and Tsao's centroid-index are as follows. For fuzzy numbers A, B, C and $-A, -B, -C$, according to Cheng's centroid-index $R(A) = \sqrt{x_0^2 + y_0^2}$, whereby the same results are obtained, that is, if $A \prec B \prec C$ then $-A \prec -B \prec -C$. This is clearly inconsistent with the mathematical logic. For Chu and Tesao's centroid-index $S(A) = x_0.y_0$, if $x_0 = 0$, then the value of $S(A) = x_0.y_0$, is a constant zero. In other words, the fuzzy numbers with centroid $(0, y_1)$ and $(0, y_2)$, $y_1 \neq y_2$ are considered the same. This is also obviously unreasonable.

In a study conducted by Wang et al. [55], the centroid formulae proposed by Cheng [22] is shown to be incorrect. Therefore to avoid many misapplication, Wang et al. [55] presented the correct centroid formulae as

$$x_0 = \frac{\int_a^b xL_A(x)\mathrm{d}x + \int_b^c x\mathrm{d}x + \int_c^d xU_A(x)\mathrm{d}x}{\int_a^b L_A(x)\mathrm{d}x + \int_b^c \mathrm{d}x + \int_c^d U_A(x)\mathrm{d}x} \tag{16}$$

and

$$y_0 = \frac{\int_0^\omega yA_U(y)\mathrm{d}y - \int_0^\omega yA_L(y)\mathrm{d}y}{\int_0^\omega A_U(y)\mathrm{d}y - \int_0^\omega A_L(y)\mathrm{d}y}. \tag{17}$$

For an arbitrary trapezoidal fuzzy number $A = (a, b, c, d; \omega)$, the centroid point (x_0, y_0) is defined as [55]

$$x_0 = \frac{1}{3}[a + b + c + d - \frac{dc - ab}{(d + c) - (a + b)}] \tag{18}$$

$$y_0 = \frac{\omega}{3}[1 + \frac{c - b}{(d + c) - (a + b)}] \tag{19}$$

In special case, when $b = c$, the trapezoidal fuzzy number is reduced to a triangular fuzzy number and formulas (19) and (20) will be simplified as follows, respectively.

$$x_0 = \frac{a + b + d}{3} \tag{20}$$

$$y_0 = \frac{1}{3} \tag{21}$$

Luu et al. [43] presented a centroid-index as follows.

$$D(A_i, G) = \sqrt{(x_{A_i} - x_{min})^2 + (y_{A_i} - y_{min})^2}, \tag{22}$$

where (x_{A_i}, y_{A_i}) are the centroid points of A_i and $G = (x_{min}, y_{min})$ is minimum point, such that

$x_{min} = infS,\ \ S = \bigcup_{i=1}^{n} S_i,\ \ S_i = \{x|A(x) > 0\},$

$y_{min} = infY,\ \ Y = \bigcup_{i=1}^{n} Y_i,\ \ Y_i = \{y|Y_{A_i}(x) < \omega\},$

The larger the value is of $D(A, G)$ the better ranking will be of A.

Allahviranloo and Saneifard [6] expressed their idea as follows.

$$Dist(A_i) = \sqrt{(x_{A_i} - \tau_{max})^2 + (y_{A_i})^2}, \tag{23}$$

where (x_{A_i}, y_{A_i}) is the centroid point of A and

$\tau_{max} = max\{x|x \in Domain(A_1, A_2, ..., A_n)\}.$

They described that smaller the value is of $Dist(A_i)$ the better ranking will be of A.

However, in some special cases, Allahviranloo and Saneifard's [6] distance also has shortcoming as Cheng's [22] and Tsao's [24] approach. In 2013, Rezvani proposed a method based on the incentre of centroid points. They used Euclidian distance as a function for ranking fuzzy numbers. It is expressed bellow.

$$R(A) = \sqrt{x_0^2 + y_0^2}, \tag{24}$$

where $I_A(x_0, y_0)$ is introduced as the incenter of fuzzy number $A = (a, b, c, d; \omega)$ and is described as follows.

$$x_0 = \frac{\alpha(\frac{a+2b}{3}) + \beta(\frac{b+c}{2}) + \gamma(\frac{2c+d}{3})}{\alpha + \beta + \gamma}, \tag{25}$$

$$y_0 = \frac{\alpha(\frac{\omega}{3}) + \beta(\frac{\omega}{2}) + \gamma(\frac{\omega}{3})}{\alpha + \beta + \gamma}, \tag{26}$$

while,

$$\alpha = \frac{\sqrt{(c - 3b + 2d)^2 + \omega^2}}{6}, \qquad \beta = \frac{\sqrt{(2c - a - 2b + d)^2}}{3}, \tag{27}$$

$$\omega = \frac{\sqrt{(3c - 2a - b)^2 + \omega^2}}{6}$$

The larger the value is of $R(A)$ the better ranking will be of A.

Consider the crisp number $A = (0, 0, 0)$ and two symmetric fuzzy numbers $B = (-1, 0, 1)$ $C = (-2, 0, 2)$. By applying Rezvani's approach the results will be $R(A) = R(B) = R(C) = 0.333$, and ranking order is $A = B = C$. In addition, to compare the crisp number $A = (1, 1, 1)$ and two symmetric fuzzy numbers $B = (0, 1, 2)$ $C = (-1, 1, 3)$. We see that $R(A) = 0.333$ $R(B) = 0.412$ $R(C) = 0.415$, the ranking order will be $A \prec B \prec C$, which is unreasonable.

To overcome the shortcoming of theses methods, we will present an improved algorithm in Section 4.

4. New Improved Method for Ranking Fuzzy Numbers by Centroid Point

In this section the centroid point of a fuzzy number corresponds to a x_0 value on the horizontal axis and y_0 value on the vertical axis. The centroid point (x_0, y_0) for a fuzzy number A is as defined [55]:

$$x_0 = \frac{\int_a^b xL_A(x)\mathrm{d}x + \int_b^c x\mathrm{d}x + \int_c^d xU_A(x)\mathrm{d}x}{\int_a^b L_A(x)\mathrm{d}x + \int_b^c \mathrm{d}x + \int_c^d U_A(x)\mathrm{d}x} \tag{28}$$

$$y_0 = \frac{\int_0^\omega yA_U(y)\mathrm{d}y - \int_0^\omega yA_L(y)\mathrm{d}y}{\int_0^\omega A_U(y)\mathrm{d}y - \int_0^\omega A_L(y)\mathrm{d}y}. \tag{29}$$

For trapezoidal fuzzy number $A = (a, b, c, d; \omega)$, the centroid point (x_0, y_0) is defined as in [55]:

$$x_0 = \frac{1}{3}[a + b + c + d - \frac{dc - ab}{(d + c) - (a + b)}] \tag{30}$$

$$y_0 = \frac{\omega}{3}[1 + \frac{c - b}{(d + c) - (a + b)}]. \tag{31}$$

Since triangular fuzzy numbers are special cases of trapezoidal fuzzy number with $b = c$ for any triangular fuzzy numbers with a piecewise linear membership function, its centroid can be determined by

$$x_0 = \frac{1}{3}(a+b+d) \tag{32}$$

$$y_0 = \frac{1}{3}\omega. \tag{33}$$

Definition 4.1. For generalized trapezoidal fuzzy number $A = (a, b, c, d; \omega)$ with the centroid point (x_0, y_0), the centroid-index associated with the ranking is defined as

$$I_{\alpha\beta} = \frac{\beta(x_0 + y_0)}{2} + (1-\beta)I_\alpha \tag{34}$$

where α,$\beta \in [0, 1]$.

$I_{\alpha\beta}$ is the modality which represents the importance of central value against the extreme values x_0, y_0 and $I_{\alpha\beta}$. Here, β represent the weight of central value and $1-\beta$ is the weight associated with the extreme values x_0 and y_0. Moreover, $I_\alpha = \alpha y_0 + (1-\alpha)x_0$ is the index of optimism which represents the degree of optimism of a decision maker. If $\alpha = 0$, we have a pessimistic decision maker's view point which is equal to the distance of the centroid point from $Y-$axis. If $\alpha = 1$, we have a optimistic decision maker's view point which is equal to the distance of the centroid point from $X-$axis, and when $\alpha = 0.5$, we have the moderate decision maker's view point and is equal to the mean of centroid point from Y and X axis. The larger value of α is, the higher the degree of the decision maker. The index of optimism is not alone sufficient to discriminate fuzzy numbers as this uses only extreme of the cicumcenter of centroid. Hence, we upgrade this by using an index.

Definition 4.2. For generalized trapezoidal fuzzy number $A = (a, b, c, d; \omega)$ with the centroid point (x_0, y_0), the ranking function of the trapezoidal fuzzy number which maps the set of all fuzzy numbers to a set of real numbers is defined as $R(A) = \sqrt{x_0^2 + y_0^2}$, which is the Euclidean distance from the centroid point and original point. Using the above definitions we define ranking between fuzzy numbers as follows

let A and B are two fuzzy numbers, then

1. $R(A) > R(B)$ if and only if $A \succ B$,
2. $R(A) < R(B)$ if and only if $A \prec B$,
3. if $R(A) = R(B)$ then in this case the discrimination of fuzzy numbers is not possible.

Case (1) let A and B are two symmetric triangular fuzzy numbers with the same core, if $R(A) + 1/\delta_A > R(B) + 1/\delta_B$ then $A \succ B$ and if $R(A) + 1/\delta_A < R(B) + 1/\delta_B$ then $A \prec B$, where δ_A, δ_B are the spreads of A and B respectively.

Case (2) In such cases we use Definitions 4.1 and 4.2 to rank fuzzy numbers as Definition 4.2 alone is not sufficient to discriminate in all cases, that is, if $I_{\alpha\beta}(A) > I_{\alpha\beta}(B)$, then $A \succ B$, and if $I_{\alpha\beta}(A) < I_{\alpha\beta}(B)$, then $A \prec B$,

Remark 4.3. For two arbitrary trapezoidal fuzzy numbers A and B, we have

$$R(A + B) = R(A) + R(B).$$

Theorem 4.4. For two symmetric triangular fuzzy numbers A and B, with the same core $A \succ B$ iff $\delta_A < \delta_B$.

Proof. Let $A \succ B$ then $R(A) > R(B)$, moreover, from case 1 of Definition 4.2 we have $R(A) + 1/\delta_A > R(B) + 1/\delta_B$. Hence, $1/\delta_A > 1/\delta_B$ and finally, $\delta_A > \delta_B$.

Consider the crisp number $A = (0, 0, 0)$ and two symmetric fuzzy numbers $B = (-1, 0, 1)$ $C = (-2, 0, 2)$. Since $\delta_A \to 0$ and $\delta_B = 1$, $\delta_C = 2$, ranking order is $R(A) > R(B) > R(C)$. In addition, to compare the crisp number $A = (1, 1, 1)$ and two symmetric fuzzy numbers $B = (0, 1, 2)$ $C = (-1, 1, 3)$, we have the same result.

Remark 4.5. For all fuzzy numbers A, B, C and D we have

1. $A \succ B$ then $A \oplus C \succ B \oplus C$
2. $A \succ B$ then $A \ominus C \succ B \ominus C$

3. $A \sim B$ then $A \oplus C \sim B \oplus C$

4. $A \succ B, C \succ D$ then $A \oplus C \succ B \oplus D$

5. Numerical Examples

This section uses three numerical examples to compare the ranking results of proposed centroid-index ranking approach with other existing ranking approaches.

Example 5.1. The two fuzzy numbers $A = (0.1, 0.2, 0.4, 0.5)$ and $B = (0.1, 0.3, 0.5)$ used in this example are adopted from Chen and Sanguansat [45]

Fig. 1 shows the graphs of the two fuzzy numbers. The results obtained by the proposed approach and other approaches are shown in Table 1. It is worth mentioning that Yager's [8] approach, Cheng's [25] approach, Chu and Tsao's [29] approach, Chen and Sanguansat's [45] cannot differentiate A and B, that is, their ranking are always the same, i.e. $A \sim B$. Note that the ranking $A \prec B$ obtained by Murakami et al.'s [11] approach, Chen and Chen's [39] approach and Chen and Chen's [44] approach, are thought of as unreasonable and not consistent with human intuition due to the fact that the center of gravity of A is larger than the center of gravity of B on the $Y-$axis.

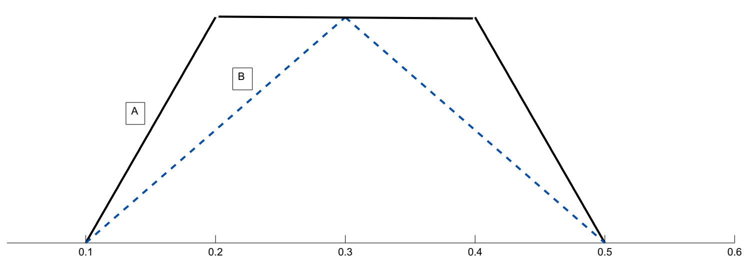

Figure 1: Fuzzy numbers A and B in Example 1.

Table 1: Comparative results of Example 5.1

Ranking approach	A	B	Result
Yager [9]	0.3	0.3	$A \sim B$
Murakami et al. [11]	0.3	0.417	$A \prec B$
Cheng [25]	0.583	0.583	$A \sim B$
Chu and Tsao [29]	0.15	0.15	$A \sim B$
Chen and Chen [39]	0.424	0.446	$A \prec B$
Chen and Chen [44]	0.254	0.258	$A \prec B$
Chen and Sanguansat [45]	0.3	0.3	$A \sim B$
Phani Bushan Rao et al. [51]	0.4711	0.5026	$A \prec B$
Luu et al. [52]	0.3333	0.2222	$A \succ B$
Proposed method	0.6690	0.4484	$A \succ B$

Example 5.2. Consider the data used in Chen and Sanguansat [45] i.e. the two triangular fuzzy numbers $A = (-0.5, -0.3, -0.1)$ and $B = (0.1, 0.3, 0.5)$ as showing in Fig. 2.

Table 2 shows the comparison results of the proposed centroid-index ranking method with other existing centroid ranking approaches. The result indicate that Cheng's approach leads to an incorrect ranking order i.e. $A \sim B$, whereas Chu and Tsao's [29] approach, Chen and Chen's approach [39], Chen and Chen's [44] approach,Chen and Sanguansat [45] approach and the proposed method get the same ranking order, i.e. $A \prec B$.

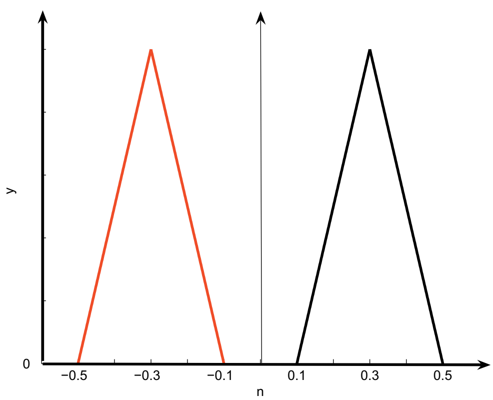

Figure 2: Fuzzy numbers A and B in Example 5.2.

Table 2: Comparative results of Example 5.2

Ranking approach	A	B	Result
Yager [8]	N_A	0.3	N_A
Murakami et al. [11]	N_A	0.3	N_A
Cheng [25]	0.583	0.583	$A \sim B$
Chu and Tsao [29]	-0.15	0.15	$A \prec B$
Chen and Chen [39]	0.446	0.747	$A \prec B$
Chen and Chen [44]	-0.258	0.258	$A \prec B$
Chen and Sanguansat [45]	-0.3	0.3	$A \prec B$
Phani Bushan Rao et al. [51]	0.0517	0.5026	$A \prec B$
Luu et al. [52]	0	0.6	$A \prec B$
Proposed method $\alpha = \frac{1}{2}, \beta = \frac{1}{2}$	0.0167	0.3167	$A \prec B$

Example 5.3. Consider the data used in Asady and Zendehnam [38] i.e. the three normal triangular fuzzy numbers $A = (5, 6, 7), B = (5.9, 6, 7)$ and $B = (6, 6, 7)$as shown in Fig. 3.

Table 3 shows the ranking results of the three triangular fuzzy numbers by using the proposed method and other approaches. It is observed that the ranking order of the three fuzzy numbers obtained by the proposed approach is consistent with the ranking order obtained by other approaches. Note that the ranking $A \succ B \succ C$ index of Chen obtained by CV [38] is thought of an unreasonable and not consistent with human intuition. This example shows the strong discrimination power of the proposed ranking approach and its advantages.

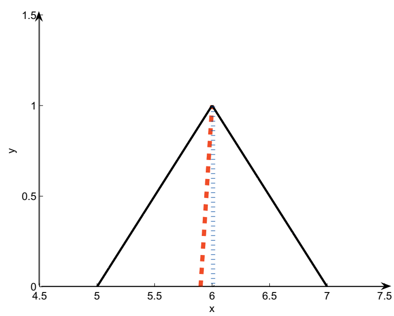

Figure 3: Fuzzy numbers A, B and C in Example 5.3.

Table 3: Comparative results of Example 5.3

Ranking approach	A	B	C	Result
Chen [13]	0.5	0.5714	0.5833	$A \prec B \prec C$
Cheng [25]	6.021	6.349	6.7519	$C \prec B \prec A$
Abbasbandy and Asady [36] (Sign distance $P = 1$)	6.12	12.45	12.5	$A \prec B \prec C$
Abbasbandy and Asady [36] (Sign distance $P = 2$)	8.25	8.82	8.85	$A \prec B \prec C$
Abbasbandy and Hajjari [40]	6	6.075	6.0834	$A \prec B \prec C$
Wang and Luo [41]	0.5	0.571	0.583	$A \prec B \prec C$
Wang et al. [42]	0.25	0.5339	0.5625	$A \prec B \prec C$
Asady [43]	0.6667	0.8182	1	$A \prec B \prec C$
Luu et al. [52]	0.2222	0.373	0.401	$A \prec B \prec C$
Hajjari [59] "Mag_N"	0.0	4.0	8.0833	$A \prec B \prec C$
Proposed method	6.0093	6.3088	6.3421	$A \prec B \prec C$

Example 5.4. Compare the crisp number $A = (1, 1, 1)$ and two symmetric fuzzy numbers $B = (0, 1, 2)$ $C = (-1, 1, 3)$, which are taken from [59](See Fig. 4).

We know that $\delta_A = 0, \delta_B = 1$ and $\delta_C = 2$. It is clear that $\delta_A < \delta_B < \delta_C$ then by applying new approach the ranking order is $A \prec B \prec C$. Since, we expect that the crisp number should be stronger than this triangular fuzzy numbers hence, this expectation is satisfied by new method.

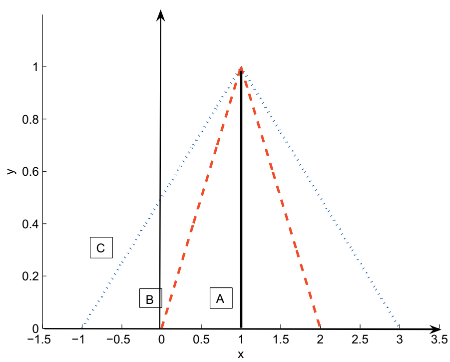

Figure 4: Fuzzy numbers A, B and C in Example 5.4.

6. Conclusions

In spite of many ranking methods, no one can rank fuzzy numbers with human intuition consistently in all cases. Here, we pointed out the shortcoming of some recent centroid-index methods and presented a new centroid-index method for ranking fuzzy numbers. Particulary, the problem of ranking of evaluations on triangular fuzzy number sensitive ti their spread has been analyzed. The proposed formulae are simple and have consistent expression on the horizontal axis and vertical axis and also be used for some especial cases in many centroid-index methods. The paper herein presents several comparative examples to illustrate the validity and advantages of proposed centroid-index ranking method. It shows that the ranking order obtained by the proposed centroid-index ranking method is more consistent with human intuitions than existing methods. Furthermore, the proposed ranking method can effectively rank a mix of various types of fuzzy numbers, which is another advantages of the proposed method over other existing ranking approaches.

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