A General Norm on Extension of a Hilbert’s Type Linear Operator

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Abstract. The main purpose of this paper is to study a general norm on extension of a Hilbert’s type linear operator in the continuous and discrete form. In addition to expressing the norm of a Hilbert’s type linear operator $T : L^2(0, \infty) \rightarrow L^2(0, \infty)$, a more general case with $\lambda > 0$, for the continuous form has been studied. By putting $\lambda = 1$ a norm of extension of Hilbert’s integral linear operator is obtained. Similar results have been expressed for series when $0 < \lambda \leq 2$.

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1. Introduction and Preliminaries

If $f(t), g(t) \geq 0$, $0 < \int_0^\infty f^2(t)dt < \infty$, and $0 < \int_0^\infty g^2(t)dt < \infty$, then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}dxdy < \pi \left\{ \int_0^\infty f^2(x)dx \right\}^{\frac{1}{2}} \left\{ \int_0^\infty g^2(x)dx \right\}^{\frac{1}{2}}, \quad (1)$$

where the constant factor $\pi$ is the best possible. Inequality (1) is named Hardy-Hilbert’s integral inequality (see [1]). Under the same condition

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of (1), we have the Hardy-Hilbert’s type inequality (see [1], Theorem 319, Theorem 341). similar to (1) that
is
\[
\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\max\{x,y\}} \, dx \, dy < 4 \left\{ \int_0^\infty f^2(x) \, dx \right\}^{\frac{1}{2}} \left\{ \int_0^\infty g^2(x) \, dx \right\}^{\frac{1}{2}}, \tag{2}
\]
where the constant factor 4 is also the best possible. The corresponding inequalities for series are:
\[
\sum_{m=1}^\infty \sum_{n=1}^\infty \frac{a_m b_n}{m+n} < \pi \left\{ \sum_{n=1}^\infty a_n^2 \right\}^{\frac{1}{2}} \left\{ \sum_{n=1}^\infty b_n^2 \right\}^{\frac{1}{2}}, \tag{3}
\]
where \(\{a_n\}\) and \(\{b_n\}\) are sequences such that \(0 < \sum_{n=1}^\infty a_n^2 < \infty\), \(0 < \sum_{n=1}^\infty b_n^2 < \infty\), and the constant factor \(\pi\) and 4 are both the best possible.

Let \(H\) be a real separable Hilbert space, and \(T : H \to H\) be a bounded self-adjoint semi-positive definite operator, then (see [8]),
\[
(a, Tb)^2 \leq \frac{\|T\|^2}{2} \left( \|a\|^2 \|b\|^2 + (a, b)^2 \right), \tag{4}
\]
where \(a, b \in H\) and \(\|a\| = \sqrt{(a,a)}\) is the norm of \(a\).
Set \(H = L^2(0, \infty) = \{ f(x) : \int_0^\infty f^2(x) \, dx < \infty \}\) and define \(T : L^2(0, \infty) \to L^2(0, \infty)\) as the following:
\[
(Tf)(y) = \int_0^\infty \frac{1}{x+y} f(x) \, dx, \tag{5}
\]
where \(y \in (0, \infty)\). It is easy to see \(T\) is a bounded operator (see [7]). By (4), one has the sharper form of Hilbert’s inequality as (see [8]),
\[
\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} \, dx \, dy \leq \frac{\pi}{\sqrt{2}} \left\{ \int_0^\infty f^2(x) \, dx \int_0^\infty g^2(x) \, dx + \left( \int_0^\infty f(x)g(x) \, dx \right)^2 \right\}^{\frac{1}{2}}, \tag{6}
\]
In 2006 and 2007, Yang studied the Hilbert’s inequalities by the norm of some Hilbert’s type linear operators in the continuous and discrete forms (see [5,6]), and in the end of 2007 Li and his colleagues studied the Hilbert’s type linear operators with the kernel \[ \frac{(xy)^{\frac{\lambda - 1}{2}}}{[\text{Amin}\{x, y\} + \text{Bmax}\{x, y\}]^\lambda} \],(see [3]).

The main purpose of this article is to study the norm of Hilbert’s type linear operator with the kernel \[ \frac{(xy)^{\frac{\lambda - 1}{2}}}{[\text{Amin}\{x, y\} + \text{Bmax}\{x, y\}]^\lambda} \] such that \( \lambda > 0 \) for the continuous form, and \( 0 < \lambda \leq 2 \) for the discrete form.

2. Main Results and Applications

Lemma 2.1. Consider \( x, y > 0, A \geq 0, B > 0, \lambda > 0 \), and \( 0 \leq \varepsilon < 1 \), with the weight functions:

\[
\omega_{\lambda}(\varepsilon, x) := \int_{0}^{\infty} \frac{(xy)^{\frac{\lambda - 1}{2}}}{[\text{Amin}\{x, y\} + \text{Bmax}\{x, y\}]^\lambda} \left( \frac{x}{y} \right)^{\frac{1+\varepsilon}{2}} \, dy, \quad (7)
\]

\[
\omega_{\lambda}(\varepsilon, y) := \int_{0}^{\infty} \frac{(xy)^{\frac{\lambda - 1}{2}}}{[\text{Amin}\{x, y\} + \text{Bmax}\{x, y\}]^\lambda} \left( \frac{y}{x} \right)^{\frac{1+\varepsilon}{2}} \, dx,
\]

where \( \omega_{\lambda}(0, x) = \omega_{\lambda}(x) \), that is, \( \omega_{\lambda}(\varepsilon, x) = \omega_{\lambda}(x) + o(1) \) \( (\varepsilon \to 0^+) \).

Then \( 0 < \omega_{\lambda}(x) = \omega_{\lambda}(y) < \infty \) is a constant.

Proof. For fixed \( x \), letting \( v = \frac{B}{A}(\frac{y}{x}), u = \frac{A}{B}(\frac{x}{y}) \), we get

\[
\omega_{\lambda}(\varepsilon, x) = \int_{0}^{\infty} \frac{(xy)^{\frac{\lambda - 1}{2}}}{[\text{Amin}\{x, y\} + \text{Bmax}\{x, y\}]^\lambda} \left( \frac{x}{y} \right)^{\frac{1+\varepsilon}{2}} \, dy
\]

\[
= \int_{0}^{x} \frac{(xy)^{\frac{\lambda - 1}{2}}}{(Ay + Bx)^\lambda} \left( \frac{x}{y} \right)^{\frac{1+\varepsilon}{2}} \, dy + \int_{x}^{\infty} \frac{(xy)^{\frac{\lambda - 1}{2}}}{(Ax + By)^\lambda} \left( \frac{x}{y} \right)^{\frac{1+\varepsilon}{2}} \, dy
\]

\[
= B^{\frac{(\varepsilon + \lambda)}{2}} \int_{0}^{A} u^{\frac{-2+\lambda-\varepsilon}{2}} \frac{1}{(1 + u)^\lambda} \, du + A^{\frac{(\varepsilon + \lambda)}{2}} \int_{0}^{\infty} \frac{v^{\frac{-2+\lambda-\varepsilon}{2}}}{(1 + v)^\lambda} \, dv.
\]
\[ \leq \frac{B^{-(\frac{\epsilon+\lambda}{2})}}{A^{\frac{\lambda}{2}}} \int_0^\infty \frac{u^{\left(\frac{\epsilon+\lambda}{2}\right)}}{(1+u)^{\lambda}} \, du + \frac{A^{-(\frac{\epsilon+\lambda}{2})}}{B^{\frac{\lambda}{2}}} \int_0^\infty \frac{v^{\left(\frac{\epsilon+\lambda}{2}\right)}}{(1+v)^{\lambda}} \, dv. \]

By Beta function (see [4]), one has
\[ 0 < \omega_\lambda(x) \leq \frac{2}{(AB)^{\frac{\lambda}{2}}} \beta\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) < \infty. \]

Also in the same way:
\[ \omega_\lambda(y) = \omega_\lambda(x) \leq \frac{2}{(AB)^{\frac{\lambda}{2}}} \beta\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) < \infty. \]

Hence \( 0 < \omega_\lambda(x) = \omega_\lambda(y) < \infty \) is a constant. □

**Lemma 2.2.** Consider, \( m, n \in \mathbb{N} \), \( A \geq 0 \), \( B > 0 \), \( 0 < \lambda \leq 2 \), \( 0 \leq \epsilon < 1 \), and the weight function \( w_\lambda(\epsilon, n) \) for discrete forms as:
\[ w_\lambda(\epsilon, n) := \sum_{m=1}^{\infty} \frac{(mn)^{\frac{\lambda-1}{2}}}{A \min\{m, n\} + B \max\{m, n\}} \left( \frac{n}{m} \right)^{\frac{1+\epsilon}{2}}. \] (9)

Then
\[ w_\lambda(n) < \omega_\lambda(n). \] (10)

**Proof.** It is obvious. □

**Note 2.3.** If \( \lambda = 1 \) then \( \omega_\lambda(x) = D(A, B) \) given in ([3] lemma 1.2.).

### 3. A General Norm on Extension of a Hilbert’s Type Linear Operator in the Continuous Forms

**Theorem 3.1.** Consider, \( A \geq 0 \), \( B > 0 \), \( \lambda > 0 \), \( T : L^2(0, \infty) \to L^2(0, \infty) \), and define:
\[ (Tf)(y) := \int_0^\infty \frac{(xy)^{\frac{\lambda-1}{2}}}{[A \min\{x, y\} + B \max\{x, y\}]} f(x) \, dx \quad (y \in (0, \infty)), \] (11)
Then, \( \|T\| = \omega(x) \) is the general norm, and for any \( f(x), g(x) \geq 0 \) such that \( f, g \in L^2(0, \infty) \), one has \( (Tf, g) < \omega(x)\|f\|_2\|g\|_2 \), that is

\[
\int_0^\infty \int_0^\infty \frac{(xy)^{\lambda - \frac{1}{2}}}{[A \min\{x, y\} + B \max\{x, y\}]^\lambda} f(x)g(y)dxdy < \omega(x) \left\{ \int_0^\infty f^2(x)dx \right\}^{\frac{1}{2}} \left\{ \int_0^\infty g^2(x)dx \right\}^{\frac{1}{2}},
\]

where the constant factor \( \omega(x) \) is the best possible.

**Proof.** For \( A \geq 0, B > 0 \), applying Holder’s inequality, we obtain

\[
(Tf, g) = \left( \int_0^\infty \frac{(xy)^{\lambda - \frac{1}{2}}f(x)}{[A \min\{x, y\} + B \max\{x, y\}]^\lambda} dx, g(y) \right)
\]

\[
= \int_0^\infty \left[ \int_0^\infty \frac{(xy)^{\lambda - \frac{1}{2}}f(x)}{[A \min\{x, y\} + B \max\{x, y\}]^\lambda} dx \right] g(y)dy
\]

\[
\leq \left\{ \int_0^\infty \left[ \int_0^\infty \frac{(xy)^{\lambda - \frac{1}{2}}}{[A \min\{x, y\} + B \max\{x, y\}]^\lambda} \left( \frac{x}{y} \right)^{\frac{1}{2}} \right] f^2(x)dx \right\}^{\frac{1}{2}}
\]

\[
\times \left\{ \int_0^\infty \left[ \int_0^\infty \frac{(xy)^{\lambda - \frac{1}{2}}}{[A \min\{x, y\} + B \max\{x, y\}]^\lambda} \left( \frac{y}{x} \right)^{\frac{1}{2}} \right] g^2(y)dy \right\}^{\frac{1}{2}}
\]

\[
= \left\{ \int_0^\infty \omega(x)f^2(x)dx \right\}^{\frac{1}{2}} \left\{ \int_0^\infty \omega(y)g^2(y)dy \right\}^{\frac{1}{2}}
\]

\[
= \omega(x)\|f\|_2\|g\|_2
\]

and hence \( \|T\| \leq \omega(x) \). If (13) takes the form of the equality, then there exist constants \( \alpha \) and \( \beta \), not both zero such that (see [2])

\[
\alpha f^2(x) \left( \frac{x}{y} \right)^{\frac{1}{2}} = \beta g^2(y) \left( \frac{y}{x} \right)^{\frac{1}{2}}
\]

Therefore, we have
\[ \alpha f^2(x)x = \beta g^2(y)y \quad \text{a.e. on } (0, \infty) \times (0, \infty). \]

Hence there exists a constant \( c \), such that
\[ \alpha f^2(x)x = \beta g^2(y)y = c \quad \text{a.e. on } (0, \infty) \times (0, \infty). \]

Without losing the generality, suppose \( \alpha \neq 0 \), then we obtain \( f^2(x) = \frac{c}{\alpha x} \), a.e. on \((0, \infty)\), which contradicts the fact that \( 0 < \int_0^\infty f^2(x)dx < \infty \). Hence (13) takes the form of a strict inequality, and we obtain (3.2).

For any \( a, b \geq 1 \), \( \varepsilon > 0 \) sufficiently small, set
\[ f_\varepsilon(x) = \begin{cases} a \varepsilon x^{-(1+\varepsilon)} & \text{if } x \in [a, \infty) \ , \\ 0 & \text{if } x \in (0, a) \end{cases} \]
and
\[ g_\varepsilon(y) = \begin{cases} b \varepsilon y^{-(1+\varepsilon)} & \text{if } y \in [b, \infty) \ , \\ 0 & \text{if } y \in (0, b) \end{cases} \]
Assume that the constant factor \( \omega_\lambda(x) \) in (12) is not the best possible, then there exists a positive real number \( k \) with \( k < \omega_\lambda(x) \) such that (12) is valid by changing \( \omega_\lambda(x) \) to \( k \).

On one hand,
\[
\int_0^\infty \int_0^\infty \frac{(xy)^{1-\frac{1}{\lambda}}}{[A \min\{x, y\} + B \max\{x, y\}]^\lambda} f_\varepsilon(x)g_\varepsilon(y)dxdy < k \left\{ \int_0^\infty f_\varepsilon^2(x)dx \right\}^{\frac{1}{2}} \left( \int_0^\infty g_\varepsilon^2(x)dx \right)^{\frac{1}{2}} = \frac{k}{\varepsilon}. \quad (15)
\]

On the other hand, setting \( t = \frac{y}{x} \), we have
\[
\int_0^\infty \int_0^\infty \frac{(xy)^{1-\frac{1}{\lambda}}}{[A \min\{x, y\} + B \max\{x, y\}]^\lambda} f_\varepsilon(x)g_\varepsilon(y)dxdy
= (ab)^{\frac{1}{2}} \int_a^\infty x^{-(1+\varepsilon)} \int_0^\infty \frac{t^{1-\frac{1}{\lambda} - \frac{1+\varepsilon}{2}}}{[A \min\{1, t\} + B \max\{1, t\}]^\lambda} dt dx
= (ab)^{\frac{1}{2}} \int_a^\infty x^{-(1+\varepsilon)} \int_0^\infty \frac{t^{1-\frac{1}{\lambda} - \frac{1+\varepsilon}{2}}}{[A \min\{1, t\} + B \max\{1, t\}]^\lambda} dt dx
- (ab)^{\frac{1}{2}} \int_a^\infty x^{-(1+\varepsilon)} \int_0^{\frac{b}{x}} \frac{t^{1-\frac{1}{\lambda} - \frac{1+\varepsilon}{2}}}{[A \min\{1, t\} + B \max\{1, t\}]^\lambda} dt dx.
\]

For \( x \geq b \) and \( 0 < \varepsilon < 1 \), we get
\[
\int_0^{\frac{b}{x}} \frac{t^{1-\frac{1}{\lambda} - \frac{1+\varepsilon}{2}}}{[A \min\{1, t\} + B \max\{1, t\}]^\lambda} dt = \int_0^{\frac{b}{x}} \frac{t^{(1-\frac{1}{\lambda} - \frac{1+\varepsilon}{2})}}{[At + B]^\lambda} dt, \quad (16)
\]
Thus
\[
0 < (ab)^\frac{\epsilon}{2} \int_a^\infty \int_0^h x^{-(1+\epsilon)} \int_0^b t^{\frac{\lambda-1}{2}-\frac{1+\epsilon}{2}} \frac{t^{\lambda-1} - \frac{1+\epsilon}{2}}{[A \min\{1, t\} + B \max\{1, t\}]^\lambda} dt dx
\]
\[
< \left( \frac{4}{B^\lambda} \right) \left( \frac{a^{2-\frac{\epsilon}{2}} b^{1+\frac{\epsilon}{2}}}{(\lambda-1)^2} \right) < \infty
\]  
(17)

Note that
\[
(ab)^\frac{\epsilon}{2} \int_a^\infty x^{-(1+\epsilon)} \int_0^h t^{\frac{\lambda-1}{2}-\frac{1+\epsilon}{2}} \frac{t^{\lambda-1} - \frac{1+\epsilon}{2}}{[A \min\{1, t\} + B \max\{1, t\}]^\lambda} dt dx = O(1),
\]  
(18)

So we have
\[
\int_0^\infty \int_0^\infty (xy)^{\frac{\lambda-1}{2}} \frac{f(x)g(y)dx dy}{[A \min\{x, y\} + B \max\{x, y\}]^\lambda}
\]
\[
= \frac{a^{\frac{\epsilon}{2}-\frac{1}{2}} b^{\frac{\epsilon}{2}}}{\epsilon} [\omega(x) + o(1)] - O(1)
\]
\[
= \frac{a^{\frac{\epsilon}{2}} b^{\frac{\epsilon}{2}}}{\epsilon} [\omega(x) + o(1)].
\]  
(19)

Now from (15) and (19) we get \(\frac{a^{\frac{\epsilon}{2}} b^{\frac{\epsilon}{2}}}{\epsilon} [\omega(x) + o(1)] < \frac{k}{\epsilon}\), that is, \(\omega(x) < k\) when \(\epsilon\) is sufficiently small and \(a, b \geq 1\), which contradicts the hypothesis. Hence the constant factor \(\omega(x)\) in (12) is the best possible and \(\|T\|_2 = \omega(x)\). This completes the proof. □

**Theorem 3.2.** If \(A \geq 0, B > 0, \lambda > 0\), and \(0 < \int_0^\infty f^2(x)dx < \infty\), then
\[
\int_0^\infty \left[ \int_0^\infty \frac{(xy)^{\frac{\lambda-1}{2}} f(x)}{[A \min\{x, y\} + B \max\{x, y\}]^\lambda} \right]^2 dy < \omega^2(x) \int_0^\infty f^2(x)dx,
\]  
(20)
where the constant factor $\omega_\lambda^2(x)$ is the best possible. Inequality (20) is equivalent to (12).

**Proof.** Let

$$g(y) = \int_0^\infty \frac{(xy)^{\frac{\lambda-1}{2}} f(x)}{[A \min\{x, y\} + B \max\{x, y\}]^\lambda} \, dx.$$  

Then, by (12), we get

$$0 < \int_0^\infty g^2(y) \, dy = \int_0^\infty \left[ \int_0^\infty \frac{(xy)^{\frac{\lambda-1}{2}} f(x)}{[A \min\{x, y\} + B \max\{x, y\}]^\lambda} \, dx \right]^2 \, dy$$

$$= \int_0^\infty \int_0^\infty \frac{(xy)^{\frac{\lambda-1}{2}}}{[A \min\{x, y\} + B \max\{x, y\}]^\lambda} f(x) g(y) \, dxdy$$

$$\leq \omega_\lambda(x) \left\{ \int_0^\infty f^2(x) \, dx \right\}^{\frac{1}{2}} \left\{ \int_0^\infty g^2(y) \, dy \right\}^{\frac{1}{2}}, \quad (21)$$

Hence, we obtain

$$0 < \int_0^\infty g^2(y) \, dy = \omega_\lambda(x) \left\{ \int_0^\infty f^2(x) \, dx \right\} < \infty. \quad (22)$$

By (12), both (21) and (22) take the form of a strict inequality, so we have (20). On the other hand, suppose that (20) is valid. By Holder’s inequality, we find

$$\int_0^\infty \int_0^\infty \frac{(xy)^{\frac{\lambda-1}{2}} f(x) g(y)}{[A \min\{x, y\} + B \max\{x, y\}]^\lambda} \, dxdy$$

$$= \int_0^\infty \left[ \int_0^\infty \frac{(xy)^{\frac{\lambda-1}{2}} f(x)}{[A \min\{x, y\} + B \max\{x, y\}]^\lambda} \, dx \right] g(y) \, dy$$

$$\leq \left\{ \int_0^\infty \left[ \int_0^\infty \frac{(xy)^{\frac{\lambda-1}{2}} f(x)}{[A \min\{x, y\} + B \max\{x, y\}]^\lambda} \, dx \right]^2 \, dy \right\}^{\frac{1}{2}} \left\{ \int_0^\infty g^2(y) \, dy \right\}^{\frac{1}{2}}.$$
By (20), we have (12). Thus (12) and (20) are equivalent. If the constant \( \omega_\lambda^2(x) \) in (20) is not the best possible, then the constant \( \omega_\lambda(x) \) in (12) is not the best possible. This completes the proof. □

**Note 3.3.** If \( A = B = 1 \) and \( \lambda = 1 \) then by Theorem 3.2., one has
\[
\int_0^\infty \left[ \int_0^\infty \frac{f(x)}{x+y} \, dx \right]^2 \, dy < \pi^2 \int_0^\infty f^2(x) \, dx. \tag{23}
\]
If \( A = 0, B = 1 \) and \( \lambda = 1 \), then one has
\[
\int_0^\infty \left[ \int_0^\infty \frac{f(x)}{\max\{x,y\}} \, dx \right]^2 \, dy < 16 \int_0^\infty f^2(x) \, dx,
\]
where the constant factors \( \pi^2 \) and 16 are both the best possible. Inequality (23) is Hilbert’s inequality in continuous form.

**4. The Corresponding Theorem for Series**

**Theorem 4.1.** Suppose that, \( a_n, b_n \geq 0, \ A \geq 0, \ B > 0, \ 0 < \lambda \leq 2 \)
and \( 0 < \sum_{n=1}^{\infty} a_n^2 < \infty, \ 0 < \sum_{n=1}^{\infty} b_n^2 < \infty \), then
\[
\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{(mn)^{\lambda-1/2}}{[A \min\{m,n\} + B \max\{m,n\}]^\lambda} a_m b_n < \omega_{\lambda}(n) \left( \sum_{n=1}^{\infty} a_n^2 \right)^{1/2} \left( \sum_{n=1}^{\infty} b_n^2 \right)^{1/2}, \tag{24}
\]
\[
\sum_{n=1}^{\infty} \left[ \sum_{m=1}^{\infty} \frac{(mn)^{\lambda-1/2}}{[A \min\{m,n\} + B \max\{m,n\}]^\lambda} a_m \right]^2 < \omega_{\lambda}(n) \sum_{n=1}^{\infty} a_n^2, \tag{25}
\]
where the constant factors \( \omega_{\lambda}(n) \) and \( \omega_{\lambda}^2(n) \) are both the best possible and inequality (24) is equivalent to (25).

**Proof.** Using a method similar to Theorem 3.1., and applying Holder’s inequality, we obtain
\[
\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{(mn)^{\lambda-1/2}}{[A \min\{m,n\} + B \max\{m,n\}]^\lambda} a_m b_n < \left\{ \sum_{n=1}^{\infty} w_{\lambda}(n) a_n^2 \right\}^{1/2} \left\{ \sum_{n=1}^{\infty} w_{\lambda}(n) b_n^2 \right\}^{1/2},
\]
By (25), we obtain (24).

For any \(a, b \geq 1, \varepsilon > 0\) sufficiently small, setting \(\tilde{a}_m = \left\{a\varepsilon^{\frac{1}{2}} m^{-(1+\varepsilon)}\right\}_{m=a}^\infty\),
\[
\tilde{b}_n = \left\{b\varepsilon^{\frac{1}{2}} n^{-(1+\varepsilon)}\right\}_{n=b}^\infty,
\]
then
\[
\propto \sum_{m=1}^\infty \sum_{n=1}^\infty \frac{(mn)^{\frac{\lambda-1}{2}}}{\left[A \min\{m, n\} + B \max\{m, n\}\right]^\lambda} \tilde{a}_m \tilde{b}_n > \int_{1}^{\infty} \int_{1}^{\infty} \frac{(xy)^{\frac{\lambda-1}{2}}}{\left[A \min\{x, y\} + B \max\{x, y\}\right]^\lambda} f(x)g(y) \, dx \, dy,
\]
where \((26)\) and
\[
\left\{\sum_{n=1}^\infty \tilde{a}_n^2\right\}^{\frac{1}{2}} \left\{\sum_{n=1}^\infty \tilde{b}_n^2\right\}^{\frac{1}{2}} = \sum_{n=a}^\infty \frac{a\varepsilon}{n^{1+\varepsilon}} < 1 + \int_{a}^{\infty} \frac{a\varepsilon}{t^{1+\varepsilon}} \, dt = 1 + \frac{1}{\varepsilon},
\]
(27)
If the constant factor \(\omega_\lambda(n)\) in (24) is not the best possible, then applying the result of Theorem 3.1., we have a contradiction. Let
\[
b_n = \sum_{m=1}^\infty \frac{(mn)^{\frac{\lambda-1}{2}}}{\left[A \min\{m, n\} + B \max\{m, n\}\right]^\lambda} a_m.
\]
We can obtain the following relation:
\[
\sum_{n=1}^\infty \left[\sum_{m=1}^\infty \frac{(mn)^{\frac{\lambda-1}{2}} a_m}{\left[A \min\{m, n\} + B \max\{m, n\}\right]^\lambda}\right]^2 = \sum_{n=1}^\infty b_n^2
\]
\[
= \sum_{n=1}^\infty \sum_{m=1}^\infty \frac{(mn)^{\frac{\lambda-1}{2}} a_m b_n}{\left[A \min\{m, n\} + B \max\{m, n\}\right]^\lambda}.
\]
Applying (24) and a method similar to Theorem 3.2., we conclude that (25), and (25) are equivalent to (24) with the best constant. \(\square\)

**Note 4.2.** If \(A = B = 1\) and \(\lambda = 1\) then by Theorem 4.1., one has Hilbert’s inequality in co-diskette form, as:
\[
\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{a_m b_n}{m + n} < \pi \left\{\sum_{n=1}^\infty a_n^2\right\}^{\frac{1}{2}} \left\{\sum_{n=1}^\infty b_n^2\right\}^{\frac{1}{2}}.
\]
If $A = 0$, $B = 1$ and $\lambda = 1$, then one has

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{\max\{m, n\}} < 4 \left\{ \sum_{n=1}^{\infty} a_n^2 \right\}^{\frac{1}{2}} \left\{ \sum_{n=1}^{\infty} b_n^2 \right\}^{\frac{1}{2}},$$

where the constant factors $\pi$ and 4 are both the best possible.

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