Fourier-Dunkl Dini Lipschitz Functions in the Space $L_{\alpha,n}^p$

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Abstract. In this paper, we consider the generalized Fourier-Dunkl transform associated with the Dunkl operator on $\mathbb{R}$ and we give condition of quite different kind for a function to have a transform belonging to certain $L_p$-classes.

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1. Introduction

Theorems 5.1 and 5.2 in Younis [5] characterized the set of functions in $L^2(\mathbb{R})$ satisfying the Cauchy Lipschitz condition by means of an asymptotic estimate growth of the norm of their Fourier transforms, namely we have the following theorem.

**Theorem 1.1.** ([5]) Let $f \in L^2(\mathbb{R})$. Then the following are equivalents

(a) $\|f(x + h) - f(x)\| = o\left((\log \frac{1}{h})^{-1}\right)$, as $h \to 0$,

(b) $\int_{|\lambda| \geq r} |\hat{f}(\lambda)|^2 d\lambda = o\left((\log r)^{-1}\right)$, as $r \to \infty$,
where \( \hat{f} \) stands for the Fourier transform of \( f \).

**Theorem 1.2.** ([5]) Let \( f \in L^2(\mathbb{R}) \). Then the following are equivalents

(a) \[ \| f(x+h) - f(x) \| = O \left( \frac{h^\delta}{(\log \frac{1}{h})^\gamma} \right), \quad \text{as} \quad h \to 0, 0 < \delta < 1, \gamma \geq 0 \]

(b) \[ \int_{|\lambda| \geq r} |\hat{f}(\lambda)|^2 d\lambda = O \left( \frac{r^{-2\delta}}{(\log r)^{2\gamma}} \right), \quad \text{as} \quad r \to \infty, \]

where \( \hat{f} \) stands for the Fourier transform of \( f \).

In this paper, we consider a first-order singular differential-difference operator \( \Lambda \) on \( \mathbb{R} \) which generalizes the Dunkl operator \( \Lambda_\alpha \). We prove an analog of Theorems 1.1 and 1.2 in the generalized Fourier-Dunkl transform associated to \( \Lambda \) in \( L^p_{\alpha,n} := L^p(\mathbb{R}, |x|^{2\alpha+2n(2-p)+1} dx) \). For this purpose, we use a generalized translation operator.

In this section, we develop some results from harmonic analysis related to the differential-difference operator \( \Lambda \). Further details can be found in [1] and [6]. In all what follows assume where \( \alpha > -1/2 \) and \( n \) a non-negative integer.

Consider the first-order singular differential-difference operator \( \Lambda \) defined on \( \mathbb{R} \) by

\[
\Lambda f(x) = f'(x) + \left( \alpha + \frac{1}{2} \right) \frac{f(x) - f(-x)}{x} - 2n \frac{f(-x)}{x}.
\]

For \( n = 0 \), we define the differential-difference operator \( \Lambda_\alpha \) by

\[
\Lambda_\alpha f(x) = f'(x) + \left( \alpha + \frac{1}{2} \right) \frac{f(x) - f(-x)}{x},
\]

which is referred to as the Dunkl operator of index \( \alpha + 1/2 \) associated with the reflection group \( \mathbb{Z}_2 \) on \( \mathbb{R} \). Such operators have been introduced by Dunkl (see [3], [4]) in connection with a generalization of the classical theory of spherical harmonics.

Define \( L^p_{\alpha,n}, 1 \leq p \leq \infty \), as the class of measurable functions \( f \) on \( \mathbb{R} \) for which \( \| f \|_{p,\alpha,n} < \infty \), where

\[
\| f \|_{p,\alpha,n} = \left( \int_\mathbb{R} |f(x)|^p x^{2\alpha+2n(2-p)+1} \right)^{1/p}, \quad \text{if} \quad p < \infty,
\]
and \( \|f\|_{\infty,a,n} = \|f\|_{\infty} = \text{ess sup}_{x \in \mathbb{R}} |f(x)| \).

If \( p = 2 \), then we have \( L^2_{a,n} = L^2(\mathbb{R}, |x|^{2\alpha+1}) \).

The one-dimensional Dunkl kernel is defined by

\[
e_\alpha(z) = j_\alpha(iz) + \frac{z}{2(\alpha + 1)} j_{\alpha+1}(iz), z \in \mathbb{C},
\]

(1)

where

\[
j_\alpha(z) = \Gamma(\alpha + 1) \sum_{m=0}^{\infty} \frac{(-1)^m (z/2)^{2m}}{m! \Gamma(m + \alpha + 1)}, z \in \mathbb{C},
\]

(2)

is the normalized spherical Bessel function of index \( \alpha \). It is well-known that the functions \( e_\alpha \) are the solutions of the differential-difference equation

\[
\Lambda_\alpha u = \lambda u, u(0) = 1.
\]

From (2) we see that

\[
\lim_{z \to 0} \frac{j_\alpha(z) - 1}{z^2} \neq 0.
\]

(3)

Hence, there exists \( c > 0 \) and \( \eta > 0 \) satisfying

\[
|z| \leq \eta \Rightarrow |j_\alpha(z) - 1| \geq c|z|^2.
\]

**Lemma 1.3.** For \( x \in \mathbb{R} \) the following inequalities are fulfilled

(i) \( |j_\alpha(x)| \leq 1 \),

(ii) \( |1 - j_\alpha(x)| \leq x^2/2 \),

(iii) \( |1 - j_\alpha(x)| \geq c \) with \( |x| \geq 1 \), where \( c > 0 \) is a certain constant which depends only on \( \alpha \).

**Proof.** Similarly as the proof of Lemma 2.9 in [2]. \( \square \)

For \( \lambda \in \mathbb{C} \), and \( x \in \mathbb{R} \), put

\[
\varphi_\lambda(x) = x^{2n} e_{\alpha+2n}(i\lambda x).
\]

where \( e_{\alpha+2n} \) is the Dunkl kernel of index \( \alpha + 2n \) given by (1).
Proposition 1.4. (i) $\varphi_\lambda$ satisfies the differential equation

$$\Lambda \varphi_\lambda = i\lambda \varphi_\lambda.$$

(ii) For all $\lambda \in \mathbb{C}$, and $x \in \mathbb{R}$

$$|\varphi_\lambda(x)| \leq |x|^{2n} e^{|Im\lambda||x|}.$$

The generalized Fourier-Dunkl transform that we call it the integral transform is defined by

$$\mathcal{F}_\Lambda f(\lambda) = \int_{\mathbb{R}} f(x) \varphi_{-\lambda}(x) |x|^{2\alpha+1} dx, \lambda \in \mathbb{R}, f \in L^1_{\alpha,n}.$$

Let $f \in L^1_{\alpha,n}$ such that $\mathcal{F}_\Lambda(f) \in L^1_{\alpha+2n} = L^1(\mathbb{R}, |x|^{2\alpha+4n+1} dx)$. Then the inverse generalized Fourier-Dunkl transform is given by the formula

$$f(x) = \int_{\mathbb{R}} \mathcal{F}_\Lambda(f)(\lambda) \varphi_\lambda(x) d\mu_{\alpha+2n}(\lambda),$$

where

$$d\mu_{\alpha+2n}(\lambda) = a_{\alpha+2n}|\lambda|^{2\alpha+4n+1} d\lambda, \quad a_\alpha = \frac{1}{2^{2\alpha+2}(\Gamma(\alpha+1))^2}.$$

Proposition 1.5. (i) For every $f \in L^p_{\alpha,n}$,

$$\mathcal{F}_\Lambda(\Lambda f)(\lambda) = i\lambda \mathcal{F}_\Lambda(f)(\lambda).$$

(ii) For every $f \in L^1_{\alpha,n} \cap L^2_{\alpha,n}$ we have the Plancherel formula

$$\int_{\mathbb{R}} |f(x)|^2 |x|^{2\alpha+1} dx = \int_{\mathbb{R}} |\mathcal{F}_\Lambda(f)(\lambda)|^2 d\mu_{\alpha+2n}(\lambda).$$

(iii) The generalized Fourier-Dunkl transform $\mathcal{F}_\Lambda$ extends uniquely to an isometric isomorphism from $L^2_{\alpha,n}$ onto $L^2(\mathbb{R}, \mu_{\alpha+2n})$.

By Plancherel equality and Marcinkiewics interpolation Theorem (see [7]) we get for $f \in L^p_{\alpha,n}$ with $1 \leq p \leq 2$ and $q$ such that $\frac{1}{p} + \frac{1}{q} = 1$,

$$\|\mathcal{F}_\Lambda(f)\|_{q,\alpha+2n} \leq K \|f\|_{p,\alpha,n}, \quad (4)$$
where $K$ is a positive constant.
The generalized translation operators $\tau^x$, $x \in \mathbb{R}$, tied to $\Lambda$ are defined by

\[
\tau^x f(y) = \frac{(xy)^{2n}}{2} \int_{-1}^{1} f(\sqrt{x^2 + y^2 - 2xy} t) \left( 1 + \frac{x - y}{\sqrt{x^2 + y^2 - 2xy}} \right) A(t) dt
\]

\[
+ \frac{(xy)^{2n}}{2} \int_{-1}^{1} f(-\sqrt{x^2 + y^2 - 2xy} t) \left( 1 - \frac{x - y}{\sqrt{x^2 + y^2 - 2xy}} \right) A(t) dt,
\]

where

\[
A(t) = \frac{\Gamma(\alpha + 2n + 1)}{\sqrt{\pi} \Gamma(\alpha + 2n + 1/2)} (1 + t)(1 - t^2)^{\alpha + 2n - 1/2}.
\]

**Proposition 1.6.** Let $f$ be in $L^p_{\alpha,n}$, $1 \leq p \leq \infty$. Then for all $x \in \mathbb{R}$, the function $\tau^x f$ belongs to $L^p_{\alpha,n}$, and

\[
\|\tau^x f\|_{p,\alpha,n} \leq 2x^{2n} \|f\|_{p,\alpha,n}.
\]

Furthermore,

\[
\mathcal{F}_\Lambda(\tau^x f)(\lambda) = x^{2n} e^{\alpha+2n(i\lambda x)} \mathcal{F}_\Lambda(f)(\lambda).
\]

(5)

## 2. Dini-Lipschitz Condition

**Definition 2.1.** Let $f \in L^p_{\alpha,n}$, $1 \leq p \leq \infty$, and define

\[
\|\tau^h f(x) + \tau^{-h} f(x) - 2h^{2n} f(x)\|_{p,\alpha,n} \leq C \left( \frac{h^{\eta + 2n}}{\ln \frac{1}{h}} \right)^\gamma, \quad \eta > 0, \gamma \geq 0,
\]

i.e.,

\[
\|\tau^h f(x) + \tau^{-h} f(x) - 2h^{2n} f(x)\|_{p,\alpha,n} = O \left( \frac{h^{\eta + 2n}}{\ln \frac{1}{h}} \right)^\gamma,
\]

for all $x \in \mathbb{R}$ and for all sufficiently small $h, C$ being a positive constant. Then we say that $f$ satisfies a Dini-Lipschitz of order $\eta$, or $f$ belongs to $\text{Lip}(\eta, \gamma, p)$. 
Definition 2.2. If however
\[ \frac{\|\tau^h f(x) + \tau^{-h} f(x) - 2h^{2n} f(x)\|_{p,a,n}}{h^{\eta+2n}/(\ln \frac{1}{h})^{\gamma}} \to 0, \quad \text{as} \quad h \to 0, \gamma \geq 0, \]
then \( f \) is said to be belong to the little Dini-Lipschitz class \( \text{lip}(\eta, \gamma, p) \).

Remark 2.3. Let \( \eta > 1 \). If \( f \in \text{Lip}(\eta, \gamma, p) \), then \( f \in \text{lip}(1, \gamma, p) \).

Proof. For \( x \in \mathbb{R}, \) small and \( f \in \text{Lip}(\eta, \gamma, p) \), we have
\[ \|\tau^h f(x) + \tau^{-h} f(x) - 2h^{2n} f(x)\|_{p,a,n} \leq Ch^{\eta+2n}/(\ln \frac{1}{h})^{\gamma}. \]
Then
\[ (\log \frac{1}{h})^{\gamma}\|\tau^h f(x) + \tau^{-h} f(x) - 2h^{2n} f(x)\|_{p,a,n} \leq C h^{\eta+2n}. \]
Therefore
\[ (\log \frac{1}{h})^{\gamma}\|\tau^h f(x) + \tau^{-h} f(x) - 2h^{2n} f(x)\|_{p,a,n} \leq C h^{\eta-1}, \]
which tends to zero with \( h \to 0 \). Thus
\[ (\log \frac{1}{h})^{\gamma}\|\tau^h f(x) + \tau^{-h} f(x) - 2h^{2n} f(x)\|_{p,a,n} \to 0, \quad h \to 0. \]
Then \( f \in \text{lip}(1, \gamma, p) \). \( \square \)

Remark 2.4. If \( \eta < \nu \), then \( \text{Lip}(\eta, 0, p) \supset \text{Lip}(\nu, 0, p) \) and \( \text{lip}(\eta, 0, p) \supset \text{lip}(\nu, 0, p) \).

Proof. We have \( 0 \leq h \leq 1 \) and \( \eta < \nu \), then \( h^\nu \leq h^\eta \).
Then the proof of theorem is immediate. \( \square \)

3. New Results on Dini-Lipschitz Class

Theorem 3.1. Let \( \eta > 2 \) and \( 1 \leq p \leq 2 \). If \( f \) belongs to the Dini-Lipschitz class, i.e.,
\[ f \in \text{Lip}(\eta, \gamma, p), \quad \eta > 2, \gamma \geq 0, 1 \leq p \leq 2. \]
Then $f$ is null almost everywhere on $\mathbb{R}$.

**Proof.** Assume that $f \in Lip(\eta, \gamma, p)$. Then we have

$$\|\tau^h f(x) + \tau^{-h} f(x) - 2h^{2n} f(x)\|_{p,\alpha,n} \leq C \frac{h^{\eta+2n}}{(\ln \frac{1}{h})^\gamma}, \quad \gamma \geq 0.$$  

By using the formulas (1), (2), and (5) we have the generalized Fourier-Dunkl transform of $\tau^h f(x) + \tau^{-h} f(x) - 2h^{2n} f(x)$ is $2h^{2n}(j_{\alpha+2n}(\lambda h) - 1)F_\Lambda f(\lambda)$.

By formula (4), we get

$$2^q h^{2q} \int_\mathbb{R} |j_{\alpha+2n}(\lambda h) - 1|^q |F_\Lambda f(\lambda)|^q d\mu_{\alpha+2n}(\lambda) \leq K^q C^q \frac{h^{\eta+2q}}{(\ln \frac{1}{h})^q \gamma}.$$  

Therefore

$$\int_\mathbb{R} |j_{\alpha+2n}(\lambda h) - 1|^q |F_\Lambda f(\lambda)|^q d\mu_{\alpha+2n}(\lambda) \leq K^q C^q \frac{h^{\eta}}{2^q} \frac{h^{\eta}}{(\ln \frac{1}{h})^{q\gamma}}.$$  

Then

$$\frac{\int_\mathbb{R} |j_{\alpha+2n}(\lambda h) - 1|^q |F_\Lambda f(\lambda)|^q d\mu_{\alpha+2n}(\lambda)}{h^{2q}} \leq K^q C^q \frac{h^{\eta-2q}}{2^q} \frac{h^{\eta-2q}}{(\ln \frac{1}{h})^{q\gamma}}.$$  

Since $\eta > 2$ we have

$$\lim_{h \to 0} \frac{h^{\eta-2q}}{(\ln \frac{1}{h})^{q\gamma}} = 0.$$  

Thus

$$\lim_{h \to 0} \int_\mathbb{R} \left( \frac{|1 - j_{\alpha+2n}(\lambda h)|}{\lambda^2 h^2} \right)^q \lambda^{2q} |F_\Lambda f(\lambda)|^q d\mu_{\alpha+2n}(\lambda) = 0.$$  

And also from the formula (3) and Fatou theorem, we obtain

$$\int_\mathbb{R} \lambda^{2q} |F_\Lambda f(\lambda)|^q d\mu_{\alpha+2n}(\lambda) = 0.$$  

Hence $\lambda^2 F_\Lambda f(\lambda) = 0$ for all $\lambda \in \mathbb{R}$, and so $f(x)$ is the null function.  \( \square \)
Theorem 3.2. Let \( f \in L^p_{\alpha,n}, 1 \leq p \leq 2 \). If \( f \) belongs to \( \text{lip}(2,0,p) \), i.e.,
\[
\| \tau^h f(x) + \tau^{-h} f(x) - 2h^{2n} f(x) \|_{p,\alpha,n} = O(h^{2+2n}), \quad \text{as} \quad h \to 0.
\]
Then \( f \) is null almost everywhere on \( \mathbb{R} \).

Proof. Assume that \( f \in \text{lip}(2,0,p) \). Then we have
\[
\| \tau^h f(x) + \tau^{-h} f(x) - 2h^{2n} f(x) \|_{p,\alpha,n} \to 0, \quad \text{as} \quad h \to 0
\]
By using the formulas (1), (2) and (5) we have the generalized Fourier-Dunkl transform of \( \tau^h f(x) + \tau^{-h} f(x) - 2h^{2n} f(x) \) is \( 2h^{2n} (j_{\alpha+2n}(\lambda h) - 1) \mathcal{F}_\Lambda f(\lambda) \).

By formula (4), we get
\[
2^q h^{2qn} \int_\mathbb{R} \frac{|j_{\alpha+2n}(\lambda h) - 1|^q |\mathcal{F}_\Lambda f(\lambda)|^q d\mu_{\alpha+2n}(\lambda)}{h^{2q+2n}} \to 0, \quad \text{as} \quad h \to 0
\]
Thus
\[
\lim_{h \to 0} \int_\mathbb{R} \left( \frac{|1 - j_{\alpha+2n}(\lambda h)|}{\lambda^2 h^2} \right)^q \lambda^{2q} |\mathcal{F}_\Lambda f(\lambda)|^q d\mu_{\alpha+2n}(\lambda) = 0.
\]
And also from the formula (3) and Fatou theorem, we obtain
\[
\int_\mathbb{R} \lambda^{2q} |\mathcal{F}_\Lambda f(\lambda)|^q d\mu_{\alpha+2n}(\lambda) = 0.
\]
Hence \( \lambda^2 \mathcal{F}_\Lambda f(\lambda) = 0 \) for all \( \lambda \in \mathbb{R} \), and so \( f(x) \) is the null function. \( \square \)

Now, we give another the main result of this paper analog of Theorem 1.2.

Theorem 3.3. Let \( f \in L^p_{\alpha,n} \). If \( f(x) \) belongs to \( \text{Lip}(\eta,\gamma,p) \). Then
\[
\int_{|\lambda| \geq r} |\mathcal{F}_\Lambda f(\lambda)|^q d\mu_{\alpha+2n}(\lambda) = O \left( \frac{r^{1-q\eta}}{(\ln r)^{q\gamma}} \right), \quad \text{as} \quad r \to \infty,
\]
where \( 1 \leq p \leq 2 \) and \( q \) such that \( \frac{1}{p} + \frac{1}{q} = 1 \).
\textbf{Proof.} Let \( f \in \text{Lip}(\eta, \gamma, p) \). Then we have

\[ \| \tau^h f(x) + \tau^{-h} f(x) - 2h^{2n} f(x) \|_{p,\alpha,n} = O \left( \frac{h^{n+2n}}{(\ln \frac{1}{h})^q} \right) \text{ as } h \to 0. \]

From formulas (1), (2) and (5) we have the generalized Fourier-Dunkl transform of \( \tau^h f(x) + \tau^{-h} f(x) - 2h^{2n} f(x) \) is \( 2h^{2n} (j_{\alpha+2n}(\lambda h) - 1) \mathcal{F}_\lambda f(\lambda) \).

By formula (4), we obtain

\[ 2^q h^{2qn} \int_{\mathbb{R}} |j_{\alpha+2n}(\lambda h) - 1|^q |\mathcal{F}_\lambda f(\lambda)|^q d\mu_{\alpha+2n}(\lambda) \leq K^q \| \tau^h f(x) + \tau^{-h} f(x) - 2h^{2n} f(x) \|^q_{p,\alpha,n}. \]

If \( |\lambda| \in \left[ \frac{1}{h}, \frac{2}{h} \right] \), then \( |\lambda h| \geq 1 \) and (iii) of Lemma 1.3 implies that

\[ 1 \leq \frac{1}{c^q} |j_{\alpha+2n}(\lambda h) - 1|^q. \]

Then

\[
\begin{align*}
\int_{\frac{1}{h} \leq |\lambda| \leq \frac{2}{h}} |\mathcal{F}_\lambda f(\lambda)|^q d\mu_{\alpha+2n}(\lambda) & \leq \frac{1}{c^q} \int_{\frac{1}{h} \leq |\lambda| \leq \frac{2}{h}} |j_{\alpha+2n}(\lambda h) - 1|^q |\mathcal{F}_\lambda f(\lambda)|^q d\mu_{\alpha+2n}(\lambda) \\
& \leq \frac{1}{c^q} \int_{-\infty}^{+\infty} |j_{\alpha+2n}(\lambda h) - 1|^q |\mathcal{F}_\lambda f(\lambda)|^q d\mu_{\alpha+2n}(\lambda) \\
& \leq \frac{h^{-2qn} K^q}{2^q c^q} \| \tau^h f(x) + \tau^{-h} f(x) - 2h^{2n} f(x) \|^q_{p,\alpha,n} \\
& = O \left( \frac{h^{qq}}{(\ln \frac{1}{h})^q} \right).
\end{align*}
\]

So we obtain

\[ \int_{r \leq |\lambda| \leq 2r} |\mathcal{F}_\lambda f(\lambda)|^q d\mu_{\alpha+2n}(\lambda) \leq C \frac{r^{-qn}}{(\ln r)^q}, \quad r \to \infty. \]

where \( C \) is a positive constant. Now, we have

\[
\begin{align*}
\int_{|\lambda| \geq r} |\mathcal{F}_\lambda f(\lambda)|^q d\mu_{\alpha+2n}(\lambda) &= \sum_{i=0}^{\infty} \int_{2^i r \leq |\lambda| \leq 2^{i+1} r} |\mathcal{F}_\lambda f(\lambda)|^q d\mu_{\alpha+2n}(\lambda) \\
& \leq C \left( \frac{r^{-qn}}{(\ln r)^q} + \frac{(2r)^{-qn}}{(\ln 2r)^q} + \frac{(4r)^{-qn}}{(\ln 4r)^q} + \cdots \right) \\
& \leq C \frac{r^{-qn}}{(\ln r)^q} \left( 1 + 2^{-qn} + (2^{-qn})^2 + (2^{-qn})^3 + \cdots \right) \\
& \leq K \eta \frac{r^{-qn}}{(\ln r)^q},
\end{align*}
\]
where $K_\eta = C(1 - 2^{\eta})^{-1}$ since $2^{\eta} > 1$.

Consequently
\[
\int_{|\lambda| \geq r} |{\mathcal F}_\Lambda f(\lambda)|^q d\mu_{\alpha+2n}(\lambda) = O\left(\frac{r^{-\eta}}{(\ln r)^{\eta}}\right), \text{ as } r \to \infty,
\]
and this completes the proof. \(\square\)

**Theorem 3.4.** Let $f \in L^2_{\alpha,n}$, $0 < \eta < 1$ and $\gamma \geq 0$. If
\[
\int_{|\lambda| \geq r} |{\mathcal F}_\Lambda f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) = O\left(\frac{r^{-2\eta}}{(\log r)^{2\gamma}}\right), \text{ as } r \to \infty,
\]
then $f \in \text{Lip}(\eta, \gamma, 2)$.

**Proof.** Suppose that
\[
\int_{|\lambda| \geq r} |{\mathcal F}_\Lambda f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) = O\left(\frac{r^{-2\eta}}{(\log r)^{2\gamma}}\right), \text{ as } r \to \infty,
\]
and write
\[
\|\tau^h f(x) + \tau^{-h} f(x) - 2h^{2n} f(x)\|_{2,\alpha,n}^2 = 4h^{4n}(I_1 + I_2),
\]
where
\[
I_1 = \int_{|\lambda| < \frac{1}{h}} |j_{\alpha+2n}(\lambda h) - 1|^2 |{\mathcal F}_\Lambda f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda),
\]
and
\[
I_2 = \int_{|\lambda| \geq \frac{1}{h}} |j_{\alpha+2n}(\lambda h) - 1|^2 |{\mathcal F}_\Lambda f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda).
\]
Firstly, we use the formulas $|j_{\alpha+2n}(\lambda)| \leq 1$ and
\[
I_2 \leq 4 \int_{|\lambda| \geq \frac{1}{h}} |{\mathcal F}_\Lambda f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) = \left(\frac{h^{2\eta}}{(\log \frac{1}{h})^{2\gamma}}\right), \text{ as } h \to 0.
\]
Set
\[
\phi(x) = \int_x^{+\infty} |{\mathcal F}_\Lambda f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda).
\]
Integrating by parts we obtain
\[
\int_0^x \lambda^2 |\mathcal{F}\alpha f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) = \int_0^x -\lambda^2 \phi'(\lambda) d\lambda = -x^2 \phi(x) + 2 \int_0^x \lambda \phi(\lambda) d\lambda
\]
\[
\leq C_1 \int_0^x \lambda \lambda^{-2\eta}(\log \lambda)^{-2\gamma} d\lambda = O(x^{2-2\eta}(\log x)^{-2\gamma}),
\]
where \(C_1\) is a positive constant.
We use the formula (ii) of Lemma 1.2
\[
\int_{-\infty}^{+\infty} |j_{\alpha+2n}(\lambda h) - 1|^2 |\mathcal{F}\alpha f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) = O \left( h^2 \int_{|\lambda| < \frac{1}{h}} \lambda^2 |\mathcal{F}\alpha f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) \right)
\]
\[
+ \left( \frac{h^{2\eta}}{(\log \frac{1}{h})^{2\gamma}} \right)
\]
\[
= O \left( h^{2} \frac{h^{2\eta-2}}{(\log \frac{1}{h})^{2\gamma}} \right) + O \left( \frac{h^{2\eta}}{(\log \frac{1}{h})^{2\gamma}} \right)
\]
\[
= O \left( \frac{h^{2\eta}}{(\log \frac{1}{h})^{2\gamma}} \right),
\]
and this ends the proof. \(\square\)

By analogy with the proof of the Theorems 3.3 and 3.4, we can establish the following results.

**Theorem 3.5.** Let \(f \in L^p_{\alpha,n}\). If
\[
\|r^h f(x) + r^{-h} f(x) - 2 h^{2n} f(x)\|_{p,\alpha,n} = o \left( h^{2n}(\ln \frac{1}{h})^{-1} \right), \quad \text{as} \quad h \to 0,
\]
then
\[
\int_{|\lambda| \geq r} |\mathcal{F}\alpha f(\lambda)|^q d\mu_{\alpha+2n}(\lambda) = o \left( (\ln r)^{-q} \right), \quad \text{as} \quad r \to \infty,
\]
where \(1 \leq p \leq 2\) and \(q\) such that \(\frac{1}{p} + \frac{1}{q} = 1\).

**Theorem 3.6.** Let \(f \in L^2_{\alpha,n}\). If
\[
\int_{|\lambda| \geq r} |\mathcal{F}\alpha f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) = o \left( (\ln r)^{-2} \right), \quad \text{as} \quad r \to \infty,
\]
then

\[ \| \tau^h f(x) + \tau^{-h} f(x) - 2h^{2n} f(x) \|_{2, \alpha, n} = o \left( h^{2n} \left( \ln \frac{1}{h} \right)^{-1} \right), \quad \text{as} \quad h \to 0. \]

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