Cyclicity of the Shift Operator on Analytic Function Spaces

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Abstract. In this paper we characterize some sufficient conditions for a vector in the Hilbert spaces of analytic functions to be cyclic for the backward shift operator.

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1. Introduction

Let $H$ be a Hilbert space of complex-valued analytic functions on the open unit disc $\mathbb{D}$ such that point evaluations are bounded linear functionals on $H$. Then for every $w \in \mathbb{D}$ there exists a function $k_w$ in $H$ such that $f(w) = \langle f, k_w \rangle$ for all $f \in H$. Now if we define $K : \mathbb{D} \times \mathbb{D} \rightarrow \mathbb{C}$ by $K(z, w) = k_w(z)$, then $K$ is a positive definite function with the
reproducing property
\[
f(w) = (f(\cdot), K(\cdot, w))
\]
for every \( w \in \mathbb{D} \) and \( f \in H \). The function \( K \) is called the reproducing kernel for \( H \).

Recall that a function \( K : \mathbb{D} \times \mathbb{D} \rightarrow \mathbb{C} \) is positive definite (denoted \( K \geq 0 \)) provided
\[
\sum_{j,k=1}^{n} a_j \bar{a}_k K(w_j, w_k) \geq 0
\]
for any finite set of complex numbers \( a_1, \ldots, a_n \) and any finite subset \( w_1, \ldots, w_n \) of \( \mathbb{D} \). Conversely, if \( K : \mathbb{D} \times \mathbb{D} \rightarrow \mathbb{C} \) is positive definite then
\[
\left\{ \sum_{j=1}^{n} a_j K(\cdot, w_j) : a_1, \ldots, a_n \in \mathbb{C} \text{ and } w_1, \ldots, w_n \in \mathbb{D} \right\}
\]
has dense linear span in a Hilbert space \( H(K) \) of functions with
\[
\left\| \sum_{j=1}^{n} a_j K(\cdot, w_j) \right\|^2 = \sum_{j,k=0}^{n} a_j \bar{a}_k K(w_j, w_k)
\]
and
\[
f(w) = (f(\cdot), K(\cdot, w))
\]
for every \( w \) in \( \mathbb{D} \) and \( f \) in \( H(K) \). Thus evaluation at \( w \) is a bounded linear functional for each \( w \) in \( \mathbb{D} \). Note also that convergence in \( H(K) \) implies uniform convergence on compact subsets of \( \mathbb{D} \).

Now if \( K \) is a kernel on \( \mathbb{D} \times \mathbb{D} \) which is analytic in the first variable and consequently coanalytic in the second variable, then \( K(z, \bar{w}) \) is an
analytic function on $\mathbb{D} \times \mathbb{D}$ in the two variables $z$ and $w$. Hence $K(z, w)$ can be represented by the double power series

$$
\sum_{j,k=0}^{\infty} a_{jk} z^j \bar{w}^k.
$$

If $C$ denotes the matrix $[a_{jk}]$, then such a $K$ can be written more compactly in the form

$$
K(z, w) = \mathbf{Z}^*C\mathbf{W} = \langle \mathbf{C}\mathbf{W}, \mathbf{Z} \rangle_{l_+^2},
$$

where $\mathbf{Z}$ denotes the column vector whose transpose is $(1, z, z^2, \ldots)$. (Here $l_+^2$ denotes the usual space of all square sumable sequences.) It is well known that $K \gg 0$ if and only if $C > 0$. Henceforth for positive matrices $C, H(C)$ will denote the space $H(K)$ where $K = \mathbf{Z}^*C\mathbf{W}$. For more information about reproducing kernels the reader is referred to [2]. Some sources on spaces of analytic functions are [3;4;5;7;8;9;10;11].

**Theorem 1.1([1]).** If $C = B^*B$ for some bounded operator $B$ on $l_+^2$, then the operator $V$ from $(\ker B^*)^\perp$ into $H(C)$ defined by

$$
V(f)(z) = \langle B^* f, \mathbf{Z} \rangle_{l_+^2}
$$

is unitary.

**Corollary 1.2.** If $C = B^*B$ and $\{f_n\}$ is an orthonormal basis for $(\ker B^*)^\perp$, then $\{\langle B^* f_n, \mathbf{Z} \rangle_{l_+^2} \}$ is an orthonormal basis for $H(C)$.

We can construct a basis for $H(C)$ by using the Cholesky decompo-
sition of the nonnegative matrix $C$ into the product $U^*U$, where $U$ is upper triangular. For more details the reader is referred to [6].

Throughout this paper $H$ is a Hilbert space of analytic functions on $\mathbb{D}$ such that $1 \in H$, $zH \subset H$ and point evaluations are bounded for every $w \in \mathbb{D}$. If the set $\{1, z, z^2, \ldots\}$ is an orthogonal basis for $H$ and

$$f = \sum_{n=0}^{\infty} f_n z^n \in H,$$

then by boundedness of point evaluations, the power series expansion of $f$ can be written as

$$f(z) = \sum_{n=0}^{\infty} f_n z^n.$$

The backward shift operator on $H$ is denoted by $L$ that is defined by

$$L(\sum_{n=0}^{\infty} f_n z^n) = \sum_{n=0}^{\infty} f_{n+1} z^n.$$

We assume that $L$ is a bounded operator on $H$.

We say that a vector $f$ in a Hilbert space $H$ is a cyclic vector of a bounded operator $A$ on $H$ if

$$H = \overline{\text{span}\{A^n f : n = 0, 1, 2, \ldots\}}.$$ 

Here $\overline{\text{span}\{\cdot\}}$ is the closed linear span of the set $\{\cdot\}$.

2. Main Result

In the main theorem of this paper we give sufficient conditions for a vector $f$ in $H(K)$ to be cyclic for the backward shift operator on $H(K)$.
Theorem 2.1. Let $H$ have the reproducing kernel
\[ K(z, w) = \frac{1 - zw}{(1 - z)(1 - w)} \sum_{i=0}^{\infty} a_i (zw)^i, \]
where $\{a_i\}_i$ is a nondecreasing sequence of positive numbers. Suppose that for sufficiently large positive integer $N$,

(i) $J_m = \sup\{(a_{k+n} - a_{k+n-1})/[(a_{m+n} - a_{m+n-1})(a_{k+N} - a_{k+N-1})] : k \geq N + m, n \geq 1\} < \infty$ for any positive integer $m$ and

(ii) $\{(k + N + 1)(a_{k+N} - a_{k+N-1})/(a_k - a_{k-1})\}_k \in \ell^1$.

If $f$ is a vector in $H$ with infinitely many $f^{(n)}(0) \neq 0$, then $f$ is a cyclic vector of $L$.

Proof. We have
\[ K(z, w) = \left(\frac{z}{1 - z} + \frac{1}{1 - w}\right) \sum_{i=0}^{\infty} a_i (zw)^i \]
\[ = \sum_{i=0}^{\infty} \sum_{m=1}^{\infty} a_i z^i w^{i+m} + \sum_{i=0}^{\infty} \sum_{n=0}^{\infty} a_i z^i w^{i+n}. \]

If we denote the matrix of $K(z, w)$ by $A = (a_{ij})_{i,j=0}^{\infty}$, then
\[ a_{i,i+m} = a_i, \quad i = 0, 1, 2, \ldots, \quad m = 1, 2, \ldots, \]
\[ a_{i+n,i} = a_i, \quad i = 0, 1, 2, \ldots, \quad n = 0, 1, 2, \ldots. \]

Hence $a_{ij} = a_i$ for $j \geq i$ and $A$ is symmetric. Now by Corollary 1.2 we can see that the set $\{e_n\}_{n=0}^{\infty}$ is an orthogonal basis for $H$ where $e_n = z^n w_0$ and
\[ w_0 = \sum_{i=0}^{\infty} z^i. \]
Let
\[ f = \sum_{n=0}^{\infty} f_n e_n \]
and put
\[ M = \text{span}\{ L^n f : n = 0, 1, 2, \ldots \}. \]
If \( M \neq H \), then there is a nonzero
\[ g = \sum_{n=0}^{\infty} g_n e_n \]
in \( H \) such that \( \langle L^n f, g \rangle = 0 \) for all \( n = 0, 1, 2, \ldots \). Put
\[ m = \min\{ k : g_k \neq 0 \}. \]
Since
\[ L^n f = \sum_{k=0}^{\infty} f_{n+k} e_k, \]
we get
\[ 0 = \langle L^n f, g \rangle = \sum_{k=m}^{\infty} f_{n+k} \bar{g}_k \| e_k \|^2. \]
Therefore
\[ f_{m+n} \bar{g}_m \| e_m \|^2 = -\sum_{k=m+1}^{\infty} f_{n+k} \bar{g}_k \| e_k \|^2. \]
Now choosing \( N \) as in conditions of the theorem, we can choose \( g \) such that \( g_k = 0 \) for \( m < k < m + 2N \). Hence we obtain
\[ |f_{m+n}||e_{m+n}| \leq \frac{1}{\|e_m\|^2 \|g_m\|} \sum_{k \geq m+2N} \langle (|f_{n+k}||e_{k+n}|)(|\bar{g}_k||e_k|) \rangle \frac{\|e_k\|\|e_{m+n}\|}{\|e_{k+n}\|}. \]
Since $\|e_n\|^{-2} = a_n - a_{n-1}$, by condition (i) we have

$$\frac{\|e_{m+n}\|\|e_k\|}{\|e_{k+n-N}\|} \leq \left( \frac{a_{k+n-N} - a_{k+n-N-1}}{(a_k - a_{k-1})(a_{m+n} - a_{m+n-1})} \right)^{\frac{1}{2}} \leq J_m^{\frac{1}{2}}$$

for $k \geq m + 2N$. Therefore

$$|f_{m+n}\|e_{m+n}\| \leq \frac{J_m^{\frac{1}{2}}\|f\|}{\|g_m\||e_m|^2} \sum_{k \geq m+2N} \left( \frac{\|e_{k+n-N}\|}{\|e_{k+n}\|} \right)$$

By the Hölder inequality we have

$$|f_{m+n}\|e_{m+n}\| \leq \frac{J_m^{\frac{1}{2}}\|f\|\|g\|}{\|g_m\||e_m|^2} \left( \sum_{k \geq m+2N} \left( \frac{\|e_{k+n-N}\|}{\|e_{k+n}\|} \right)^2 \right)^{\frac{1}{2}}.$$

Now if

$$h = L^i f = \sum_{n=0}^{\infty} h_n e_n,$$

then

$$h_n = (L^i f)_n = f_{n+i}$$

and $\langle L^h, g \rangle = 0$ for all $n = 0, 1, 2, \ldots$. By the same manner as we used in the above calculations, by replacing $f$ by $h$, we obtain

$$|h_{m+n}\|e_{m+n}\| \leq c_{m+n}\|h\|,$$

where

$$c_{m+n} = \frac{J_m^{\frac{1}{2}}\|g\|}{\|g_m\||e_m|^2} \left( \sum_{k \geq m+2N} \left( \frac{\|e_{k+n-N}\|}{\|e_{k+n}\|} \right)^2 \right)^{\frac{1}{2}}.$$

So for any vector

$$h = \sum_{n=0}^{\infty} h_n e_n$$
in $M$, we have

$$|h_{m+n}||e_{m+n}| \leq c_{m+n}||h||,$$

where the constants $c_{m+n}$ do not depend upon the choice of $h$ in $M$. If

$$\alpha_i = \begin{cases} c_i & i \geq m \\ 1 & i < m \end{cases},$$

then

$$|h_i||e_i| \leq \alpha_i||h|| \quad (\ast)$$

for all

$$h = \sum_{i=0}^{\infty} h_i e_i$$

in $M$. Now we prove that $\{\alpha_i\} \in \ell^2$. Note that for $i \geq m$,

$$\alpha_i = \frac{J_m^{1/2} ||g||}{||g_m|| ||e_m||^2} \left( \sum_{k \geq 2N} \frac{||e_{k+i-N}||^2}{||e_{k+i}||^2} \right)^{1/2}.$$

Put

$$\gamma_i = \left( \sum_{k \geq 2N} \frac{||e_{k+i-N}||}{||e_{k+i}||} \right)^{1/2}, \quad i \geq m.$$

It is sufficient to show that

$$\sum_{i \geq m} \gamma_i^2 < \infty.$$
We have
\[
\sum_{i \geq m} \gamma_i^2 = \sum_{i \geq m} \sum_{k \geq 2N} \left( \frac{\|e_{k+i-N}\|}{\|e_{k+i}\|} \right)^2
\]
\[
= \sum_{k=0}^{\infty} (k + 1) \left( \frac{\|e_{m+k+N}\|}{\|e_{2N+m+k}\|} \right)^2
\]
\[
= \sum_{k \geq 2N+m} (k + 1 - 2N - m) \left( \frac{\|e_{k-N}\|}{\|e_k\|} \right)^2
\]
\[
\lesssim \sum_{k \geq 2N} (k + 1) \left( \frac{\|e_{k-N}\|}{\|e_k\|} \right)^2.
\]
But \( \|e_n\|^{-2} = a_n - a_{n-1} \), thus
\[
\sum_{i \geq m} \gamma_i^2 \leq \sum_{k \geq N} (k + 1) \frac{a_k - a_{k-1}}{a_{k-N} - a_{k-N-1}}
\]
\[
= \sum_{k=0}^{\infty} (k + N + 1) \frac{a_{k+N} - a_{k+N-1}}{a_k - a_{k-1}}
\]
that is finite by condition (ii). Now, by inequality (*), we can see that \( M \) is finite dimensional, which contradicts our assumption that \( f^{(n)}(0) \neq 0 \) for infinitely many \( n \). This implies that \( M = H \) and so \( f \) is a cyclic vector of \( L \). This completes the proof. \( \square \)

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