On Characterizing Pairs of Non-Abelian Nilpotent and Filiform Lie Algebras by their Schur Multipliers

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Abstract. Let $L$ be an $n$-dimensional non-abelian nilpotent Lie algebra. Niroomand and Russo (2011) proved that $\dim M(L) = \frac{1}{2}(n-1)(n-2) + 1 - s(L)$, where $M(L)$ is the Schur multiplier of $L$ and $s(L)$ is a non-negative integer. They also characterized the structure of $L$, when $s(L) = 0$. Assume that $(N, L)$ is a pair of finite dimensional nilpotent Lie algebras, in which $L$ is non-abelian and $N$ is an ideal in $L$ and also $M(N, L)$ is the Schur multiplier of the pair $(N, L)$. If $N$ admits a complement $K$ say, in $L$ such that $\dim K = m$, then $\dim M(N, L) = \frac{1}{2}(n^2 + 2nm - 3n - 2m + 2) + 1 - (s(L) - t(K))$, where $t(K) = \frac{1}{2}m(m-1) - \dim M(K)$. In the present paper, we characterize the pairs $(N, L)$, for which $0 \leq t(K) \leq s(L) \leq 3$. In particular, we classify the pairs $(N, L)$ such that $L$ is a non-abelian filiform Lie algebra and $0 \leq t(K) \leq s(L) \leq 17$.

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1. Introduction

All Lie algebras are considered over a fixed field $\Lambda$ and $[,]$ denotes the Lie bracket. Let $(N,L)$ be a pair of Lie algebras, where $N$ is an ideal in $L$. Then we define the Schur multiplier of the pair $(N,L)$ to be the abelian Lie algebra $\mathcal{M}(N,L)$ appearing in the following natural exact sequence of Lie algebras,

$$H_3(L) \rightarrow H_3(L/N) \rightarrow \mathcal{M}(N,L) \rightarrow \mathcal{M}(L) \rightarrow \mathcal{M}(L/N),$$

where $\mathcal{M}(-)$ and $H_3(-)$ denote the Schur multiplier and the third homology of a Lie algebra, respectively. Ellis [4] proved that $\mathcal{M}(N,L) \cong \ker(N \wedge L \rightarrow L)$, in which $N \wedge L$ denotes the non-abelian exterior product of Lie algebras. Also using the above sequence, one can easily see that if the ideal $N$ possesses a complement in $L$, then $\mathcal{M}(L) = \mathcal{M}(N,L) \oplus \mathcal{M}(L/N)$. In this case, if the free presentation $0 \rightarrow R \rightarrow F \rightarrow L \rightarrow 0$ of $L$, $\mathcal{M}(N,L)$ is isomorphic to the factor Lie algebra $(R \cap [S,F])/[R,F]$, where $S$ is an ideal in the free Lie algebra $F$ such that $S/R \cong N$. In particular, if $N = L$, then the Schur multiplier of $(N,L)$ will be $\mathcal{M}(L) = (R \cap F^2)/[R,F]$ (see [2] for more detail), which is analogous to the Schur multiplier of a group (see [8]).

In 1994, Moneyhun [9] proved the Lie algebra analogue of Green’s result (1956), which states that for an $n$-dimensional Lie algebra $L$, we have $\dim \mathcal{M}(L) = \frac{1}{2} n(n-1) - t(L)$, where $t(L) \geq 0$. Characterization of finite dimensional nilpotent Lie algebras by their Schur multipliers has been already studied by many authors. In [2, 6, 7], all nilpotent Lie algebras are characterized, when $t(L) = 0, 1, 2, \ldots , 8$.

Moreover, Saeedi et al. [12] proved that in a pair $(N,L)$ of finite dimensional nilpotent Lie algebras, if $N$ admits a complement $K$ say, in $L$ with $\dim K = m$, then $\dim \mathcal{M}(N,L) = \frac{1}{2} n(n+2m-1) - t(N,L)$, for a non-negative integer $t(N,L)$. This actually gives us the Moneyhun’s result, if $m = 0$. The first author and colleagues [1] characterized the pairs $(N,L)$ of finite dimensional nilpotent Lie algebras, for which $t(N,L) = 0, 1, 2, 3, 4$. Also in a special case, when $L$ is a filiform Lie algebra, they determined all pairs $(N,L)$, for $t(N,L) = 0, 1, \ldots , 10$.

Niroomand and Russo [11] presented an upper bound for the dimension of the Schur multiplier of an $n$-dimensional non-abelian nilpotent Lie algebra $L$ and showed that $\dim \mathcal{M}(L) = \frac{1}{2} (n-1)(n-2) + 1 - s(L)$, where $s(L) \geq 0$. They also classified the structure of $L$, when $s(L) = 0$.

Now, let $(N,L)$ be a pair of finite dimensional nilpotent Lie algebras, in which $L$ is non-abelian of dimension $n$. If $N$ possesses a complement $K$ in $L$ such that $\dim K = m$, then by using $\mathcal{M}(L) = \mathcal{M}(N,L) \oplus \mathcal{M}(L/N)$, we get $\dim \mathcal{M}(N,L) =$
A nilpotent Lie algebra is defined in Theorem 2.1, and in this paper, we intend to classify the pairs \((N, L)\) such that \(0 \leq t(K) \leq s(L) \leq 3\). One should notice that if \(m = 0\), then the above bound for the dimension of \(\mathcal{M}(N, L) = \mathcal{M}(L)\) is actually \(\frac{1}{2}(n - 1)(n - 2) + 1\), which is discussed in [11]. While the results of [1] depend on the Moneyhun’s bound \(\frac{1}{2}n(n - 1)\). This clarifies the differences of our results from the ones in [1].

In particular, we characterize the pairs \((N, L)\) with \(0 \leq t(K) \leq s(L) \leq 17\), where \(L\) is a finite dimensional non-abelian filiform Lie algebra. Note that our method in the proof of Theorem B of this paper, uses the Ganea’s exact sequence [8], which is different from the applied technique in [1].

## 2. Preliminary Results

In this section, we discuss some preliminary results which will be used in the next section.

**Theorem 2.1.** Let \(L\) be a finite dimensional nilpotent Lie algebra. Then

- (a) \(t(L) = 0\) if and only if \(L\) is abelian;
- (b) \(t(L) = 1\) if and only if \(L \cong H(1)\);
- (c) \(t(L) = 2\) if and only if \(L \cong H(1) \oplus A(1)\);
- (d) \(t(L) = 3\) if and only if \(L \cong H(1) \oplus A(2)\);
- (e) \(t(L) = 4\) if and only if \(L \cong H(1) \oplus A(3), L(3, 4, 1, 4)\) or \(L(4, 5, 2, 4)\);
- (f) \(t(L) = 5\) if and only if \(L \cong H(1) \oplus A(4)\) or \(H(2)\);
- (g) \(t(L) = 6\) if and only if \(L \cong H(1) \oplus A(5), H(2) \oplus A(1), L(4, 5, 1, 6), L(3, 4, 1, 4) \oplus A(1)\) or \(L(4, 5, 2, 4) \oplus A(1)\);
- (h) \(t(L) = 7\) if and only if \(L \cong H(1) \oplus A(6), H(2) \oplus A(2), H(3), L(7, 5, 2, 7), L(7, 5, 1, 7), L' (7, 5, 1, 7), L(7, 6, 2, 7)\) or \(L(7, 6, 2, 7, \beta_1, \beta_2)\);
- (i) \(t(L) = 8\) if and only if \(L \cong H(1) \oplus A(7), H(2) \oplus A(3), H(3) \oplus A(1), L(4, 5, 1, 6) \oplus A(1), L(3, 4, 1, 4) \oplus A(2)\) or \(L(4, 5, 2, 4) \oplus A(2)\).

The \(H(m)\) denotes the Heisenberg Lie algebra of dimension \(2m + 1\), \(A(n)\) is an \(n\)-dimensional abelian Lie algebra and \(L(a, b, c, d)\) denotes the algebra discovered for the case \(t(L) = a\), where \(b = \dim L, c = \dim Z(L)\) and \(d = t(L)\).

Table I describes the nilpotent Lie Algebras which are referred in the above theorem (see [2, 6, 7]).

A filiform Lie algebra is an algebra with maximal nilpotent index and defined by Vergne in [13].

**Definition 2.2.** Let \(L\) be a Lie algebra of dimension \(n\) over a field \(C\) of characteristic zero, and \(C^1L = L, C^2L = [L, L], \ldots, \) and \(C^nL = [L, C^{n-1}L], \ldots, \) be a
descending sequence of $L$. Then $L$ is called a filiform Lie algebra if $\dim C^q L = n - q$, for $2 \leq q \leq n$.

The above definition implies that if $n = q$, then $L$ is nilpotent. Also $\dim C^{n-1} L/C^q L = 1$, for $q = 2, \ldots, n - 1$ and $\dim C^1 L/C^2 L = 2$. In [5], filiform Lie algebras are classified up to dimension 11.

Recall that in [11], Niroomand and Russo proved that if $L$ is an $n$-dimensional non-abelian nilpotent Lie algebra, then $\dim M(L) = \frac{1}{2}(n-1)(n-2) + 1 - s(L)$, where $s(L) \geq 0$. They also classified the structure of $L$, when $s(L) = 0$. Later in [10], Niroomand characterized all finite dimensional non-abelian nilpotent Lie algebras with $s(L) = 1, 2, 3$. In the following, we have collected all the above results.

**Theorem 2.3.** Let $L$ be an $n$-dimensional non-abelian nilpotent Lie algebra. Then

(a) $s(L) = 0$ if and only if $L \cong H(1) \oplus A(n - 3)$;
(b) $s(L) = 1$ if and only if $L \cong L(4,5,2,4)$;
(c) $s(L) = 2$ if and only if $L$ is isomorphic to one of the following Lie algebras

\[ L(3,4,1,4), \ L(4,5,2,4) \oplus A(1), \ H(m) \oplus A(n - 2m - 1), \ m \geq 2; \]

(d) $s(L) = 3$ if and only if $L$ is isomorphic to one of the following Lie algebras

\[ L(4,5,1,6), \ L(5,6,2,7), \ L'(5,6,2,7), \ L(7,6,2,7), \ L'(7,6,2,7), \]
\[ L(3,4,1,4) \oplus A(1), \ L(4,5,2,4) \oplus A(2). \]

Now, suppose that $(N, L)$ is a pair of finite dimensional nilpotent Lie algebras, in which $L$ is non-abelian of dimension $n$ and $N$ is a non-trivial ideal in $L$. Also, assume that $K$ is the complement of $N$ in $L$ such that $\dim K = m$. Then it follows from $M(L) = M(N, L) \oplus M(L/N)$, $\dim M(L) = \frac{1}{2}(n-1)(n-2) + 1 - s(L)$ and $\dim M(K) = \frac{1}{2}m(m - 1) - t(K)$ that $\dim M(N, L) = \frac{1}{2}(n^2 + 2nm - 3n - 2m + 2) + 1 - (s(L) - t(K))$.

In the following theorem, we characterize the pairs $(N, L)$, for which $0 \leq t(K) \leq s(L) \leq 3$.

**Theorem A.** If a pair $(N, L)$ satisfies the above assumptions, then

(a) $(s(L), t(K)) = (0, 0)$ if and only if $(N, L) \cong (H(1) \oplus A(j), H(1) \oplus A(n - 3))$, $0 \leq j \leq n - 4$.
(b) There is not any pair $(N, L)$, for $(s(L), t(K))$ with $s(L) = 1$ and $t(K) = 0, 1$. 


(c) \((s(L), t(K)) = (2, 0)\) if and only if \((N, L)\) is isomorphic to one of the following,

\[
(L(4, 5, 2, 4), L(4, 5, 2, 4) \oplus A(1)), (H(m) \oplus A(j), H(m) \oplus A(n - 2m - 1))
\]

such that \(0 \leq j \leq n - 2m - 2, \ m \geq 2\).

(d) There is not any pair \((N, L)\), for \((s(L), t(K))\) with \(s(L) = 2\) and \(t(K) = 1, 2\).

(e) \((s(L), t(K)) = (3, 0)\) if and only if \((N, L)\) is isomorphic to one of following,

\[
(L(3, 4, 1, 4), L(4, 5, 1, 6)), (L(3, 4, 1, 4), L(3, 4, 1, 4) \oplus A(1)),
\]

\[
(L(4, 5, 2, 4), L(4, 5, 2, 4) \oplus A(2)).
\]

(f) \((s(L), t(K)) = (3, 1)\) if and only if \((N, L)\) is isomorphic to one of following,

\[
(H(1), L'(5, 6, 2, 7)), (H(1), L(7, 6, 2, 7)).
\]

(g) There is not any pair \((N, L)\), for \((s(L), t(K))\) with \(s(L) = 3\) and \(t(K) = 2, 3\).

Also in the following result, we classify the pairs \((N, L)\) with \(0 \leq t(K) \leq s(L) \leq 17\), where \(L\) is a finite dimensional non-abelian filiform Lie algebra.

**Theorem B.** Let \((N, L)\) be a pair of finite dimensional filiform Lie algebras, in which \(L\) is non-abelian and \(N\) is a non-trivial ideal in \(L\). Then

(a) \((s(L), t(K)) = (7, 0)\) if and only if \((N, L) \cong (N, L'(11, 6, 1, 11))\) such that \(\{x, y, z, c, r, t\}\) is a basis for \(L'(11, 6, 1, 11)\) and

\[
N = \langle y, z, c, r, t | [z, c] = -t, [y, r] = t, [y, z] = r + \alpha t \rangle
\]

is an ideal of \(L'\) \((\alpha \in \Lambda)\).

(b) There is not any pair \((N, L)\) for \((s(L), t(K))\) such that \(s(L) \in \{0, 1, 2, \cdots, 17\}\) \(-\{7\}\) and \(t(K) \in \{1, 2, \cdots, 17\}\).

### 3. Proof of the Main Theorems

The following results are needed for proving Theorem A and B (see [2, 3, 9]).

**Lemma 3.1.** Let \(A(n)\) be an \(n\)-dimensional abelian Lie algebra and \(H(m)\) the Heisenberg Lie algebra of dimension \(2m + 1\). Then

(a) \(\dim \mathcal{M}(A(n)) = \frac{1}{2} n(n - 1)\);
Lemma 3.2. Let $L = A \oplus B$. Then
\[
\dim \mathcal{M}(L) = \dim \mathcal{M}(A) + \dim \mathcal{M}(B) + \dim(A/A^2 \otimes B/B^2).
\]

Theorem 3.3. If $L$ is a finite dimensional nilpotent Lie algebra of dimension greater than 1 and class $c$, then $\mathcal{M}(L) \neq 0$.

By using [11], we may easily prove the following results. Observe that the results discussed in [11] are based on the equality $t(K) = \frac{1}{2}n(n-1) - \dim \mathcal{M}(L)$ in [9]. But our similar results depend on $s(L) = \frac{1}{2}(n-1)(n-2) + 1 - \dim \mathcal{M}(L)$.

Proposition 3.4. Let $L$ be a non-abelian filiform Lie algebra of dimension $n$, $K = Z(L)$ and $H = L/K$. Then $s(H) + \dim L^2 \leq s(L) + 1$.

Proposition 3.5. Let $L$ be a non-abelian filiform Lie algebra of dimension $n$. Then $\dim L^2 \leq s(L) + 1$.

Theorem 3.6. Let $L$ be an $n$-dimensional non-abelian filiform Lie algebra and $\dim L^2 = c$. Then $c^2 - c \leq 2s(L)$.

Theorem 3.7. Let $L$ be an $n$-dimensional non-abelian filiform Lie algebra. Then

(a) There is no filiform Lie algebra, with $s(L) = 1, 3, 4, 5, 6, 9, 10, 11, 14, 15, 16, 17$;
(b) $s(L) = 0$ if and only if $L \cong H(1)$;
(c) $s(L) = 2$ if and only if $L \cong L(3, 4, 1, 4)$;
(d) $s(L) = 4$ if and only if $L \cong L(11, 6, 1, 11)$ or $L'(11, 6, 1, 11)$;
(e) $s(L) = 8$ if and only if $L \cong L(11, 6, 1, 12)$ or $L'(11, 6, 1, 12)$;
(f) $s(L) = 12$ if and only if $L \cong L(17, 7, 1, 17)$ or $L'(17, 7, 1, 17)$;
(g) $s(L) = 13$ if and only if $L \cong L(17, 7, 1, 18)$.

In Table II, we describe the filiform Lie Algebras which are referred in the above theorem, using [2, 6, 7].

Now, we are ready to prove our main theorems.

Proof of Theorem A. Case $s(L) = 0$. In this case $t(K) = 0$ and Theorem 2.1 and 2.3 imply that $L \cong H(1) \oplus A(n-3)$ and $K$ is an abelian subalgebra of $L$. According to the structure of $L$ by choosing a suitable subalgebra $K$ and an ideal $N$ in $L$, similar to the proof of Theorems A and B in [1], one can easily verify that

$$(N, L) \cong (H(1) \oplus A(j), H(1) \oplus A(n-3)),$$
where $0 \leq j \leq n - 4$.

Case $s(L) = 1$. In this case $t(K) = 0, 1$ and by Theorem 2.3, $L \cong L(4, 5, 2, 4)$. Let $t(K) = 0$. Thus $K$ is abelian. Now if $\dim N = 4$ and $\dim K = 1$, then

$$N = \langle x_1, x_3, x_4, x_5 \mid [x_1, x_4] = x_5 \rangle \cong H(1) \oplus A(1),$$

such that $\{x_1, x_2, x_3, x_4, x_5\}$ is a basis for $L$. Hence Theorem 2.1 implies that $t(N) = 2$ and $\dim \mathcal{M}(N) = 4$. But by Lemma 3.2, we have $\dim \mathcal{M}(N) = 3$, which is a contradiction. Also if $\dim N = 3$ and $\dim K = 2$ or $\dim N = 2$ and $\dim K = 3$, then there do not exist any ideal and subalgebra satisfying the assumptions of the theorem. If $t(K) = 1$, then $K \cong H(1)$ and there do not exist any ideal and subalgebra satisfying in the theorem.

Case $s(L) = 2$. In this case $t(K) = 0, 1, 2$. By using Theorem 2.3, we get $L \cong L(3, 4, 1, 4), L(4, 5, 2, 4) \oplus A(1)$ or $H(m) \oplus A(n - 2m - 1)$ such that $m \geq 2$.

First suppose that $L \cong L(3, 4, 1, 4)$ and $t(K) = 0$, then $K$ is abelian. If $\dim N = 3$ and $\dim K = 1$ or $\dim K = \dim N = 2$, then choosing a suitable subalgebra $K$ and an ideal $N$ in $L$ and Lemma 3.2 imply that there is no such a pair. If $t(K) = 1$, then $K \cong H(1)$. By choosing a suitable ideal $N$ in $L$ and Lemma 3.2, there is not any desired pair. And finally if $t(K) = 2$, then ideal $N$ is trivial.

Now, let $L \cong L(4, 5, 2, 4) \oplus A(1)$. If $t(K) = 0$, then $K$ is abelian. Therefore $(N, L) \cong (L(4, 5, 2, 4), L(4, 5, 2, 4) \oplus A(1))$. If $t(K) = 1, 2$, then Theorem 2.1, Lemma 3.2 and choosing a suitable subalgebra $K$ and an ideal $N$ of $L$ imply that there is not any pair.

Finally, let $L \cong H(m) \oplus A(n - 2m - 1)$ such that $m \geq 2$. If $t(K) = 0$, then similar to the first case, we can easily see that

$$(N, L) \cong (H(m) \oplus A(j), H(m) \oplus A(n - 2m - 1)),$$

where $0 \leq j \leq n - 2m - 2$ and $m \geq 2$. If $t(K) = 1$, then there is not any suitable ideal $N$ in $L$, and if $t(K) = 2$, then by Theorem 2.1 and Lemma 3.2, there is no such a pair.

Case $s(L) = 3$. By using Theorem 2.3, we have $L \cong L(4, 5, 1, 6), L(3, 4, 1, 4) \oplus A(1), L(4, 5, 2, 4) \oplus A(2), L'(5, 6, 2, 7)$ or $L(7, 6, 2, 7)$. By an analogous manner in [1] and the previous cases, we can show that $(N, L)$ is isomorphic to one of the following pairs

$$(L(3, 4, 1, 4), L(4, 5, 1, 6)), \quad (L(3, 4, 1, 4), L(3, 4, 1, 4) \oplus A(1)),
\quad (L(4, 5, 2, 4), L(4, 5, 2, 4) \oplus A(2)), \quad (H(1), L'(5, 6, 2, 7)),
\quad (H(1), L(7, 6, 2, 7)).$$

$\square$
Proof of Theorem B. By Theorem 3.7 there is not any filiform Lie algebra, with \( s(L) = 1, 3, 4, 5, 6, 9, 10, 11, 14, 15, 16, 17 \). Similar to the proof of Theorem A, one can easily check that in cases \( s(L) = 0,2 \), there is not any suitable pair. Thus consider only the cases \( s(L) = 7,8,12,13 \).

Case \( s(L) = 7 \). In this case \( t(K) = 0,1,2,\ldots,7 \). Theorem 3.7 implies that \( L \cong L(11,6,1,11) \) or \( L'(11,6,1,11) \).

Let \( L \cong L(11,6,1,11) \) and \( t(K) = 0 \). Now if \( \dim N = 5 \), \( \dim K = 1 \) and \( \{ x,y,z,c,r,t \} \) is a basis for \( L(11,6,1,11) \), then \( N = \langle x,z,c,r,t \rangle \) such that \( [x,z] = c, [x,c] = r, [x,r] = at \) and \( [z,c] = t \) or \( N = \langle y,z,c,r,t \rangle \) such that \( [y,z] = \beta t, [y,r] = t \) and \( [z,c] = -t \), where \( \alpha, \beta \in \Lambda \). If \( N = \langle x,z,c,r,t \rangle \), then \( \dim \mathcal{M}(N) = 10 - l \), for a non-negative integer \( 0 \leq l \leq 10 \). If \( l = 0,1,2,\ldots,8 \) or \( 10 \), then by Theorem 2.1 and Theorem 3.3, we get a contradiction. Therefore, \( l = 9 \) and \( \dim \mathcal{M}(N) = 1 \). But Lemma 3.2 implies that \( \dim \mathcal{M}(N) = 2 \), which is impossible. Now, assume that \( N = \langle y,z,c,r,t \rangle \), then by Lemma 3.2, \( \dim \mathcal{M}(N) = 0 \), which is a contradiction by Theorem 3.3. In the other possibilities for the dimensions of \( N \) and \( K \), one may easily see that there do not exist any ideal and subalgebra satisfying in the hypothesis of the theorem. Similar to previous cases, we may check that if \( t(K) = 1,2,3,4,5,6,7 \), then there is not any pair.

Let \( L \cong L'(11,6,1,11) \) and \( t(K) = 0 \). Hence \( K \) is abelian. Suppose that \( \dim N = 5 \), \( \dim K = 1 \) and \( \{ x,y,z,c,r,t \} \) is a basis for \( L'(11,6,1,11) \). If \( N = \langle x,z,c,r,t \rangle \) such that \( [x,z] = c, [x,c] = r, [x,r] = at \) and \( [z,c] = -t \), where \( \alpha \in \Lambda \), then by Theorem 2.1 and Lemma 3.2, we get a contradiction. So assume that \( N = \langle y,z,c,r,t \rangle \) such that \( [y,z] = \beta t, [y,r] = t, [z,c] = -t \), where \( \beta \in \Lambda \). Thus \( \dim \mathcal{M}(N) = 1 \) and we have \( (N,K) \cong (N,K') \). In the other cases for the dimensions of \( N \) and \( K \), one may check that there do not exist any ideal and subalgebra satisfying in the theorem. Also, if \( t(K) = 1,2,3,4,5,6,7 \), then there is not any pair.

Case \( s(L) = 8 \). In this case, similar to the previous cases, there is not any pair.

Case \( s(L) = 12 \). By using Theorem 3.7, we have \( L \cong (17,7,1,17) \) or \( L'(17,7,1,17) \). First, let \( L \cong L(17,7,1,17) \) and \( t(K) = 0 \). Then \( K \) is abelian. Suppose that \( \dim N = 6 \) and \( \dim K = 1 \) and \( \{ x,y,z,c,r,t,u \} \) is a basis for \( L(17,7,1,17) \). Now if \( N = \langle x,z,c,r,t,u \rangle \) such that \( [y,z] = \gamma u \), where \( \gamma = \alpha \), then by Lemma 3.2, we have \( \dim \mathcal{M}(N) = 0 \), which is a contradiction. If \( N = \langle x,z,c,r,u,t \rangle \) such that \( [y,z] = c, [x,r] = t \), then Lemma 3.2 implies that \( \dim \mathcal{M}(N) = 1 \). Now, consider the Lie algebra analogue of Ganea’s exact sequence (see [8], Theorem 2.6.5) as follows

\[
Z \otimes N/N^2 \rightarrow \mathcal{M}(N) \rightarrow \mathcal{M}(N/Z) \rightarrow N^2 \cap Z \rightarrow 0,
\]
where \( Z \) is a central ideal of \( N \). Thus \( Z(N) = \langle u, t \rangle \) and \( \dim \mathcal{M}(N/Z) \leq \dim \mathcal{M}(N) + \dim(N^2 \cap Z) \). If \( Z = \langle u \rangle \), then \( \dim \mathcal{M}(N/\langle u \rangle)) = 2 - l \), for some non-negative integer \( l \). Now if \( l = 0 \), then Theorem 2.1 implies that \( N/\langle u \rangle \) is abelian, which is impossible. If \( l = 1, 2 \), then \( N/\langle u \rangle \cong H(1) \) or \( H(1) \oplus A(1) \), which is a contradiction and also if \( Z = \langle u, t \rangle \), then by an analogous manner, we can get a contradiction. For the other cases of dimensions of \( N \) and \( K \), there are no ideal and subalgebra satisfying in the conditions of the theorem. Also, if \( t(K) = 1, 2, \ldots, 8 \), then there is not any pair.

Now, assume that \( t(K) = 9 \). If \( \dim K = 1, 2, 3 \), then \( K \) is abelian or \( H(1) \). Therefore \( t(K) = 0 \) or \( 1 \), which is a contradiction. If \( \dim K = 4 \), then \( \dim \mathcal{M}(K) < 0 \), which is impossible. If \( \dim K = 5 \) and \( \dim N = 2 \), then there are no ideal and subalgebra satisfying in the theorem. And finally, if \( \dim K = 6 \) and \( \dim N = 1 \), then \( \dim \mathcal{M}(N, L) < 0 \), which is impossible. In the case \( t(K) = 10, 11, 12 \) by a similar method, we may get a contradiction.

Let \( L \cong L'(17, 7, 1, 7) \) and \( t(K) = 0 \). Similarly (the case \((s(L), t(K)) = (12, 0) \) and by using Ganea’s exact sequence), there is not any pair \((N, L) \). Also, there are no ideal and subalgebra satisfying in the theorem, for \( t(K) = 1, 2, \ldots, 12 \).

Case \( s(L) = 13 \). In this case \( L \cong L(17, 7, 1, 18) \) and we can check that there is not any pair. The proof is complete. \( \square \)
Table I:

<table>
<thead>
<tr>
<th>t(L)</th>
<th>dim L</th>
<th>Non Zero Multiplication</th>
<th>Nilpotent Lie algebra</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td></td>
<td>Abelian</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
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<td>$H(1)$</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>$x_1, x_2 = x_3$</td>
<td>$H(1) \oplus A(1)$</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
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<td>$H(1) \oplus A(2)$</td>
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<td>4</td>
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<td>$x_1, x_2 = x_3$</td>
<td>$H(1) \oplus A(3)$</td>
</tr>
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<td>5</td>
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</tr>
<tr>
<td>5</td>
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<td>$x_1, x_2 = x_3$</td>
<td>$H(1) \oplus A(4)$</td>
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<td>5</td>
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<tr>
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<td>$x_1, x_2 = x_5 + \beta_1 x_6, x_3, x_4 = x_5$</td>
<td>$L(7, 6, 2, 7, \beta_1, \beta_2)$</td>
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<td>7</td>
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<td>$H(2) \oplus A(2)$</td>
</tr>
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<td>7</td>
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<td>$H(3)$</td>
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<tr>
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<td>$L(3, 4, 1, 4) \oplus A(2)$</td>
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<td>8</td>
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<td>$H(2) \oplus A(3)$</td>
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<td>8</td>
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Table II:

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<th>dim L</th>
<th>Non Zero Multiplication</th>
<th>Filiform Lie algebra</th>
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<tbody>
<tr>
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<td>4</td>
<td>$[x, y] = z$, $[x, z] = r$</td>
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<tr>
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<td>5</td>
<td>$[x, y] = z$, $[x, z] = c$, $[x, c] = r$</td>
<td>$L(7, 5, 1, 7)$</td>
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<tr>
<td>11</td>
<td>6</td>
<td>$[x, y] = z$, $[x, z] = c$, $[x, c] = r$</td>
<td>$L'(7, 5, 1, 7)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$[x, r] = \alpha_4 t$, $[y, z] = \alpha_5 t$</td>
<td></td>
</tr>
<tr>
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<td>$[y, r] = t$, $[z, c] = -t$</td>
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<td>$L'(7, 5, 1, 7)$</td>
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<td>$[y, c] = \beta_6 u$</td>
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<tr>
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<td>$L(17, 7, 1, 17)$</td>
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<td>$\beta_6 = \alpha_5$</td>
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<td>$[y, r] = \beta_7 u$, $[z, c] = \beta_8 u$</td>
<td>$\beta_8 = 1 - \beta_7$</td>
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<td>18</td>
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<td>$[x, r] = t$, $[y, z] = \alpha_5 t + \beta_5 u$</td>
<td>$\beta_7 = -\beta_8$</td>
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<td>$[y, c] = t + \beta_6 u$, $[y, r] = \beta_7 u$</td>
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<td></td>
<td></td>
<td>$[z, c] = \beta_8 u$</td>
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</tbody>
</table>

References


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