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Measures of Comparative Growth Analysis of Composite Entire and Meromorphic Functions

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Abstract. In this paper we establish some newly developed results related to the growth rates of composite entire and meromorphic functions on the basis of their generalized relative order and generalized relative lower order respectively denoted by $\rho_g^{[l]}(f)$ and $\lambda_g^{[l]}(f)$ where l is any positive integer with meromorphic f and entire g.

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1. Introduction

Let f be an entire function defined in the finite complex plane $\mathbb{C}$. The maximum modulus function corresponding to entire f is defined as

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$M_f(r) = \max\{|f(z)| : |z| = r\}$. In this connection we state the following property:

Property (A) [2]: A non-constant entire function f is said have the Property (A) if for any $\sigma > 1$ and for all sufficiently large values of r, $[M_f(r)]^2 \leqslant M_f(r^\sigma)$ holds.

For examples of functions with or without the Property (A), one may see [2].
It is to be noted that whenever f is constant, the equality in $[M_f(r)]^2 \leqslant M_f(r^\sigma)$ as mentioned in Property (A) holds.
When f is meromorphic, $M_f(r)$ can not be defined as f is not analytic. In this case one may define another function $T_f(r)$ known as Nevanlinna's Characteristic function of f, playing the same role as maximum modulus function in the following manner:

$$T_f(r) = N_f(r) + m_f(r),$$

where the function $N_f(r,a)$ $\left(\bar{N}_f(r,a)\right)$ known as counting function of a-points (distinct a-points) of meromorphic f is defined as

$$N_f(r,a) = \int_0^r \frac{n_f(t,a) - n_f(0,a)}{t} dt + \bar{n}_f(0,a)\log r$$

$$\left(\bar{N}_f(r,a) = \int_0^r \frac{\bar{n}_f(t,a) - \bar{n}_f(0,a)}{t} dt + \bar{n}_f(0,a)\log r\right).$$

Moreover we denote by $n_f(r,a)$ $\left(\bar{n}_f(r,a)\right)$ the number of a-points (distinct a-points) of f in $|z| \leqslant r$ and an ∞ -point is a pole of f. In many occasions $N_f(r,\infty)$ and $\bar{N}_f(r,\infty)$ are denoted by $N_f(r)$ and $\bar{N}_f(r)$ respectively.
Also the function $m_f(r,\infty)$ alternatively denoted by $m_f(r)$ known as the proximity function of f is defined as follows:

$$m_f(r) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left| f\left(re^{i\theta}\right)\right| d\theta,$$

where $\log^+ x\& = \max(\log x, 0)$for all $x \geqslant 0$.
Also we may denote $m\left(r, \frac{1}{f-a}\right)$ by $m_f(r, a)$.
If f is entire function, then the Nevanlinna's Characteristic function $T_f(r)$ of f is defined as

$$T_f(r) = m_f(r).$$

For any two entire functions f and g, the ratio $\frac{M_f(r)}{M_g(r)}$ as $r \to \infty$ is called the growth of f with respect to g in terms of their maximum moduli. Also the ratio $\frac{T_f(r)}{T_g(r)}$ as $r \to \infty$ is called the growth of f with respect to g in terms of the Nevanlinna's Characteristic functions when f and g are both meromorphic functions. Accordingly the study of comparative growth properties of entire and meromorphic functions which is one of the prominent branch as of the value distribution theory of entire and meromorphic functions is the prime concern of the paper. We do not explain the standard definitions and notations in the theory of entire and meromorphic functions as those are available in [13] and [16]. In the sequel the following two notations are used:

$$\begin{aligned} \log^{[k]} x &= \log\left(\log^{[k-1]} x\right) \text{ for } k = 1, 2, 3, \cdots; \\ \log^{[0]} x &= x \end{aligned}$$

and

$$\begin{aligned} \exp^{[k]} x &= \exp\left(\exp^{[k-1]} x\right) \text{ for } k = 1, 2, 3, \cdots; \\ \exp^{[0]} x &= x. \end{aligned}$$

Taking this into account the *generalized order* (respectively, *generalized lower order*) of an entire function f as introduced by Sato [15] is given by:

$$\rho_f^{[l]} = \limsup_{r\to\infty} \frac{\log^{[l]} M_f(r)}{\log\log M_{\exp z}(r)} = \limsup_{r\to\infty} \frac{\log^{[l]} M_f(r)}{\log r}$$

$$\left(\text{respectively } \lambda_f^{[l]} = \liminf_{r\to\infty} \frac{\log^{[l]} M_f(r)}{\log\log M_{\exp z}(r)} = \liminf_{r\to\infty} \frac{\log^{[l]} M_f(r)}{\log r}\right)$$

where $l \geqslant 1$.
When f is meromorphic function, one can easily verify that

$$\rho_f^{[l]} = \limsup_{r\to\infty} \frac{\log^{[l-1]} T_f(r)}{\log T_{\exp z}(r)} = \limsup_{r\to\infty} \frac{\log^{[l-1]} T_f(r)}{\log\left(\frac{r}{\pi}\right)} = \limsup_{r\to\infty} \frac{\log^{[l-1]} T_f(r)}{\log r + O(1)}$$

$$\left(\text{respectively } \lambda_f^{[l]} = \liminf_{r\to\infty} \frac{\log^{[l-1]} T_f(r)}{\log T_{\exp z}(r)} = \liminf_{r\to\infty} \frac{\log^{[l-1]} T_f(r)}{\log r + O(1)}\right)$$

where $l \geqslant 1$.
These definitions extend the definitions of *order* ρ_f and *lower order* λ_f of an entire or meromorphic function f since for $l = 2$, these correspond to the particular case $\rho_f^{[2]} = \rho_f$ and $\lambda_f^{[2]} = \lambda_f$.
Given a non-constant entire function g defined in the open complex plane $\mathbb{C}$, its maximum modulus function $M_g(r)$ and Nevanlinna's Characteristic function $T_g(r)$ are both strictly increasing and continuous functions of r. Also their inverses $M_g^{-1}(r) : (|g(0)|, \infty) \to (0, \infty)$ and $T_g^{-1} : (T_g(0), \infty) \to (0, \infty)$ exists respectively and are such that $\lim_{s\to\infty} M_g^{-1}(s) = \infty$ and $\lim_{s\to\infty} T_g^{-1}(s) = \infty$.
Extending the idea of relative order of entire functions as established by Bernal $\{[1], [2]\}$, Lahiri and Banerjee [14] introduced the definition of relative order of a meromorphic function f with respect to another entire function g, denoted by $\rho_g(f)$ to avoid comparing growth just with $\exp z$ as follows:

$$\begin{aligned} \rho_g(f) &= \inf\{\mu > 0 : T_f(r) < T_g(r^\mu) \text{ for all large } r\} \\ &= \limsup_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log r}. \end{aligned}$$

The definition coincides with the classical one if $g(z) = \exp z$ $\{$cf. [14] $\}$.

Likewise, one can define the relative lower order of a meromorphic function f with respect to an entire function g denoted by $\lambda_g(f)$ as follows:

$$\lambda_g(f) = \liminf_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log r}.$$

Further, Banerjee and Jana [4] gave a more generalized concept of relative order a meromorphic function with respect to an entire function in the following way:

Definition 1.1. [4] *If $l \geqslant 1$ is a positive integer, then the l-* th generalized relative order *of a meromorphic function f with respect to an entire function g, denoted by $\rho_g^{[l]}(f)$ is defined by*

$$\rho_g^{[l]}(f) = \limsup_{r\to\infty} \frac{\log^{[l]} T_g^{-1} T_f(r)}{\log r}.$$

Likewise one can define the generalized relative lower order *of a meromorphic function f with respect to an entire function g denoted by $\lambda_g^{[l]}(f)$ as*

$$\lambda_g^{[l]}(f) = \liminf_{r\to\infty} \frac{\log^{[l]} T_g^{-1} T_f(r)}{\log r}.$$

For entire and meromrophic functions, the notions of thier growth indicators such as *order* is classical in complex analysis and during the past decades, several researchers have already been exploring their studies in the area of comparative growth properties of composite entire and meromorphic functions in different directions using the classical growth indicators. But at that time, the concepts of *relative orders* and consequently the *generalized relative orders* of entire and meromorphic functions with respect to another entire function and as well as their technical advantages of not comparing with the growths of $\exp z$ are not at all known to the researchers of this area. Therefore the growth of composite entire and meromorphic functions needs to be modified on the basis of their *relative order* some of which has been explored in [6], [7], [8], [9], [10], [11] and [12]. In this paper we establish some newly developed results related to the growth rates of composite entire and meromorphic functions on the basis of their *generalized relative orders (* respectively *generalized relative lower orders).*

2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 2.1. [5] *If f and g are two entire functions then for all sufficiently large values of r,*

$$M_f\left(\frac{1}{8}T_g\left(\frac{r}{2}\right)-|g(0)|\right)\leqslant M_{f\circ g}(r).$$

Lemma 2.2. [3] *Let f be meromorphic and g be entire then for all sufficiently large values of r,*

$$T_{f\circ g}(r)\leqslant\{1+o(1)\}\frac{T_g(r)}{\log M_g(r)}T_f(M_g(r)).$$

Lemma 2.3. [12] *Let f be an entire function which satisfies the Property (A), $\beta>0$, $\delta>1$ and $\alpha>2$. Then*

$$\beta T_f(r)<T_f\left(\alpha r^{\delta}\right).$$

3. Main Results

In this section we present the main results of the paper.

Theorem 3.1. *If f be meromorphic and g, h be any two entire functions such that $\lambda_g^{[m]}<\lambda_h^{[l]}(f)\leqslant\rho_h^{[l]}(f)<\infty$ where l and m are integers with $l>1$ and $m>2$. Also let h satisfies the Property (A). Then*

$$\liminf_{r\to\infty}\frac{\log^{[l]}T_h^{-1}T_{f\circ g}(r)}{\log^{[l-m+1]}T_h^{-1}T_f(r)}=0\ .$$

Proof. Let $\beta>2$ and $\delta>1$. Since $T_h^{-1}(r)$ is an increasing function of r, it follows from Lemma 2.2 and Lemma 2.3, for all sufficiently large values of r that

$$\begin{aligned}T_h^{-1}T_{f\circ g}(r)&\leqslant T_h^{-1}\left[\{1+o(1)\}T_f(M_g(r))\right]\\ i.e.,\ T_h^{-1}T_{f\circ g}(r)&\leqslant\beta\left[T_h^{-1}T_f(M_g(r))\right]^{\delta}\\ i.e.,\ \log^{[l]}T_h^{-1}T_{f\circ g}(r)&\leqslant\log^{[l]}T_h^{-1}T_f(M_g(r))+O(1)\ .\end{aligned}$$

From above we get for a sequence of values of r tending to infinity,

$$\log^{[l]} T_h^{-1} T_{f\circ g}(r) \leqslant \left(\rho_h^{[l]}(f) + \varepsilon\right) \log M_g(r) + O(1) \tag{1}$$

$$i.e., \ \log^{[l]} T_h^{-1} T_{f\circ g}(r) \leqslant \left(\rho_h^{[l]}(f) + \varepsilon\right) \exp^{[m-2]} r^{\lambda_g^{[m]}+\varepsilon} + O(1)\ . \tag{2}$$

Again from the definition of relative order, we obtain for all sufficiently large values of r that

$$\log^{[l-m+1]} T_h^{-1} T_f(r) \geqslant \exp^{[m-2]} r^{\left(\lambda_h^{[l]}(f)-\varepsilon\right)}. \tag{3}$$

In view of (2) and (3), we get for a sequence of values of r tending to infinity that

$$\frac{\log^{[l]} T_h^{-1} T_{f\circ g}(r)}{\log^{[l-m+1]} T_h^{-1} T_f(r)} < \frac{\left(\rho_h^{[l]}(f) + \varepsilon\right) \exp^{[m-2]} r^{\lambda_g^{[m]}+\varepsilon} + O(1)}{\exp^{[m-2]} r^{\left(\lambda_h^{[l]}(f)-\varepsilon\right)}}. \tag{4}$$

Now as $\lambda_g^{[m]} < \lambda_h^{[l]}(f)$, we can choose $\varepsilon\,(>0)$ in such a way that $\lambda_g^{[m]}+\varepsilon < \lambda_h^{[l]}(f) - \varepsilon$ and the theorem follows from (4). □

Remark 3.2. *If we take* $\rho_g^{[m]} < \lambda_h^{[l]}(f) \leqslant \rho_h^{[l]}(f) < \infty$ *instead of* $\lambda_g^{[m]} < \lambda_h^{[l]}(f) \leqslant \rho_h^{[l]}(f) < \infty$ *and the other conditions remain the same, the conclusion of Theorem 3.1 remains valid with "limit inferior " replaced by " limit " as we see in the following theorem.*

Theorem 3.3. *If f be meromorphic and g, h be any two entire functions such that* $\rho_g^{[m]} < \lambda_h^{[l]}(f) \leqslant \rho_h^{[l]}(f) < \infty$ *where l and m are integers with $l > 1$ and $m > 2$. Also let h satisfy the Property (A). Then*

$$\lim_{r\to\infty} \frac{\log^{[l]} T_h^{-1} T_{f\circ g}(r)}{\log^{[l-m+1]} T_h^{-1} T_f(r)} = 0\ .$$

Proof. Let us consider $\beta > 2$ and $\delta > 1$. As $T_h^{-1}(r)$ is an increasing function of r, in view of Lemma 2.2 we get from(1) for all sufficiently large values of r that

$$\log^{[l]} T_h^{-1} T_{f\circ g}(r) \leqslant \left(\rho_h^{[l]}(f) + \varepsilon\right) \exp^{[m-2]} r^{\rho_g^{[m]}+\varepsilon} + O(1). \tag{5}$$

Now combining (3) and (5), it follows for all sufficiently large values of r,

$$\frac{\log^{[l]} T_h^{-1} T_{f\circ g}(r)}{\log^{[l-m+1]} T_h^{-1} T_f(r)} \leqslant \frac{\left(\rho_h^{[l]}(f)+\varepsilon\right)\exp^{[m-2]} r^{\rho_g^{[m]}+\varepsilon}+O(1)}{\exp^{[m-2]} r^{\left(\lambda_h^{[l]}(f)-\varepsilon\right)}}. \qquad (6)$$

As $\rho_g^{[m]} < \lambda_h^{[l]}(f)$ we can choose $\varepsilon\,(>0)$ in such a manner that $\rho_g^{[m]}+\varepsilon < \lambda_h^{[l]}(f)-\varepsilon$ and thus the theorem follows from(6). □

Theorem 3.4. *Let g, q, h and k be any four entire functions such that h satisfies the Property (A), $\lambda_h^{[l]}(q)>0$ and $\rho_g^{[m]} < \lambda_k^{[m]}$ where l and m are integers with $l>1$ and $m>2$. Then for every meromorphic function f with $0<\rho_h^{[l]}(f)<\infty$,*

$$\lim_{r\to\infty}\frac{\log^{[l]} T_h^{-1} T_{q\circ k}(r)}{\log^{[l]} T_h^{-1} T_{f\circ g}(r)} = \infty.$$

Proof. Since $\rho_g^{[m]} < \lambda_k^{[m]}$ we can choose $\varepsilon\ (>0)$ in such a way that

$$\rho_g^{[m]}+\varepsilon < \lambda_k^{[m]}-\varepsilon. \qquad (7)$$

Now in view of Lemma 2.1 and in view of the inequality $T_n(r) \leqslant \log M_n(r) \leqslant 3T_n(2r)$ {cf. [5]}for any entire n , we get for all sufficiently large values of r that

$$\begin{aligned} M_{q\circ k}(r) &\geqslant M_q\left(\frac{1}{16}M_k\left(\frac{r}{2}\right)\right) \\ i.e.,\ 3T_{q\circ k}(r) &\geqslant T_q\left(\frac{1}{32}M_k\left(\frac{r}{2}\right)\right). \end{aligned}$$

Since $T_h^{-1}(r)$ is an increasing function of r, we obtain from above for any $\beta>2$, $\delta>1$ and for all sufficiently large values of r that

$$
\begin{aligned}
T_h^{-1}\left[3T_{q\circ k}(r)\right] &\geqslant T_h^{-1}T_{ql}\left(\frac{1}{32}M_k\left(\frac{r}{2}\right)\right)\\
i.e.,\ T_h^{-1}T_{q\circ k}(r) &\geqslant \frac{1}{\beta}\left[T_h^{-1}T_q\left(\frac{1}{32}M_k\left(\frac{r}{2}\right)\right)\right]^{\frac{1}{\delta}}\\
i.e.,\ \log^{[l]}T_h^{-1}T_{q\circ k}(r) &\geqslant \log^{[l]}T_h^{-1}T_q\left(\frac{1}{32}M_k\left(\frac{r}{2}\right)\right)+O(1)\\
i.e.,\ \log^{[l]}T_h^{-1}T_{q\circ k}(r) &\geqslant \left(\lambda_h^{[l]}(q)-\varepsilon\right)\log M_k\left(\frac{r}{2}\right)+O(1) \qquad (8)\\
i.e.,\ \log^{[l]}T_h^{-1}T_{q\circ k}(r) &\geqslant \left(\lambda_h^{[l]}(q)-\varepsilon\right)\exp^{[m-2]}\left(\frac{r}{2}\right)^{\lambda_k^{[m]}-\varepsilon}+O(1). \qquad (9)
\end{aligned}
$$

Now from (5), (7) and (9) it follows for all sufficiently large values of r that

$$
\frac{\log^{[l]}T_h^{-1}T_{q\circ k}(r)}{\log^{[l]}T_h^{-1}T_{f\circ g}(r)} \geqslant \frac{\left(\lambda_h^{[l]}(q)-\varepsilon\right)\exp^{[m-2]}\left(\frac{r}{2}\right)^{\lambda_k^{[m]}-\varepsilon}+O(1)}{\left(\rho_h^{[l]}(f)+\varepsilon\right)\exp^{[m-2]}r^{\rho_g^{[m]}+\varepsilon}+O(1)}
$$

$$
i.e.,\ \liminf_{r\to\infty}\frac{\log^{[l]}T_h^{-1}T_{q\circ k}(r)}{\log^{[l]}T_h^{-1}T_{f\circ g}(r)}=\infty,
$$

from which the theorem follows. □

Theorem 3.5. *Let g, q, h and k be any four entire functions such that h satisfy the Property (A), $\lambda_h^{[l]}(q)>0$ and $\rho_g^{[m]}<\lambda_k^{[m]}$ where l and m are integers with $l>1$ and $m>2$. Then for every meromorphic function f with $\rho_h^{[l]}(f)<\infty$,*

$$
\lim_{r\to\infty}\frac{\log^{[l-1]}T_h^{-1}T_{q\circ k}(r)}{\log^{[l-1]}T_h^{-1}T_{f\circ g}(r)\cdot\log^{[l-1]}T_h^{-1}T_f(r)}=\infty.
$$

Proof. For any $\delta>1$, we obtain from (9) and (5) for all sufficiently large values of r that

$$
\log^{[l-1]}T_h^{-1}T_{q\circ k}(r)\geqslant\exp\left[\left(\lambda_h^{[l]}(q)-\varepsilon\right)\exp^{[m-2]}\left(\frac{r}{2}\right)^{\lambda_k^{[m]}-\varepsilon}+O(1)\right] \qquad (10)
$$

and

$$\log^{[l-1]} T_h^{-1} T_{f\circ g}(r) \leqslant \exp\left[\left(\rho_h^{[l]}(f)+\varepsilon\right)\exp^{[m-2]} r^{\rho_g^{[m]}+\varepsilon} + O(1)\right]. \qquad (11)$$

Again from the definition of relative order we have for all sufficiently large values of r that

$$\log^{[l]} T_h^{-1} T_f(r) \leqslant \left(\rho_h^{[l]}(f)+\varepsilon\right)\log r$$

$$i.e., \ \log^{[l-1]} T_h^{-1} T_f(r) \leqslant r^{\left(\rho_h^{[l]}(f)+\varepsilon\right)}. \qquad (12)$$

From (11) and (12), it follows for all sufficiently large values of r that

$$\log^{[l-1]} T_h^{-1} T_{f\circ g}(r) \cdot \log^{[l-1]} T_h^{-1} T_f(r)$$
$$\leqslant r^{\left(\rho_h^{[l]}(f)+\varepsilon\right)} \cdot \exp\left[\left(\rho_h^{[l]}(f)+\varepsilon\right)\exp^{[m-2]} r^{\rho_g^{[m]}+\varepsilon} + O(1)\right]. \qquad (13)$$

Combining (10) and (13), we get for all sufficiently large values of r that

$$\frac{\log^{[l-1]} T_h^{-1} T_{q\circ k}(r)}{\log^{[l-1]} T_h^{-1} T_{f\circ g}(r) \cdot \log^{[l-1]} T_h^{-1} T_f(r)}$$
$$\geqslant \frac{\exp\left[\left(\lambda_h^{[l]}(q)-\varepsilon\right)\exp^{[m-2]}\left(\frac{r}{2}\right)^{\lambda_k^{[m]}-\varepsilon} + O(1)\right]}{r^{\left(\rho_h^{[l]}(f)+\varepsilon\right)} \cdot \exp\left[\left(\rho_h^{[l]}(f)+\varepsilon\right)\exp^{[m-2]} r^{\rho_g^{[m]}+\varepsilon} + O(1)\right]}. \qquad (14)$$

Since $\rho_g^{[m]} < \lambda_k^{[m]}$, we can choose $\varepsilon\,(>0)$ in such a manner that

$$\rho_g^{[m]} + \varepsilon < \lambda_k^{[m]} - \varepsilon. \qquad (15)$$

Thus the theorem follows from (14) and (15). □

Remark 3.6. *If we consider* $\rho_g^{[m]} < \rho_k^{[m]}$ *instead of* $\rho_g^{[m]} < \lambda_k^{[m]}$ *and the other conditions remain the same, the conclusion of Theorem 3.5 remains valid with "limit superior " replaced by " limit " as we see in the following theorem.*

Theorem 3.7. *Let* g, q, h *and* k *be four entire functions such that* h *satisfy the Property (A),* $\lambda_h^{[l]}(q) > 0$ *and* $\rho_g^{[m]} < \rho_k^{[m]}$ *where* l *and* m *are* *integers with $l > 1$ and $m > 2$. Then for every meromorphic function f with $\rho_h^{[l]}(f) < \infty$,*

$$\limsup_{r\to\infty} \frac{\log^{[l-1]} T_h^{-1} T_{q\circ k}(r)}{\log^{[l-1]} T_h^{-1} T_{f\circ g}(r) \cdot \log^{[l-1]} T_h^{-1} T_f(r)} = \infty.$$

Proof. As $\rho_g^{[m]} < \rho_k^{[m]}$, we can choose $\varepsilon\,(> 0)$ in such a manner that

$$\rho_g^{[m]} + \varepsilon < \lambda_k^{[m]} - \varepsilon. \tag{16}$$

Now for any $\delta > 1$, we get from (8) for a sequence of values of r tending to infinity that

$$\log^{[l]} T_h^{-1} T_{q\circ k}(r) \geqslant \left(\lambda_h^{[l]}(q) - \varepsilon\right) \exp^{[m-2]} \left(\frac{r}{2}\right)^{\rho_k^{[m]}-\varepsilon} + O(1)$$

$$i.e.,\ \log^{[l-1]} T_h^{-1} T_{q\circ k}(r) \geqslant \exp\left[\left(\lambda_h^{[l]}(q) - \varepsilon\right) \exp^{[m-2]} \left(\frac{r}{2}\right)^{\rho_k^{[m]}-\varepsilon} + O(1)\right]. \tag{17}$$

Therefore combining (13) and (17), we obtain for a sequence of values of r tending to infinity that

$$\frac{\log^{[l-1]} T_h^{-1} T_{q\circ k}(r)}{\log^{[l-1]} T_h^{-1} T_{f\circ g}(r) \cdot \log^{[l-1]} T_h^{-1} T_f(r)} \geqslant \frac{\exp\left[\left(\lambda_h^{[l]}(q) - \varepsilon\right) \exp^{[m-2]} \left(\frac{r}{2}\right)^{\rho_k^{[m]}-\varepsilon} + O(1)\right]}{r^{\left(\rho_h^{[l]}(f)+\varepsilon\right)} \cdot \exp\left[\left(\rho_h^{[l]}(f) + \varepsilon\right) \exp^{[m-2]} r^{\rho_g^{[m]}+\varepsilon} + O(1)\right]}. \tag{18}$$

Thus in view of (16), the theorem follows from (18). □

In the line of Theorem 3.5 and Theorem 3.7, the following two theorems can be carried out. Hence their proofs are omitted.

Theorem 3.8. *Let g, p, q, h and k be five entire functions such that q and h both satisfy the Property (A), $\lambda_q^{[n]}(p) > 0$ and $\rho_h^{[l]}(g) < \infty$ where*

l and n are integers with $l > 1$ and $n > 1$. Then for every meromorphic function f with $\rho_h^{[l]}(f) < \infty$,

$$(i)\ \lim_{r\to\infty} \frac{\log^{[n-1]} T_q^{-1} T_{p\circ k}(r)}{\log^{[l-1]} T_h^{-1} T_{f\circ g}(r) \cdot \log^{[l-1]} T_h^{-1} T_f(r)} = \infty$$

and

$$(ii)\ \lim_{r\to\infty} \frac{\log^{[n-1]} T_q^{-1} T_{p\circ k}(r)}{\log^{[l-1]} T_h^{-1} T_{f\circ g}(r) \cdot \log^{[l-1]} T_h^{-1} T_g(r)} = \infty$$

when $\rho_g^{[m]} < \lambda_k^{[m]}$ for any integer $m > 2$.

Theorem 3.9. *Let g, p, q, h and k be five entire functions such that q and h both satisfy the Property (A), $\lambda_q^{[n]}(p) > 0$ and $\rho_h^{[l]}(g) < \infty$ where l and n are integers with $l > 1$ and $n > 1$. Then for every meromorphic function f with $\rho_h(f) < \infty$,*

$$(i)\ \limsup_{r\to\infty} \frac{\log^{[n-1]} T_q^{-1} T_{p\circ k}(r)}{\log^{[l-1]} T_h^{-1} T_{f\circ g}(r) \cdot \log^{[l-1]} T_h^{-1} T_f(r)} = \infty$$

and

$$(ii)\ \limsup_{r\to\infty} \frac{\log^{[n-1]} T_q^{-1} T_{p\circ k}(r)}{\log^{[l-1]} T_h^{-1} T_{f\circ g}(r) \cdot \log^{[l-1]} T_h^{-1} T_g(r)} = \infty$$

when $\rho_g^{[m]} < \rho_k^{[m]}$ for any integer $m > 2$.

Theorem 3.10. *Let h be an entire funtion satisfying the Property (A) and f be a meromorphic function such that $0 < \lambda_h^{[l]}(f) \leqslant \rho_h^{[l]}(f) < \infty$. Then for any entire function g with $\rho_g^{[m]} < \infty$,*

$$\limsup_{r\to\infty} \frac{\log^{[l+m-1]} T_h^{-1} T_{f\circ g}(r)}{\log^{[l]} T_h^{-1} T_f(r)} \leqslant \frac{\rho_g^{[m]}}{\lambda_h^{[l]}(f)},$$

where l and m are integers with $l > 1$ and $m > 2$.

Proof. From (1), it follows for all sufficiently large values of r that

$$\log^{[l+m-1]} T_h^{-1} T_{f\circ g}(r) \leqslant \log^{[m]} M_g(r) + O(1)$$

$$i.e., \quad \frac{\log^{[l+m-1]} T_h^{-1} T_{f\circ g}(r)}{\log^{[l]} T_h^{-1} T_f(r)} \leqslant \frac{\log^{[m]} M_g(r) + O(1)}{\log r} \cdot \frac{\log r}{\log^{[l]} T_h^{-1} T_f(r)},$$

$$i.e., \quad \limsup_{r\to\infty} \frac{\log^{[l+m-1]} T_h^{-1} T_{f\circ g}(r)}{\log^{[l]} T_h^{-1} T_f(r)} \leqslant \limsup_{r\to\infty} \frac{\log^{[m]} M_g(r) + O(1)}{\log r} \cdot \limsup_{r\to\infty} \frac{\log r}{\log^{[l]} T_h^{-1} T_f(r)},$$

$$i.e., \quad \limsup_{r\to\infty} \frac{\log^{[l+m-1]} T_h^{-1} T_{f\circ g}(r)}{\log^{[l]} T_h^{-1} T_f(r)} \leqslant \rho_g^{[m]} . \frac{1}{\lambda_h^{[l]}(f)} = \frac{\rho_g^{[m]}}{\lambda_h^{[l]}(f)}.$$

This proves the theorem. □

Theorem 3.11. *Let f be a meromorphic function and g, h be two entire functions satisfying (i) $\rho_h^{[l]}(f) < \infty$, (ii) $\lambda_h^{[l]}(g) > 0$ and (iii) $\rho_g^{[m]} < \infty$ where l and m are integers with $l > 1$ and $m > 2$. Then*

$$\limsup_{r\to\infty} \frac{\log^{[l+m-1]} T_h^{-1} T_{f\circ g}(r)}{\log^{[l]} T_h^{-1} T_g(r)} \leqslant \frac{\rho_g^{[m]}}{\lambda_h^{[l]}(g)},$$

when h follows the Property (A).

The proof of Theorem 3.11 is omitted as it can be carried out in the line of Theorem 3.10.

Theorem 3.12. *Let f be meromorphic and g, h be any two entire functions such that h satisfy the Property (A) and $0 < \lambda_h^{[l]}(f) \leqslant \rho_h^{[l]}(f) < \infty$. Then for any entire g with $\rho_g^{[m]} < \infty$ where l and m are integers with $l > 1$ and $m > 2$,*

$$\lim_{r\to\infty} \frac{\log^{[l]} T_h^{-1} T_f\left(\exp^{[m-1]} r^{\mu}\right)}{\log^{[l]} T_h^{-1} T_{f\circ g}(r)} = \infty ,$$

where $\rho_g^{[m]} < \mu < \infty$.

Proof. From the definition of the generalized relative lower order, we obtain for all sufficiently large values of r that

$$\log^{[l]} T_h^{-1} T_f \left(\exp^{[m-1]} r^{\mu}\right) \geqslant \left(\lambda_h^{[l]}(f) - \varepsilon\right) \log\left\{\exp^{[m-1]} r^{\mu}\right\}$$

$$i.e., \ \log^{[l]} T_h^{-1} T_f \left(\exp^{[m-1]} r^{\mu}\right) \geqslant \left(\lambda_h^{[l]}(f) - \varepsilon\right) \exp^{[m-2]} r^{\mu}. \tag{19}$$

Now from (5) and (19), it follows for all sufficiently large values of r that

$$\frac{\log^{[l]} T_h^{-1} T_f \left(\exp^{[m-1]} r^{\mu}\right)}{\log^{[l]} T_h^{-1} T_{f\circ g}(r)} \geqslant \frac{\left(\lambda_h^{[l]}(f) - \varepsilon\right) \exp^{[m-2]} r^{\mu}}{\left(\rho_h^{[l]}(f) + \varepsilon\right) \exp^{[m-2]} r^{\rho_g^{[m]} + \varepsilon} + O(1)} . \tag{20}$$

As $\rho_g^{[m]} < \mu$, we can choose $\varepsilon\,(> 0)$ in such a way that

$$\rho_g^{[m]} + \varepsilon < \mu . \tag{21}$$

Thus from (20) and (21), we obtain that

$$\lim_{r\to\infty} \frac{\log^{[l]} T_h^{-1} T_f \left(\exp^{[m-1]} r^{\mu}\right)}{\log^{[l]} T_h^{-1} T_{f\circ g}(r)} = \infty .$$

Thus the theorem follows. □

In the line of Theorem 3.12, we may state the following theorem without its proof.

Theorem 3.13. *Let f be meromorphic and g, h be any two entire functions such that h satisfy the Property (A), $\lambda_h^{[l]}(g) > 0$ and $\rho_h^{[l]}(f) < \infty$. Then for every μ with $\rho_g^{[m]} < \mu < \infty$,*

$$\lim_{r\to\infty} \frac{\log^{[l]} T_h^{-1} T_g \left(\exp^{[m-1]} r^{\mu}\right)}{\log^{[l]} T_h^{-1} T_{f\circ g}(r)} = \infty$$

where l and m are integers with $l > 1$ and $m > 2$.

Corollary 3.14. *Under the assumptions of Theorem 3.12,*

$$\lim_{r\to\infty} \frac{T_h^{-1} T_f \left(\exp^{[m-1]} r^{\mu}\right)}{T_h^{-1} T_{f\circ g}(r)} = \infty, \qquad \rho_g^{[m]} < \mu < \infty.$$

Proof. In view of Theorem 3.12, we get for all sufficiently large values of r that

$$\begin{aligned} \log^{[l]} T_h^{-1} T_f \left(\exp r^{\mu}\right) &\geqslant A \log^{[L]} T_h^{-1} T_{f \circ g}\left(r\right) \text{ for } A > 1 \\ i.e., \ \log^{[l-1]} T_h^{-1} T_f \left(\exp r^{\mu}\right) &\geqslant \left[\log^{[l-1]} T_h^{-1} T_{f \circ g}\left(r\right)\right]^A, \end{aligned}$$

from which the corollary follows. □

Corollary 3.15. *Under the assumptions of Theorem 3.13,*

$$\lim_{r\to\infty} \frac{T_h^{-1} T_g \left(\exp^{[m-1]} r^{\mu}\right)}{T_h^{-1} T_{f \circ g}\left(r\right)} = \infty, \qquad \rho_g^{[m]} < \mu < \infty.$$

The proof of the above corollary is omitted as it may be carried out in the line of Corollary 3.14 and from Theorem 3.13 respectively.

Theorem 3.16. *Let f be meromorphic and g, h be any two entire functions such that (i) $\rho_h^{[l]}\left(f \circ g\right) < \infty$ and (ii) $\lambda_h^{[l]}\left(g\right) > 0$ where l is any integer with $l > 1$. Then*

$$\lim_{r\to\infty} \frac{\left[\log^{[l]} T_h^{-1} T_{f \circ g}\left(r\right)\right]^2}{\log^{[l-1]} T_h^{-1} T_g \left(\exp^{[l]} r\right) \cdot \log^{[l]} T_h^{-1} T_g\left(r\right)} = 0 \ .$$

Proof. For any arbitrary positive ε we have for all sufficiently large values of r that

$$\log^{[l]} T_h^{-1} T_{f \circ g}\left(r\right) \leqslant \left(\rho_h^{[l]}\left(f \circ g\right) + \varepsilon\right) \log r \tag{22}$$

and

$$\log^{[l]} T_h^{-1} T_g\left(r\right) \geqslant \left(\lambda_h^{[l]}\left(g\right) - \varepsilon\right) \log r \ . \tag{23}$$

Similarly for all sufficiently large values of r we have

$$\begin{aligned} \log^{[l]} T_h^{-1} T_g \left(\exp^{[l]} r\right) &\geqslant \left(\lambda_h^{[l]}\left(g\right) - \varepsilon\right) \exp^{[l-1]} r, \\ i.e., \ \log^{[l-1]} T_h^{-1} T_g \left(\exp^{[l]} r\right) &\geqslant \exp\left[\left(\lambda_h^{[l]}\left(g\right) - \varepsilon\right) \exp^{[l-1]} r\right], \quad (24) \end{aligned}$$

From (22) and (23), we have for all sufficiently large values of r that

$$\frac{\log^{[l]} T_h^{-1} T_{f\circ g}(r)}{\log^{[l]} T_h^{-1} T_g(r)} \leqslant \frac{\left(\rho_h^{[l]}(f\circ g)+\varepsilon\right)\log r}{\left(\lambda_h^{[l]}(g)-\varepsilon\right)\log r} .$$

As $\varepsilon\,(>0)$ is arbitrary, we obtain from above that

$$\limsup_{r\to\infty} \frac{\log^{[l]} T_h^{-1} T_{f\circ g}(r)}{\log^{[l]} T_h^{-1} T_g(r)} \leqslant \frac{\rho_h^{[l]}(f\circ g)}{\lambda_h^{[l]}(g)} . \tag{25}$$

Again from (22) and (24), we get for all sufficiently large values of r that

$$\frac{\log^{[l]} T_h^{-1} T_{f\circ g}(r)}{\log^{[l-1]} T_h^{-1} T_g\left(\exp^{[l]} r\right)} \leqslant \frac{\left(\rho_h^{[l]}(f\circ g)+\varepsilon\right)\log r}{\exp\left[\left(\lambda_h^{[l]}(g)-\varepsilon\right)\exp^{[l-1]} r\right]} .$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows from above that

$$\begin{aligned}\limsup_{r\to\infty} \frac{\log^{[l]} T_h^{-1} T_{f\circ g}(r)}{\log^{[l-1]} T_h^{-1} T_g\left(\exp^{[l]} r\right)} &= 0 \\ \textit{i.e.,}\ \lim_{r\to\infty} \frac{\log^{[l]} T_h^{-1} T_{f\circ g}(r)}{\log^{[l-1]} T_h^{-1} T_g\left(\exp^{[l]} r\right)} &= 0. \end{aligned} \tag{26}$$

Thus the theorem follows from (25) and (26). □

In view of Theorem 3.16 the following theorem can be carried out:

Theorem 3.17. *Let f be meromorphic and g, h be any two entire functions such that (i) $\rho_h^{[l]}(f\circ g)<\infty$ and (ii) $\lambda_h^{[l]}(f)>0$ where l is any integer with $l>1$. Then*

$$\lim_{r\to\infty} \frac{\left[\log^{[l]} T_h^{-1} T_{f\circ g}(r)\right]^2}{\log^{[l-1]} T_h^{-1} T_f\left(\exp^{[l]} r\right)\cdot \log^{[l]} T_h^{-1} T_f(r)} = 0 .$$

The proof is omitted.

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