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Original Research Paper

Boundedness Theorems for Composition-differentiation Operators Acting on Classical Function Spaces on Bidisk

Z. Hedayatfar

Imam Khomeini International University

A. Abkar*

Imam Khomeini International University

Abstract. We establish boundedness criteria for products of composition and differentiation operators acting from weighted Bergman space $A^2_{\alpha_1, \alpha_2}(\mathbb{D}^2)$ on the bidisk into the weighted Bergman space $A^2_\beta(\mathbb{D})$ on the unit disk. We then derive sufficient conditions on the symbol functions to guarantee the boundedness of composition-differentiation operators acting from the Hardy space $H^2(\mathbb{D}^2)$ on the bidisk into the Hardy space $H^2(\mathbb{D})$ on the unit disk.

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1 Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ denote the open unit disk. For $\alpha > -1$, the weighted Bergman space $A^2_\alpha(\mathbb{D})$ consists of analytic functions f on the

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*Corresponding Author

open unit disk that are square-integrable in the sense that

$$\int_{\mathbb{D}} |f(z)|^2 dA_{\alpha}(z) < \infty,$$

where

$$dA_{\alpha}(z) = (\alpha + 1)(1 - |z|^2)^{\alpha} dA(z), \quad dA(z) = \frac{1}{\pi} dx dy.$$

Equipped with the inner product

$$\langle f, g \rangle = \int_{\mathbb{D}} f(z) \overline{g(z)} dA_{\alpha}(z),$$

$A_{\alpha}^2(\mathbb{D})$ is a Hilbert space, and the associated norm is given by

$$\|f\|_{A_{\alpha}^2} = \langle f, f \rangle^{1/2} = \left(\int_{\mathbb{D}} |f(z)|^2 dA_{\alpha}(z) \right)^{1/2}.$$

It is well known that for every $w \in \mathbb{D}$, the evaluation functional $\phi_w : A_{\alpha}^2(\mathbb{D}) \rightarrow \mathbb{C}$ defined by $\phi_w(f) = f(w)$ is bounded. This implies that the weighted Bergman space $A_{\alpha}^2(\mathbb{D})$ is a reproducing kernel Hilbert space; that is, for every $w \in \mathbb{D}$, there exists a unique function $K_w^{\alpha} \in A_{\alpha}^2(\mathbb{D})$ such that

$$f(w) = \langle f, K_w^{\alpha} \rangle = \int_{\mathbb{D}} f(z) \overline{K_w^{\alpha}(z)} dA_{\alpha}(z) \quad \text{for all } f \in A_{\alpha}^2(\mathbb{D}).$$

The function

$$K_w^{\alpha}(z) = \frac{1}{(1 - \overline{w}z)^{\alpha+2}}$$

is called the Bergman kernel function.

Let $\mathbb{D}^2 = \mathbb{D} \times \mathbb{D}$ denote the bidisk in $\mathbb{C}^2$, and let α_1, α_2 be two numbers in the interval $(-1, \infty)$. In this paper we consider the Bergman space $A_{\alpha_1, \alpha_2}^2(\mathbb{D}^2)$ consisting of all analytic functions $f(z_1, z_2)$ on $\mathbb{D}^2$ satisfying

$$\|f\|_{A_{\alpha_1, \alpha_2}^2(\mathbb{D}^2)}^2 = \int_{\mathbb{D}^2} |f(z_1, z_2)|^2 dA_{\alpha_1}(z_1) dA_{\alpha_2}(z_2) < \infty.$$

The inner product in this space is given by

$$\langle f, g \rangle = \int_{\mathbb{D}^2} f(z_1, z_2) \overline{g(z_1, z_2)} dA_{\alpha_1}(z_1) dA_{\alpha_2}(z_2).$$

If $\alpha_1 = \alpha_2 = \alpha$, the space $A_{\alpha_1, \alpha_2}^2(\mathbb{D}^2)$ is denoted by $A_\alpha^2(\mathbb{D}^2)$; and if $\alpha_1 = \alpha_2 = 0$, the space is simply denoted by $A^2(\mathbb{D}^2)$. Indeed, the space $A_{\alpha_1, \alpha_2}^2(\mathbb{D}^2)$ generalizes the classical weighted and un-weighted Bergman spaces on the bidisk. It is easy to see that $A_{\alpha_1, \alpha_2}^2(\mathbb{D}^2)$ is a reproducing kernel Hilbert space whose kernel is given by

$$K_w^{\alpha_1, \alpha_2}(z) = \frac{1}{(1 - \overline{w_1}z_1)^{\alpha_1+2}} \cdot \frac{1}{(1 - \overline{w_2}z_2)^{\alpha_2+2}},$$

where $w = (w_1, w_2)$ and $z = (z_1, z_2)$ are elements in $\mathbb{D}^2$. For basic information on the theory of Bergman spaces we refer the reader to the books [6] and [10].

The Hardy space $H^2(\mathbb{D})$ consists of analytic functions f on the unit disk satisfying

$$\|f\|_{H^2(\mathbb{D})}^2 = \sup_{0 < r < 1} \int_{\mathbb{T}} |f(r\zeta)|^2 dm(\zeta) < \infty,$$

where $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ denotes the unit circle, and dm represents the normalized arc-length measure on $\mathbb{T}$. A fundamental property of this space is that if $f \in H^2$, then the radial limit of f exists almost everywhere on $\mathbb{T}$; this means that for almost every $\zeta \in \mathbb{T}$ we have (see [5], or [6]):

$$f(\zeta) = \lim_{r \rightarrow 1} f(r\zeta),$$

and

$$\|f\|_{H^2(\mathbb{D})}^2 = \sup_{0 < r < 1} \int_{\mathbb{T}} |f(r\zeta)|^2 dm(\zeta) = \int_{\mathbb{T}} |f(\zeta)|^2 dm(\zeta).$$

The Hardy space $H^2(\mathbb{D})$ is a Hilbert space whose reproducing kernel is given by

$$\mathcal{K}_z(w) = \frac{1}{1 - \overline{z}w}, \quad (z, w) \in \mathbb{D} \times \mathbb{D}.$$

The Hardy space on bidisk, $H^2(\mathbb{D}^2)$, consists of all holomorphic functions f in the bidisk for which

$$\|f\|_{H^2(\mathbb{D}^2)}^2 = \sup_{0 < r < 1} \int_{\mathbb{T}^2} |f(r\zeta)|^2 dm(\zeta) < \infty,$$

where

$$dm(\zeta) = \frac{dm(\zeta_1)dm(\zeta_2)}{(2\pi)^2}, \quad \zeta = (\zeta_1, \zeta_2),$$

is the normalized measure on $\mathbb{T}^2$. As in one-dimensional case, for each $f \in H^2(\mathbb{D}^2)$ the radial limit

$$f(\zeta) = \lim_{r \rightarrow 1^-} f(r\zeta)$$

exists for almost every $\zeta \in \mathbb{T}^2$, and

$$\|f\|_{H^2(\mathbb{D}^2)}^2 = \sup_{0 < r < 1} \int_{\mathbb{T}^2} |f(r\zeta)|^2 dm(\zeta) = \int_{\mathbb{T}^2} |f(\zeta)|^2 dm(\zeta).$$

We now consider two analytic self-maps $\varphi, \psi : \mathbb{D} \rightarrow \mathbb{D}$ on the unit disk, and a holomorphic mapping

$$\Phi : \mathbb{D} \rightarrow \mathbb{D}^2$$

defined by

$$\Phi(z) = (\varphi(z), \psi(z)), \quad z \in \mathbb{D}.$$

For α and β bigger than -1 , we define the composition operator

$$C_\Phi : A_\alpha^2(\mathbb{D}^2) \rightarrow A_\beta^2(\mathbb{D})$$

by the relation

$$C_\Phi(f)(z) = (f \circ \Phi)(z) = f(\varphi(z), \psi(z)).$$

In the complex plane, it is a classical result that for any analytic self-map $\varphi : \mathbb{D} \rightarrow \mathbb{D}$, the composition operator $C_\varphi : A_\alpha^2(\mathbb{D}) \rightarrow A_\alpha^2(\mathbb{D})$ defined by $C_\varphi(f) = f \circ \varphi$ is bounded. It is also known that this operator is bounded on the Hardy space $H^2(\mathbb{D})$ (see, for instance [5]). There are plenty of research articles on the study of composition operators in one-dimensional case; see for instance [7], [9], and [12]. As for composition operators acting on spaces defined on subsets of $\mathbb{C}^n$, we refer the reader to [15] and [16].

Another operator closely related to C_φ is the differentiation operator D_φ defined by

$$D_\varphi(f) = f' \circ \varphi.$$

In contrast to C_φ , the differentiation operator D_φ can be unbounded for some symbol functions φ . This operator was recently studied by several authors where its boundedness, compactness, spectral property, and Hilbert-Schmidtness were discussed (see for instance, [2, 3, 4], [8], [11], [13, 14]).

In this paper we consider

$$E_\Phi : A_{\alpha_1, \alpha_2}^2(\mathbb{D}^2) \rightarrow A_\beta^2(\mathbb{D})$$

defined by

$$(E_\Phi f)(z) = (Rf \circ \Phi)(z), \quad z \in \mathbb{D},$$

where Rf denotes the radial derivative of f . We Recall that for a holomorphic function f in the unit ball of $\mathbb{C}^2$, the radial derivative of f is defined by

$$Rf(z) = z_1 \frac{\partial f}{\partial z_1}(z) + z_2 \frac{\partial f}{\partial z_2}(z), \quad z = (z_1, z_2).$$

Radial derivative plays a prominent role in the development of the theory of analytic functions in several complex variables [17]. Thus, for $\Phi = (\varphi, \psi) : \mathbb{D} \rightarrow \mathbb{D}^2$, $f \in A_{\alpha_1, \alpha_2}^2(\mathbb{D}^2)$, and $z \in \mathbb{D}$ we have

$$\begin{aligned} (E_\Phi f)(z) &= (Rf \circ \Phi)(z) \\ &= \varphi(z) \frac{\partial f}{\partial z_1}(\varphi(z), \psi(z)) + \psi(z) \frac{\partial f}{\partial z_2}(\varphi(z), \psi(z)). \end{aligned} \quad (1)$$

In this paper, we search for sufficient conditions to ensure that E_Φ is bounded. We prove that if φ and ψ satisfy some boundedness criteria, then E_Φ is bounded. More precisely, we show that $E_\Phi : A_{\alpha_1, \alpha_2}^2(\mathbb{D}^2) \rightarrow A_\beta^2(\mathbb{D})$ is bounded provided that $\max\{\|\varphi\|_\infty, \|\psi\|_\infty\} < 1$, or alternatively, if $\|\varphi\psi\|_\infty < 1$ and the operators D_φ and D_ψ are bounded (see Theorem 2.2 and Theorem 2.4). We then take up similar problem on the Hardy space, and prove that $E_\Phi : H^2(\mathbb{D}^2) \rightarrow H^2(\mathbb{D})$ is bounded provided that the symbol functions satisfy similar criteria (see Theorems 3.3 and 3.4).

2 Boundedness On Bergman Space

We first investigate the composition-differentiation operator

$$E_\Phi : A_{\alpha_1, \alpha_2}^2(\mathbb{D}^2) \rightarrow A_\beta^2(\mathbb{D})$$

defined through the holomorphic mapping $\Phi = (\varphi, \psi) : \mathbb{D} \rightarrow \mathbb{D}^2$, where φ and ψ are analytic self-maps of the unit disk. Our results state that how the boundedness of φ and ψ affect the boundedness of E_Φ in various function spaces.

Theorem 2.1. *Let $\Phi = (\varphi, \psi)$ where φ and ψ are analytic self-maps of the open unit disk satisfying $\|\varphi\psi\|_\infty < 1$. Then $C_\Phi : A_{\alpha_1, \alpha_2}^2(\mathbb{D}^2) \rightarrow A_\beta^2(\mathbb{D})$ is bounded.*

Proof. See [3, Theorem 2.3]. $\square$

Theorem 2.2. *Let $\Phi = (\varphi, \psi) : \mathbb{D} \rightarrow \mathbb{D}^2$ where φ and ψ are analytic self-maps of the open unit disk satisfying $\max\{\|\varphi\|_\infty, \|\psi\|_\infty\} < 1$. Then $E_\Phi : A_{\alpha_1, \alpha_2}^2(\mathbb{D}^2) \rightarrow A_\beta^2(\mathbb{D})$ is bounded.*

Proof. Choose a number σ satisfying

$$\max\{\|\varphi\|_\infty, \|\psi\|_\infty\} \leq \sigma < 1.$$

For $F(z_1, z_2) \in A_{\alpha_1, \alpha_2}^2(\mathbb{D}^2)$ we write

$$F(z_1, z_2) = \sum_{n=0}^{\infty} z_1^n F_n(z_2) = \sum_{n=0}^{\infty} z_2^n G_n(z_1), \quad (2)$$

where F_n 's are in $A_{\alpha_2}^2(\mathbb{D})$, and G_n 's are in $A_{\alpha_1}^2(\mathbb{D})$. A computation shows that

$$\|F\|_{A_{\alpha_1, \alpha_2}^2(\mathbb{D}^2)}^2 = \sum_{n=0}^{\infty} k_{\alpha_1}(n) \|F_n\|_{A_{\alpha_2}^2(\mathbb{D})}^2 = \sum_{n=0}^{\infty} k_{\alpha_2}(n) \|G_n\|_{A_{\alpha_1}^2(\mathbb{D})}^2$$

where

$$k_\alpha(n) = \frac{n! \Gamma(\alpha + 2)}{\Gamma(n + \alpha + 2)}, \quad n = 0, 1, 2, \dots$$

To estimate $\|E_\Phi F\|_{A_\beta^2(\mathbb{D})}^2$ we note that

$$\begin{aligned} (E_\Phi F)(z) &= \varphi(z) \frac{\partial F}{\partial z_1}(\varphi(z), \psi(z)) + \psi(z) \frac{\partial F}{\partial z_2}(\varphi(z), \psi(z)) \\ &= \varphi(z) \sum_{n=0}^{\infty} n \varphi^{n-1}(z) F_n(\psi(z)) + \psi(z) \sum_{n=0}^{\infty} n \psi^{n-1}(z) G_n(\varphi(z)). \end{aligned} \quad (3)$$

Thus, by using the inequality

$$|z + w|^2 \leq 2(|z|^2 + |w|^2), \quad z, w \in \mathbb{C},$$

and (3) we write

$$\begin{aligned} \int_{\mathbb{D}} |E_\Phi F|^2 dA_\beta(z) &= \int_{\mathbb{D}} \left| \sum_{n=1}^{\infty} n \varphi^n(z) (C_\psi F_n)(z) \right. \\ &\quad \left. + \sum_{n=1}^{\infty} n \psi^n(z) (C_\varphi G_n)(z) \right|^2 dA_\beta(z) \\ &\leq 2 \int_{\mathbb{D}} \left| \sum_{n=1}^{\infty} n \varphi^n(z) (C_\psi F_n)(z) \right|^2 dA_\beta(z) \quad (4) \end{aligned}$$

$$+ 2 \int_{\mathbb{D}} \left| \sum_{n=1}^{\infty} n \psi^n(z) (C_\varphi G_n)(z) \right|^2 dA_\beta(z). \quad (5)$$

We now use the Cauchy-Schwartz inequality and $\|\varphi\|_\infty \leq \sigma$ to get

$$\begin{aligned} \left| \sum_{n=1}^{\infty} n \varphi^n(z) (C_\psi F_n)(z) \right| &\leq \left(\sum_{n=1}^{\infty} \frac{n^2}{k_{\alpha_1}(n)} |\varphi(z)|^{2n} \right)^{1/2} \\ &\quad \cdot \left(\sum_{n=1}^{\infty} k_{\alpha_1}(n) |C_\psi F_n(z)|^2 \right)^{1/2} \end{aligned}$$

or equivalently,

$$\left| \sum_{n=1}^{\infty} n \varphi^n(z) (C_\psi F_n)(z) \right|^2 \leq \sum_{n=1}^{\infty} \frac{n^2 \sigma^{2n}}{k_{\alpha_1}(n)} \sum_{n=1}^{\infty} k_{\alpha_1}(n) |C_\psi F_n(z)|^2.$$

Thus, an estimation for (4) is obtained:

$$\begin{aligned}
2 \int_{\mathbb{D}} \left| \sum_{n=1}^{\infty} n \varphi^n(z) (C_{\psi} F_n)(z) \right|^2 dA_{\beta}(z) \\
\leq 2 \sum_{n=1}^{\infty} \frac{n^2 \sigma^{2n}}{k_{\alpha_1}(n)} \int_{\mathbb{D}} \sum_{n=1}^{\infty} k_{\alpha_1}(n) |C_{\psi} F_n(z)|^2 dA_{\beta}(z) \\
\leq 2 S_{\alpha_1}(\sigma) \|C_{\psi}\|^2 \|F\|_{A_{\alpha_1, \alpha_2}^2(\mathbb{D}^2)}^2,
\end{aligned}$$

where

$$S_{\alpha_1}(\sigma) = \sum_{n=1}^{\infty} \frac{n^2 \sigma^{2n}}{k_{\alpha_1}(n)} = \frac{1}{\Gamma(\alpha_1 + 2)} \sum_{n=1}^{\infty} \frac{n^2 \Gamma(n + \alpha_1 + 2)}{n!} \sigma^{2n}$$

is a constant depending on α_1 and σ . As for the convergence of the above series we put

$$u_n = \frac{n^2 \Gamma(n + \alpha_1 + 2)}{n!} \sigma^{2n},$$

and use $\Gamma(x + 1) = x\Gamma(x)$ (for x positive) to get

$$\frac{u_{n+1}}{u_n} = \frac{(n+1)\Gamma(n + \alpha_1 + 3)}{n^2 \Gamma(n + \alpha_1 + 2)} \sigma^2 = \frac{(n+1)(n + \alpha_1 + 2)}{n^2} \sigma^2,$$

and eventually,

$$\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = \sigma^2 < 1.$$

Using a similar argument, we obtain the following upper bound for (5):

$$2 \int_{\mathbb{D}} \left| \sum_{n=1}^{\infty} n \psi^n(z) (C_{\varphi} G_n)(z) \right|^2 dA_{\beta}(z) \leq 2 S_{\alpha_2}(\sigma) \|C_{\varphi}\|^2 \|F\|_{A_{\alpha_1, \alpha_2}^2(\mathbb{D}^2)}^2.$$

Finally, we have

$$\int_{\mathbb{D}} |E_{\Phi} F|^2 dA_{\beta}(z) \leq 2 \left(S_{\alpha_1}(\sigma) \|C_{\psi}\|^2 + S_{\alpha_2}(\sigma) \|C_{\varphi}\|^2 \right) \|F\|_{A_{\alpha_1, \alpha_2}^2(\mathbb{D}^2)}^2,$$

which proves that $E_{\Phi} : A_{\alpha_1, \alpha_2}^2(\mathbb{D}^2) \rightarrow A_{\beta}^2(\mathbb{D})$ is bounded. $\square$

In the following theorem, we replace the assumption

$$\max\{\|\varphi\|_\infty, \|\psi\|_\infty\} < 1$$

with the weaker assumption $\|\varphi\psi\|_\infty < 1$. We further assume that the differentiation operators D_φ and D_ψ are bounded. To compare these two assumptions, we note that if $\max\{\|\varphi\|_\infty, \|\psi\|_\infty\} < 1$, then $\|\varphi\psi\|_\infty < 1$. On the other hand, for $\varphi(z) = z$, $\psi(z) = \frac{z}{2}$, we have $\frac{1}{2} = \|\varphi\psi\|_\infty < 1$ but $\max\{\|\varphi\|_\infty, \|\psi\|_\infty\} = 1$.

Lemma 2.3. *Let φ, ψ be analytic self-maps of the open unit disk that satisfy the assumption $\|\varphi\psi\|_\infty < \sigma < 1$. Then there exists a measurable partition $\{\Omega_1, \Omega_2\}$ of the open unit disk such that $|\varphi(z)| < \sqrt{\sigma}$ for $z \in \Omega_1$, and $|\psi(z)| < \sqrt{\sigma}$, for $z \in \Omega_2$.*

Proof. See [1, Lemma 2.1] where this lemma was proved for $n \geq 2$. $\square$

Theorem 2.4. *Let $\Phi = (\varphi, \psi) : \mathbb{D} \rightarrow \mathbb{D}^2$ where φ and ψ are analytic self-maps of the open unit disk satisfying $\|\varphi\psi\|_\infty < 1$. Assume further that the operators $D_\varphi : A_{\alpha_1}^2(\mathbb{D}) \rightarrow A_\beta^2(\mathbb{D})$ and $D_\psi : A_{\alpha_2}^2(\mathbb{D}) \rightarrow A_\beta^2(\mathbb{D})$ are bounded. Then $E_\Phi : A_{\alpha_1, \alpha_2}^2(\mathbb{D}^2) \rightarrow A_\beta^2(\mathbb{D})$ is bounded.*

Proof. Let us consider $F(z_1, z_2) \in A_{\alpha_1, \alpha_2}^2(\mathbb{D}^2)$, and write

$$F(z_1, z_2) = \sum_{n=0}^{\infty} z_1^n F_n(z_2),$$

where F_n 's are functions in $A_{\alpha_2}^2(\mathbb{D})$. It now follows that

$$\|F\|_{A_{\alpha_1, \alpha_2}^2(\mathbb{D}^2)}^2 = \sum_{n=0}^{\infty} \|z_1^n\|_{A_{\alpha_1}^2(\mathbb{D})}^2 \|F_n\|_{A_{\alpha_2}^2(\mathbb{D})}^2 = \sum_{n=0}^{\infty} k_{\alpha_1}(n) \|F_n\|_{A_{\alpha_2}^2(\mathbb{D})}^2.$$

To prove the theorem, we need to estimate $\|E_\Phi F\|_{A_\beta^2(\mathbb{D})}^2$. We first note that for $F \in A_{\alpha_1, \alpha_2}^2(\mathbb{D}^2)$, we have

$$\begin{aligned} (E_\Phi F)(z) &= (RF)(\varphi(z), \psi(z)) \\ &= \varphi(z) \frac{\partial F}{\partial z_1}(\varphi(z), \psi(z)) + \psi(z) \frac{\partial F}{\partial z_2}(\varphi(z), \psi(z)) \\ &= \varphi(z) \sum_{n=0}^{\infty} n \varphi^{n-1}(z) F_n(\psi(z)) + \psi(z) \sum_{n=0}^{\infty} \varphi^n(z) F'_n(\psi(z)). \end{aligned} \quad (6)$$

Thus, using the subsets Ω_j , $j = 1, 2$, introduced in Lemma 2.3 we have

$$\|E_\Phi F\|_{A_\beta^2(\mathbb{D})}^2 = \int_{\Omega_1} |E_\Phi F|^2 dA_\beta + \int_{\Omega_2} |E_\Phi F|^2 dA_\beta \quad (7)$$

For the first term on the right-hand side of (7), we use (6) to write

$$\begin{aligned} \int_{\Omega_1} |E_\Phi F|^2 dA_\beta = \\ \int_{\Omega_1} \left| \sum_{n=1}^{\infty} n\varphi^n(z)(C_\psi F_n)(z) + \psi(z) \sum_{n=1}^{\infty} \varphi^n(z)(D_\psi F_n)(z) \right|^2 dA_\beta(z), \end{aligned}$$

from which we obtain

$$\int_{\Omega_1} |E_\Phi F|^2 dA_\beta(z) \leq 2 \int_{\Omega_1} \left| \sum_{n=1}^{\infty} n\varphi^n(z)(C_\psi F_n)(z) \right|^2 dA_\beta(z) \quad (8)$$

$$+ 2 \int_{\Omega_1} \left| \sum_{n=1}^{\infty} \varphi^n(z)(D_\psi F_n)(z) \right|^2 dA_\beta(z). \quad (9)$$

It now follows from the Cauchy-Schwartz inequality that

$$\begin{aligned} \left| \sum_{n=1}^{\infty} n\varphi^n(z)(C_\psi F_n)(z) \right| \leq \\ \left(\sum_{n=1}^{\infty} \frac{n^2}{k_{\alpha_1}(n)} |\varphi(z)|^{2n} \right)^{1/2} \left(\sum_{n=1}^{\infty} k_{\alpha_1}(n) |C_\psi F_n(z)|^2 \right)^{1/2}. \end{aligned}$$

In particular, for $z \in \Omega_1$ we have $|\varphi(z)| \leq \sqrt{\sigma}$ in accordance with Lemma 2.3, so that

$$\left| \sum_{n=1}^{\infty} n\varphi^n(z)(C_\psi F_n)(z) \right|^2 \leq \sum_{n=1}^{\infty} \frac{n^2 \sigma^n}{k_{\alpha_1}(n)} \sum_{n=1}^{\infty} k_{\alpha_1}(n) |C_\psi F_n(z)|^2,$$

from which we obtain the following upper bound for (8):

$$\begin{aligned}
& 2 \int_{\Omega_1} \left| \sum_{n=1}^{\infty} n \varphi^n(z) (C_\psi F_n)(z) \right|^2 dA_\beta(z) \\
& \leq 2 \sum_{n=1}^{\infty} \frac{n^2 \sigma^n}{k_{\alpha_1}(n)} \int_{\mathbb{D}} \sum_{n=1}^{\infty} k_{\alpha_1}(n) |C_\psi F_n(z)|^2 dA_\beta(z) \\
& \leq 2 \sum_{n=1}^{\infty} \frac{n^2 \sigma^n}{k_{\alpha_1}(n)} \sum_{n=1}^{\infty} k_{\alpha_1}(n) \|C_\psi\|^2 \|F_n\|_{A_{\alpha_2}^2(\mathbb{D}^2)}^2 \\
& = 2 s_{\alpha_1}(\sigma) \|C_\psi\|^2 \|F\|_{A_{\alpha_1, \alpha_2}^2(\mathbb{D}^2)}^2,
\end{aligned}$$

where

$$s_{\alpha_1}(\sigma) = \sum_{n=1}^{\infty} \frac{n^2 \sigma^n}{k_{\alpha_1}(n)}.$$

As for (9) we note that

$$\begin{aligned}
& \left| \sum_{n=1}^{\infty} \varphi^n(z) (D_\psi F_n)(z) \right| \leq \\
& \left(\sum_{n=1}^{\infty} \frac{1}{k_{\alpha_1}(n)} |\varphi(z)|^{2n} \right)^{1/2} \left(\sum_{n=1}^{\infty} k_{\alpha_1}(n) |D_\psi F_n(z)|^2 \right)^{1/2}.
\end{aligned}$$

In particular, for $z \in \Omega_1$,

$$\left| \sum_{n=1}^{\infty} \varphi^n(z) (D_\psi F_n)(z) \right|^2 \leq \sum_{n=1}^{\infty} \frac{\sigma^n}{k_{\alpha_1}(n)} \sum_{n=1}^{\infty} k_{\alpha_1}(n) |D_\psi F_n(z)|^2,$$

from which we obtain

$$\begin{aligned}
& 2 \int_{\Omega_1} \left| \sum_{n=1}^{\infty} \varphi^n(z) (D_\psi F_n)(z) \right|^2 dA_\beta(z) \\
& \leq 2 \sum_{n=1}^{\infty} \frac{\sigma^n}{k_{\alpha_1}(n)} \sum_{n=1}^{\infty} k_{\alpha_1}(n) \|D_\psi\|^2 \|F_n\|^2 \\
& \leq 2 \sum_{n=1}^{\infty} \frac{\sigma^n}{k_{\alpha_1}(n)} \|D_\psi\|^2 \|F\|_{A_{\alpha_1, \alpha_2}^2(\mathbb{D}^2)}^2 \\
& = 2 t_{\alpha_1}(\sigma) \|D_\psi\|^2 \|F\|_{A_{\alpha_1, \alpha_2}^2(\mathbb{D}^2)}^2,
\end{aligned}$$

where

$$t_{\alpha_1}(\sigma) = \sum_{n=1}^{\infty} \frac{\sigma^n}{k_{\alpha_1}(n)}.$$

Thus,

$$\int_{\Omega_1} |E_{\Phi} F|^2 dA_{\beta} \leq 2 \left(s_{\alpha_1}(\sigma) \|C_{\psi}\|^2 + t_{\alpha_1}(\sigma) \|D_{\psi}\|^2 \right) \|F\|_{A_{\alpha_1, \alpha_2}^2(\mathbb{D}^2)}^2. \quad (10)$$

Next, we use the same pattern to estimate the second term in (7). This time, we use the expansion

$$F(z_1, z_2) = \sum_{n=0}^{\infty} z_2^n G_n(z_1),$$

where G_n 's are functions in $A_{\alpha_1}^2(\mathbb{D})$. It turns out that

$$\int_{\Omega_2} |E_{\Phi} F|^2 dA_{\beta} \leq 2 \left(s_{\alpha_2}(\sigma) \|C_{\varphi}\|^2 + t_{\alpha_2}(\sigma) \|D_{\varphi}\|^2 \right) \|F\|_{A_{\alpha_1, \alpha_2}^2(\mathbb{D}^2)}^2. \quad (11)$$

Finally, it follows from (7), (10), and (11) that $E_{\Phi} : A_{\alpha_1, \alpha_2}^2(\mathbb{D}^2) \rightarrow A_{\beta}^2(\mathbb{D})$ is bounded, and

$$\|E_{\Phi}\|^2 \leq 2 \left(s_{\alpha_1}(\sigma) \|C_{\psi}\|^2 + s_{\alpha_2}(\sigma) \|C_{\varphi}\|^2 + t_{\alpha_1}(\sigma) \|D_{\psi}\|^2 + t_{\alpha_2}(\sigma) \|D_{\varphi}\|^2 \right).$$

□

3 Boundedness On Hardy Space

In the following theorem, we replace the target space of E_{Φ} to the Hardy space. The following theorem is a consequence of the statements proved in the previous section.

Theorem 3.1. *Let $\Phi = (\varphi, \psi) : \mathbb{D} \rightarrow \mathbb{D}^2$ where φ and ψ are analytic self-maps of the open unit disk satisfying $\max\{\|\varphi\|_{\infty}, \|\psi\|_{\infty}\} < 1$. Then $E_{\Phi} : H^2(\mathbb{D}^2) \rightarrow A^2(\mathbb{D})$ is bounded.*

Proof. We first observe that $H^2(\mathbb{D}^2) \subseteq A^2(\mathbb{D}^2)$ and

$$\|F\|_{H^2(\mathbb{D}^2)}^2 \leq \|F\|_{A^2(\mathbb{D}^2)}^2.$$

To see this, we write

$$F(z, w) = \sum_{m,n=0}^{\infty} a_{m,n} z^m w^n.$$

A computation shows that

$$\|F\|_{A^2(\mathbb{D}^2)}^2 = \sum_{m,n=0}^{\infty} \frac{|a_{m,n}|^2}{(m+1)(n+1)} \leq \sum_{m,n=0}^{\infty} |a_{m,n}|^2 = \|F\|_{H^2(\mathbb{D}^2)}^2.$$

On the other hand, we have seen in Theorem 2.2 that there is a constant C depending on φ , ψ , and σ such that for each $F \in H^2(\mathbb{D}^2)$ we have

$$\|E_{\Phi}F\|_{A^2(\mathbb{D})}^2 \leq C\|F\|_{A^2(\mathbb{D}^2)}^2 \leq C\|F\|_{H^2(\mathbb{D}^2)}^2,$$

which completes the proof. $\square$

Theorem 3.2. *Let $\Phi = (\varphi, \psi) : \mathbb{D} \rightarrow \mathbb{D}^2$ where φ and ψ are analytic self-maps of the open unit disk satisfying $\|\varphi\psi\|_{\infty} < 1$. If the operators $D_{\varphi}, D_{\psi} : A^2(\mathbb{D}) \rightarrow A^2(\mathbb{D})$ are bounded, then $E_{\Phi} : H^2(\mathbb{D}^2) \rightarrow A^2(\mathbb{D})$ is bounded.*

Proof. It follows from Theorem 2.4 (for $\alpha_1 = \alpha_2 = \beta = 0$) that $E_{\Phi} : A^2(\mathbb{D}^2) \rightarrow A^2(\mathbb{D})$ is bounded. Thus there is a constant $C > 0$ such that for each $F \in H^2(\mathbb{D}^2) \subseteq A^2(\mathbb{D}^2)$,

$$\|E_{\Phi}F\|_{A^2(\mathbb{D})}^2 \leq C\|F\|_{A^2(\mathbb{D}^2)}^2 \leq C\|F\|_{H^2(\mathbb{D}^2)}^2.$$

This proves the theorem. $\square$

Theorem 3.3. *Let $\Phi = (\varphi, \psi) : \mathbb{D} \rightarrow \mathbb{D}^2$ where φ and ψ are analytic self-maps of the open unit disk satisfying $\max\{\|\varphi\|_{\infty}, \|\psi\|_{\infty}\} < 1$. Then $E_{\Phi} : H^2(\mathbb{D}^2) \rightarrow H^2(\mathbb{D})$ is bounded.*

Proof. Let $\max\{\|\varphi\|_\infty, \|\psi\|_\infty\} \leq \sigma < 1$. For $F(z_1, z_2) \in H^2(\mathbb{D}^2)$ we write

$$F(z_1, z_2) = \sum_{n=0}^{\infty} z_1^n F_n(z_2) = \sum_{n=0}^{\infty} z_2^n G_n(z_1),$$

where F_n 's and G_n 's are functions in $H^2(\mathbb{D})$. It follows that

$$\|F\|_{H^2(\mathbb{D}^2)}^2 = \sum_{n=0}^{\infty} \|F_n\|_{H^2(\mathbb{D})}^2 = \sum_{n=0}^{\infty} \|G_n\|_{H^2(\mathbb{D})}^2.$$

To estimate $\|E_\Phi F\|_{H^2(\mathbb{D})}^2$, we use (3) to write

$$\begin{aligned} & \int_{\mathbb{T}} |E_\Phi F|^2 dm \\ &= \int_{\mathbb{T}} \left| \sum_{n=1}^{\infty} n\varphi^n(z)(C_\psi F_n)(z) + \sum_{n=1}^{\infty} n\psi^n(z)(C_\varphi G_n)(z) \right|^2 dm(z), \end{aligned}$$

from which we obtain

$$\int_{\mathbb{T}} |E_\Phi F|^2 dm \leq 2 \int_{\mathbb{T}} \left| \sum_{n=1}^{\infty} n\varphi^n(z)(C_\psi F_n)(z) \right|^2 dm(z) \quad (12)$$

$$+ 2 \int_{\mathbb{T}} \left| \sum_{n=1}^{\infty} n\psi^n(z)(C_\varphi G_n)(z) \right|^2 dm(z). \quad (13)$$

Since $\|\varphi\|_\infty \leq \sigma$, it follows that

$$\left| \sum_{n=1}^{\infty} n\varphi^n(z)(C_\psi F_n)(z) \right| \leq \left(\sum_{n=1}^{\infty} n^2 |\varphi(z)|^{2n} \right)^{1/2} \left(\sum_{n=1}^{\infty} |C_\psi F_n(z)|^2 \right)^{1/2}$$

or equivalently,

$$\left| \sum_{n=1}^{\infty} n\varphi^n(z)(C_\psi F_n)(z) \right|^2 \leq \sum_{n=1}^{\infty} n^2 \sigma^{2n} \sum_{n=1}^{\infty} |C_\psi F_n(z)|^2.$$

Thus, (12) is bounded by

$$\begin{aligned} 2 \int_{\mathbb{T}} \left| \sum_{n=1}^{\infty} n \varphi^n(z) (C_{\psi} F_n)(z) \right|^2 dm(z) \\ \leq 2 \sum_{n=1}^{\infty} n^2 \sigma^{2n} \int_{\mathbb{T}} \sum_{n=1}^{\infty} |C_{\psi} F_n(z)|^2 dm(z) \\ \leq 2 \frac{\sigma^2 + \sigma^4}{(1 - \sigma^2)^3} \|C_{\psi}\|^2 \|F\|_{H^2(\mathbb{D}^2)}^2. \end{aligned}$$

Using a similar argument, we obtain the following upper bound for (13):

$$2 \int_{\mathbb{T}} \left| \sum_{n=1}^{\infty} n \psi^n(z) (C_{\varphi} G_n)(z) \right|^2 dm(z) \leq 2 \frac{\sigma^2 + \sigma^4}{(1 - \sigma^2)^3} \|C_{\varphi}\|^2 \|F\|_{H^2(\mathbb{D}^2)}^2.$$

Finally, we have

$$\|E_{\Phi} F\|_{H^2(\mathbb{D})}^2 \leq 2 \frac{\sigma^2 + \sigma^4}{(1 - \sigma^2)^3} \left(\|C_{\psi}\|^2 + \|C_{\varphi}\|^2 \right) \|F\|_{H^2(\mathbb{D}^2)}^2,$$

which proves the boundedness of $E_{\Phi} : H^2(\mathbb{D}^2) \rightarrow H^2(\mathbb{D})$. $\square$

Theorem 3.4. *Let $\Phi = (\varphi, \psi) : \mathbb{D} \rightarrow \mathbb{D}^2$ where φ and ψ are analytic self-maps of the open unit disk satisfying $\|\varphi\psi\|_{\infty} < 1$. Assume further that the operators D_{φ} and D_{ψ} acting on $H^2(\mathbb{D})$ are bounded. Then $E_{\Phi} : H^2(\mathbb{D}^2) \rightarrow H^2(\mathbb{D})$ is bounded.*

Proof. By modifying the proof of Lemma 2.3 we can find a measurable partition $\{\Gamma_1, \Gamma_2\}$ of the unit circle $\mathbb{T}$ such that

$$|\varphi(\zeta)| < \sqrt{\sigma}, \quad \text{for almost every } \zeta \in \Gamma_1,$$

$$|\psi(\zeta)| < \sqrt{\sigma}, \quad \text{for almost every } \zeta \in \Gamma_2.$$

For $F(z, w) \in H^2(\mathbb{D}^2)$, we write

$$F(z, w) = \sum_{n=0}^{\infty} z^n F_n(w) = \sum_{n=0}^{\infty} w^n G_n(z).$$

Therefore,

$$\|F\|_{H^2(\mathbb{D}^2)}^2 = \sum_{n=0}^{\infty} \|F_n\|_{H^2(\mathbb{D})}^2 = \sum_{n=0}^{\infty} \|G_n\|_{H^2(\mathbb{D})}^2. \quad (14)$$

Let σ satisfy

$$\|\varphi\psi\|_{\infty} = \sup_{z \in \mathbb{D}} |\varphi(z)\psi(z)| < \sigma < 1,$$

and write

$$\|E_{\Phi}F\|_{H^2(\mathbb{D})}^2 = \int_{\Gamma_1} |E_{\Phi}F|^2 dm + \int_{\Gamma_2} |E_{\Phi}F|^2 dm. \quad (15)$$

Using (6), we obtain

$$\begin{aligned} \int_{\Gamma_1} |E_{\Phi}F|^2 dm &\leq 2 \int_{\Gamma_1} \left| \sum_{n=1}^{\infty} n\varphi(\zeta)^n (C_{\psi}F_n)(\zeta) \right|^2 dm(\zeta) \\ &\quad + 2 \int_{\Gamma_1} \left| \sum_{n=1}^{\infty} \varphi(\zeta)^n (D_{\psi}F_n)(\zeta) \right|^2 dm(\zeta). \end{aligned} \quad (16)$$

As for the first term on the right-hand side of (16), we use (14) and the boundedness of C_{ψ} to get the following upper bound:

$$\begin{aligned} 2 \int_{\Gamma_1} \left| \sum_{n=1}^{\infty} n\varphi(\zeta)^n (C_{\psi}F_n)(\zeta) \right|^2 dm(\zeta) &\leq 2 \int_{\Gamma_1} \sum_{n=1}^{\infty} n^2 |\varphi(\zeta)|^{2n} \sum_{n=1}^{\infty} |(C_{\psi}F_n)(\zeta)|^2 dm(\zeta) \\ &\leq 2 \sum_{n=1}^{\infty} n^2 \sigma^n \int_{\mathbb{T}} \sum_{n=1}^{\infty} |(C_{\psi}F_n)(\zeta)|^2 dm(\zeta) \\ &= 2 \frac{\sigma + \sigma^2}{(1 - \sigma)^3} \|C_{\psi}\|^2 \|F\|_{H^2(\mathbb{D}^2)}^2. \end{aligned}$$

An upper bound for the second term on the right-hand side of (16) is obtained in a similar manner:

$$\begin{aligned} 2 \int_{\Gamma_1} \left| \sum_{n=1}^{\infty} \varphi(\zeta)^n (D_{\psi}F_n)(\zeta) \right|^2 dm(\zeta) &\leq 2 \sum_{n=1}^{\infty} \sigma^n \sum_{n=1}^{\infty} \|D_{\psi}\|^2 \|F_n\|^2 \\ &= \frac{2\sigma}{1 - \sigma} \|D_{\psi}\|^2 \|F\|_{H^2(\mathbb{D}^2)}^2. \end{aligned}$$

Thus,

$$\int_{\Gamma_1} |E_\Phi F|^2 dm \leq 2 \left(\frac{\sigma + \sigma^2}{(1 - \sigma)^3} \|C_\psi\|^2 + \frac{\sigma}{1 - \sigma} \|D_\psi\|^2 \right) \|F\|_{H^2(\mathbb{D}^2)}^2.$$

Likewise, we can prove that

$$\int_{\Gamma_2} |E_\Phi F|^2 dm \leq 2 \left(\frac{\sigma + \sigma^2}{(1 - \sigma)^3} \|C_\varphi\|^2 + \frac{\sigma}{1 - \sigma} \|D_\varphi\|^2 \right) \|F\|_{H^2(\mathbb{D}^2)}^2,$$

and eventually,

$$\begin{aligned} \|E_\Phi F\|_{H^2(\mathbb{D})}^2 &\leq \frac{2\sigma(1 + \sigma)}{(1 - \sigma)^3} (\|C_\varphi\|^2 + \|C_\psi\|^2) \|F\|_{H^2(\mathbb{D}^2)}^2 \\ &\quad + \frac{2\sigma}{1 - \sigma} (\|D_\varphi\|^2 + \|D_\psi\|^2) \|F\|_{H^2(\mathbb{D}^2)}^2. \end{aligned}$$

Thus,

$$\|E_\Phi\|^2 \leq \frac{2\sigma(1 + \sigma)}{(1 - \sigma)^3} (\|C_\varphi\|^2 + \|C_\psi\|^2) + \frac{2\sigma}{1 - \sigma} (\|D_\varphi\|^2 + \|D_\psi\|^2),$$

which proves the boundedness of $E_\Phi : H^2(\mathbb{D}^2) \rightarrow H^2(\mathbb{D})$. $\square$

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Zahra Hedayatfar

PhD Candidate of Mathematics
Department of Pure Mathematics
Imam Khomeini International University
Qazvin 34149, Iran
E-mail: z.hedayatfar@edu.ikiu.ac.ir

Ali Abkar

Professor of Mathematics
Department of Pure Mathematics
Imam Khomeini International University
Qazvin 34149, Iran
E-mail: abkar@sci.ikiu.ac.ir