Fuglede-Putnam Type Theorems Via the Moore-Penrose Inverse and Aluthge Transform

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Abstract. Let $A, B \in B(H)$, where $H$ is a Hilbert space. Let $\tilde{T}$ and $T^\dagger$ denote the Aluthge transform and the Moore-Penrose inverse of $T$, respectively. We show that (i) if $A^*$ is quasinormal, then $((\tilde{A})^\dagger, (\tilde{B})^\dagger)$ has the FP-property; (ii) if $(A^\dagger, B^\dagger)$ has the FP-property, then so has $((\tilde{A})^\dagger, (\tilde{B})^\dagger)$. In general, $(T)^\dagger \neq \tilde{T}^\dagger$. Finally, we give some applications to the Lambert multiplication operator $M_wEM_u$ on $L^2(\Sigma)$, where $E$ is the conditional expectation operator.

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1. Introduction

In this section our purpose is to investigate some Fuglede-Putnam properties (shortened to FP-properties) for operators acting on Hilbert spaces.
The classical Fuglede-Putnam commutativity theorem says that if $A, B$ are normal (see, e.g., [14, p. 84]), then the pair $(A, B)$ has the FP-property.

Given a complex separable Hilbert space $H$, let $B(H)$ denotes the linear space of all bounded linear operators on $H$. Let $T = U|T|$ be the polar decomposition of $T$. An operator $T$ is said to be binormal, if $[[T], [T^*]] = 0$, where $[A, B] = AB - BA$ for operators $A$ and $B$. $T$ is said to be quasinormal, if $(T^*T)T = T(T^*T)$. Associated with $T \in B(H)$, there is a useful related operator $\tilde{T} = |T|^{1/2}U|T|^{1/2}$, called the Aluthge transform of $T$. Let $CR(H)$ be the set of all bounded linear operators on $H$ with closed range. For $T \in CR(H)$, the Moore-Penrose inverse of $T$, denoted by $T^\dagger$, is the unique operator $T^\dagger \in CR(H)$ which satisfies

$$TT^\dagger T = T, \quad T^\dagger TT^\dagger = T^\dagger, \quad (TT^\dagger)^* = TT^\dagger, \quad (T^\dagger T)^* = T^\dagger T.$$ (1)

We recall that $T^\dagger$ exists if and only if $T \in CR(H)$. The Moore-Penrose inverse is designed as a measure for the invertibility of an operator. If $T = U|T|$ is invertible, then $U$ is unitary and so $|T| = (T^*T)^{1/2}$ is invertible. It is a classical fact that the polar decomposition of $T^*$ is $U^*|T^*|$. It is easy to check that $U^*|T^*|^\dagger$ and $|T^*|^\dagger U^*|T^*|^\dagger$ are the polar decomposition and Aluthge transform of $T^\dagger$, respectively. It is sufficient to show that $(T^*)^\dagger = |T^*|^\dagger U$, because $T^\dagger = ((T^*)^\dagger)^*$.

$$T^*(T^*)^\dagger T^* = U^*|T^*|(|T^*|^\dagger U)U^*|T^*|$$
$$= U^*|T^*| |T^*|^\dagger |T^*|$$
$$= U^*|T^*| = T^*,$$

$$(T^*)^\dagger T^*(T^*)^\dagger = |T^*|^\dagger UU^*|T^*||T^*|^\dagger U$$
$$= |T^*|^\dagger |T^*| |T^*|^\dagger U$$
$$= |T^*|^\dagger U = (T^*)^\dagger.$$ 

Since $T^* = U^*|T^*|$ is polar decomposition for $T^*$, so $N(T^*) = N(U^*)$. But $N(T^*) = N(T^\dagger)$. Hence $N(U^*) = N(T^\dagger)$. Also it is easy to check that $T^*(T^*)^\dagger$ and $(T^*)^\dagger T^*$ are self-adjoint operators. Therefore, we have $T^\dagger = U^*|T^*|^\dagger$ is polar decomposition. Similarly it can be shown $|T^\dagger|^{1/2}U^*|T^\dagger|^{1/2}$
is Aluthge transform of $T^\dagger$. We shall make use of the following general properties of $T^*$, $\tilde{T}$, $T^\dagger$, parts of their projections and polar decompositions. For proofs and discussions of some of these facts see [1, 2, 3, 7, 12, 13, 15, 19].

1. $P(\tilde{T}^\dagger = |T^\dagger|^{\frac{1}{2}}U^*|T^\dagger|^\frac{1}{2} , (\tilde{T}^\dagger)^\dagger = \tilde{T}(t)$;
2. $P(\text{If } T \text{ is binormal then } \tilde{T}^\dagger = (|T^\dagger|^{\frac{1}{2}}U^*(|T^\dagger|^\frac{1}{2}));$
3. $P(\text{If } T \text{ is binormal then so is } T^\dagger \text{ and } (T^\dagger)^* = (T^\dagger)^\dagger$;
4. $P(\text{If } |T^\dagger| = |T^\dagger|^\dagger \text{ and } |T^\dagger|^\frac{1}{2} = (|T^\dagger|^\frac{1}{2})^\dagger$;
5. $P(U^*(|T^\dagger|\frac{1}{2})^2 = (|T^\dagger|^\frac{1}{2})U^*$;
6. $P(|(T^\dagger)^\dagger| = |T|^\dagger$;
7. $P(U^*|T^\dagger|^\dagger \text{ and } U^*|T^\dagger|^\dagger \text{ are the polar decompositions of } T^*$ and $T^\dagger$, respectively;
8. $P(UU^*|T^\dagger|^\dagger = |T^\dagger|^\dagger$;
9. $P(|(T^\dagger)^*|^\dagger = U^*$.

The next lemmas are concerned with the Fuglede-Putnam theorem and we need them in the future.

**Lemma 1.1.** [18, 16] Let $A, B \in B(H)$. Then the following assertions are equivalent:

(i) The pair $(A, B)$ has the FP-property.
(ii) If $X \in \text{Com}(A,B)$, then $\overline{R(X)}$ reduces $A$, $(ker X)^\perp$ reduces $B$ and $A_{\overline{R(X)}}, B_{(ker X)^\perp}$ are unitarily equivalent normal operators.

**Lemma 1.2.** If $A, B \in B(H)$ are invertible, then

(i) $X \in \text{Com}(A^\dagger, B^\dagger) \Leftrightarrow |A^\dagger|^\dagger X|B^\dagger|^{-1} = UXV^*$.
(ii) $X \in \text{Com}(A^\dagger, B^\dagger) \cap \text{Com}((A^\dagger)^*, (B^\dagger)^*) \Leftrightarrow |A^\dagger|^\dagger X|B^\dagger|^{-1} = UXV^* = X$.

**Proof.**

(i) It is clear by definition.

(ii) Let $X \in \text{Com}(A^\dagger, B^\dagger) \cap \text{Com}((A^\dagger)^*, (B^\dagger)^*)$. Then $A^\dagger X = XB^\dagger, (A^\dagger)^* X = X(B^\dagger)^*$, thus we get that $U^*|A^\dagger|^\dagger X = XV^*|B^\dagger|^\dagger, |A^\dagger|^\dagger VX = X|B^\dagger|^\dagger V$. Hence, $UXV^* = |A^\dagger|^\dagger X|B^\dagger|^{-1}, (|A^\dagger|^\dagger)^2 X = (A^\dagger)^* A^\dagger X = (A^\dagger)^* (XB^\dagger) = X(B^\dagger)|B^\dagger = X(|B^\dagger|^\dagger)^2$. Then $(|A^\dagger|^\dagger)^2 X = X(|B^\dagger|^\dagger)^2$. Utilizing a sequence of polynomials uniformly converging to $f(t) = \sqrt{t}$ on $SP(|A^\dagger|^\dagger)^2 \cup SP(|B^\dagger|^\dagger)^2$ and the functional calculus we get $|A^\dagger|^\dagger X = X|B^\dagger|^\dagger$, that
is $|A^\dagger|X(|B^\dagger|)^{-1} = X$. Hence from (i) we have $UXV^* = X$. The reverse direction is trivial. □

In the following, we try to provide some results concerning this problem that we call it the Fuglede-Putnam-Moore-Penrose problem. More precisely, we prove that if $(A^\dagger, B^\dagger)$ has the FP-property, then $\text{Com}(A, B) \subseteq \text{Com}(A^\dagger, B^\dagger)$ and if, moreover, $A$ is invertible operator then, $\text{Com}(A^\dagger, B^\dagger) = \text{Com}((A)^\dagger, (B)^\dagger)$. Note that if $A = U|A|$ is invertible then $U$ is unitary and $|A|$ is also invertible.

2. Fuglede-Putnam Theorem for Moore-Penrose Inverse and Aluthge Transforms

As mentioned above, in this section we present some results concerning the Fuglede-Putnam-Moore-Penrose problem.

**Lemma 2.1.** If $A, B \in CR(H)$ are binormal, then

(i) $X \in \text{Com}(A^\dagger, B^\dagger) \Leftrightarrow (|A^\dagger|\frac{1}{2}U^*X(|B^\dagger|)^{\frac{1}{2}} \in \text{Com}((\tilde{A})^\dagger, (\tilde{B})^\dagger)$.

(ii) $X \in \text{Com}((A)^\dagger*, (B)^\dagger*) \Leftrightarrow (|A^\dagger|\frac{1}{2}XV(|B^\dagger|)^{\frac{1}{2}} \in \text{Com}(((A)^\dagger)^*, ((B)^\dagger)^*)$.

**Proof.** (i) If $A^\dagger = U^*|A^\dagger|^\frac{1}{2}, B^\dagger = V^*|B^\dagger|^\frac{1}{2}$ be polar decompositions, since $U^*(|T^\dagger|^\frac{1}{2}) = (|T^\dagger|U^*)$ then we have,

$$X \in \text{Com}(A^\dagger, B^\dagger) \Rightarrow U^*|A^\dagger|^\frac{1}{2}X = XV^*|B^\dagger|^\frac{1}{2} \Rightarrow |A^\dagger|U^*X = X|B^\dagger|V^*.$$

Hence,

$$(\tilde{A})^\dagger(|A|\frac{1}{2}U^*X|B|^\frac{1}{2}) = |A|\frac{1}{2}U^*(|A|\frac{1}{2}U^*X|B|^\frac{1}{2})$$

$$= |A|\frac{1}{2}U^*(X|B|^\frac{1}{2}V^*)|B|^\frac{1}{2}$$

$$= (|A|\frac{1}{2}U^*X|B|^\frac{1}{2})(\tilde{B})^\dagger.$$

The converse obviously holds.

(ii) It could be proved in a similar way (i). □

**Corollary 2.2.** If $A, B, \tilde{A}, \tilde{B} \in CR(H)$ and $A, B$ are binormal. Then
\(|A^\dagger|^{\frac{1}{2}} XV (|B^\dagger|)^{\frac{1}{2}} \in \text{Com}(\tilde{A}^\dagger, \tilde{B}^\dagger) \) if and only if \(|A^\dagger|^{\frac{1}{2}} U^* X (|B^\dagger|)^{\frac{1}{2}} \in \text{Com}(\tilde{A}^\dagger, (\tilde{B}^\dagger)^\dagger)\).

**Proof.** It follows from P(3) and Lemma 2.1. \(\Box\)

**Theorem 2.3.** Let \(A, B, \tilde{A}, \tilde{B} \in CR(H)\) and \(A, B\) are binormal, \(A^*\) is quasinormal. Then the pair \(((\tilde{A})^\dagger, (\tilde{B})^\dagger)\) has the FP-property.

**Proof.** Let \(A = U|A|\) and \(B = V|B|\) be the polar decompositions. We show that \(\text{Com}((\tilde{A})^\dagger, (\tilde{B})^\dagger) \subseteq \text{Com}(((\tilde{A})^\dagger)^*, ((\tilde{B})^\dagger)^*)\) if and only if \(U^2 X = XV^2\), for any \(X \in \text{Com}(A^\dagger, B^\dagger)\). First, we prove that the FP-property for \(\text{Com}((\tilde{A})^\dagger, (\tilde{B})^\dagger)\) is equivalent to following requirement,

\[
U^* X V^* \in \text{Com}((A^\dagger)^*, (B^\dagger)^*), \quad (X \in \text{Com}(A^\dagger, B^\dagger)). \tag{2}
\]

Let \(((\tilde{A})^\dagger, (\tilde{B})^\dagger)\) have the FP-property and \((X \in \text{Com}(A^\dagger, B^\dagger))\). By Lemma 2.1(i), \(|A^*|^\frac{1}{2} U^* X (|B^*|^\frac{1}{2}) \in \text{Com}(((\tilde{A})^\dagger)^*, ((\tilde{B})^\dagger)^*)\).

Since \(\text{Com}((\tilde{A})^\dagger, (\tilde{B})^\dagger)\) has the FP-property then, \(|A^*|^\frac{1}{2} U^* X (|B^*|^\frac{1}{2}) \in \text{Com}((\tilde{A})^\dagger, (\tilde{B})^\dagger)\)). By Lemma 2.1(ii),

\[
(|A^\dagger|^\frac{1}{2} (|A^\dagger|^{\frac{1}{2}} U^* X (|B^\dagger|)^{\frac{1}{2}} (|B^\dagger|)^{\frac{1}{2}} (|B^\dagger|)^{\frac{1}{2}} V^* \in \text{Com}((A^\dagger)^*, (B^\dagger)^*). \]

Then, \(U^* X V^* \in \text{Com}((A^\dagger)^*, (B^\dagger)^*)\).

To prove the reverse, assume that (2.1) holds and let \(X \in \text{Com}((\tilde{A})^\dagger, (\tilde{B})^\dagger)\).

It follows from Lemma 2.1(i), that \(U(|A^\dagger|^{\frac{1}{2}} X (|B^\dagger|)^{\frac{1}{2}} \in \text{Com}(A^\dagger, B^\dagger)\).

Then by (2.1) we have \(U^* U(|A^\dagger|^{\frac{1}{2}} X (|B^\dagger|)^{\frac{1}{2}} V^* \in \text{Com}((A^\dagger)^*, (B^\dagger)^*))\).

and by Lemma 2.1(ii), implies that \(X \in \text{Com}((A^\dagger)^*, (B^\dagger)^*)\), therefore \(\text{Com}(A^\dagger, B^\dagger)\) has the FP-property.

Let (2.1) hold, then by Lemma 2.1(i) for any \(X \in \text{Com}(A^\dagger, B^\dagger)\), it follows that the \(|A^*|^\dagger X (|B^*|^\dagger)^{-1} = UXV^*\). By using (2.1), we obtain

\[
(A^\dagger) U^* X V^* = U^* X V^* (B^\dagger)^* \Rightarrow |A^*|^\dagger X V^* = U^* X V^* |B^*|^\dagger V =
\]

\[
(U^2)^2 U^* X V^* |B^*|^\dagger V.
\]

Then by Lemma 2.1(i) we get that, \(|A^*|^\dagger X (V^*)^2 = (U^*)^2 |A^*|^\dagger X\). Since \(A^*\) is quasinormal then, \(X (V^*)^2 = (U^*)^2 X\). Thus \(XV^2 = U^2 X\). The converse can be proved in the same way. \(\Box\)
Corollary 2.4. If \((A^\dagger, B^\dagger)\) has the FP-property, then so has \(((\tilde{A})^\dagger, (\tilde{B})^\dagger)\).

Proof. If \((A^\dagger, B^\dagger)\) has the FP-property, then by Lemma 2.1(ii), \(UX = XV\) for any \(X \in (A^\dagger, B^\dagger)\). Thus \(U^2X = U(XV) = (UX)V = XV^2\). Therefore by the theorem 2.3, \(((\tilde{A})^\dagger, (\tilde{B})^\dagger)\) has the FP-property. □

Corollary 2.5. If \((A, B)\) has the FP-property, then so has \((\tilde{A}(\dagger), \tilde{B}(\dagger))\).

Proof. By corollary 2.4 and P(1) the proof is completes. □

Corollary 2.6. If \(A, B\) be binormal, \((A, B)\) has the FP-property, then so has \((\tilde{A}(\ast), \tilde{B}(\ast))\).

Proof. By corollary 2.4 and P(3) the proof is completes. □

Theorem 2.7. Let \(A, B \in CR(H)\) and let \((A^\dagger, B^\dagger)\) have the FP-property. Then \(\text{Com}(A^\dagger, B^\dagger) \subseteq \text{Com}((\tilde{A})^\dagger, (\tilde{B})^\dagger)\).

Proof. Let \(A^\dagger = U^\ast|A^\ast|^\dagger, B^\dagger = V^\ast|B^\ast|^\dagger\) be the polar decomposition and let \(\{f_n\}\) be a sequence of polynomials with no constant term such that \(\{f_n(t)\} \to t^{\frac{1}{2}}\) as \(t \to \infty\). Now let \(X \in \text{Com}(A^\dagger, B^\dagger)\), then by the hypothesis, \((A^\dagger)X = X(B^\dagger),(A^\dagger)^\ast X = X(B^\dagger)^\ast\), thus \(|A|^2X = X(|B|^2)^{\frac{1}{2}}\). Using the same argument we get \(|A|^2X = X(|B|^2)^{\frac{1}{2}}\). Also \(U^\ast|A^\ast|^\dagger X = XV^\ast|B^\ast|^\dagger\), then \(|A|^\dagger U^\ast X = X(|B|^\dagger)^{\frac{1}{2}} V^\ast\), and the same argument above show that, \(|A|^\dagger \frac{1}{2} U^\ast X = X(|B|^\dagger)^{\frac{1}{2}} V^\ast\).

Therefore,

\[
(\tilde{A})^\dagger X = (|A|^\dagger)^{\frac{1}{2}} U^\ast (|A|^\dagger)^{\frac{1}{2}} X = X(|B|^\dagger)^{\frac{1}{2}} V^\ast (|B|^\dagger)^{\frac{1}{2}} = X(\tilde{B})^\dagger.
\]

Lemma 2.8. Let \(A \in CR(H)\), then

(i) \(|(A)^\dagger|^q = U^\ast(|A^\ast|^\dagger)^q U^\ast\).

(ii) \(A^\dagger\) is quasinormal if and only if \(U^\ast|A^\ast|^\dagger = |A^\ast|^\dagger U^\ast\).

Proof. (i) By P(4), P(7), P(8) and P(9) we have

\[
(|A|^\dagger)^2 = (|A^\ast|^\dagger)^2 = (|A^\dagger|^\ast)^2 = A^\dagger (A^\dagger)^\ast = U^\ast |A^\ast|^\dagger |A^\ast|^\dagger U
\]

\[
= U^\ast |A^\ast|^\dagger U U^\ast |A^\ast|^\dagger U = (U^\ast |A^\ast|^\dagger U)^2.
\]
Let \( \{p_n\} \) be a sequence of polynomials with no constant term such that 
\[ p_n(t) \to \sqrt{t} \quad \text{uniformly on a certain compact subset of } \mathbb{R}^+ \quad \text{as } n \to \infty. \]
It follows that 
\[ p_n((|A|)^2) = p_n((U^*|A^*|U)^2), \]
and so 
\[ |A|^t = U^*|A^*|^t U. \]
By induction, 
\[ (|A|^t)^m = U^*(|A^t|^t)^m U \]
holds for each \( m, n \in \mathbb{N} \). Now, by using of the functional calculus, 
\[ (|A|^t)^q = U^*(|A^t|^q)^q U. \]
(ii) It is a classical fact that \( A \) is quasinormal if and only if \( U|A| = |A|U \)
(see [7, Theorem 3]. Now, the desired conclusion follows from this and \( P(7) \). \( \square \)

**Theorem 2.9.** Let \( A \in B(H) \) be onto. Then the following statements are equivalent:

(i) \( A^* \) is quasinormal.

(ii) \( A^\dagger \) is quasinormal.

**Proof.** (i)\( \iff \) (ii) By (1.1) we have 
\[ |A^*||A^\dagger|A^*| = |A^*|, \]
then we get that 
\[ A^* \text{is quasinormal} \iff U^*|A^*| = |A^*|U^* \]
\[ \iff U^*|A^*||A^\dagger|A^*| = |A^*||A^\dagger|U^*|A^*|U^* \]
\[ \iff |A^*|U^*|A^\dagger|A^*| = |A^*||A^\dagger|U^*|A^*| \]
\[ \iff |A^*|(U^*|A^\dagger| - |A^\dagger|U^*)|A^*| = 0. \]
By hypothesis, \( N(|A^*|) = N(A^*) = \{0\} \). Hence \( (U^*|A^\dagger| - |A^\dagger|U^*)|A^*| = 0 \), and so 
\[ U^*|A^\dagger| = |A^\dagger|U^* \quad \text{on } R(|A^*|) \] 
On the other hand, \( U^*|A^\dagger| = |A^\dagger|U^* \quad \text{on } N(|T^*|^t) = N(U^*). \)
Thus, \( U^*|A^\dagger| = |A^\dagger|U^* \) on \( H \). Consequently, by Lemma 2.8, (i)\( \iff \) (ii) holds. \( \square \)

**Theorem 2.10.** Let \( A, B \in CR(H) \) be onto and \( A^*, B^* \) be quasinormal. 
If \((A^\dagger, B^\dagger)\) has the FP-property then 
\( Com(A^\dagger, B^\dagger) = Com((\tilde{A})^\dagger, (\tilde{B})^\dagger) \).

**Proof.** According to Theorem 2.7, it is sufficient to prove 
\( Com(A^\dagger, B^\dagger) \supseteq \) 
\( Com((\tilde{A})^\dagger, (\tilde{B})^\dagger) \). Let 
\( X \in Com((\tilde{A})^\dagger, (\tilde{B})^\dagger) \), 
\( \Lambda = (|A|^t)^{\frac{1}{2}} X (|B|^t)^{\frac{1}{2}}. \)
Since \((\tilde{A})^\dagger X = X(\tilde{B})^\dagger \) Then,
\[ (|A|^t)^{\frac{1}{2}} (\tilde{A})^\dagger X (|B|^t)^{\frac{1}{2}} = (|A|^t)^{\frac{1}{2}} X (\tilde{B})^\dagger (|B|^t)^{\frac{1}{2}}. \]
\[ \Rightarrow U^* (|A|^t)^{\frac{1}{2}} X (|B|^t)^{\frac{1}{2}} = (|A|^t)^{\frac{1}{2}} X (|B|^t)^{\frac{1}{2}} V^* B^\dagger. \]
\[ \Rightarrow U^* (|A|^t)(|A|^t)^{\frac{1}{2}} X (|B|^t)^{\frac{1}{2}} = \Lambda V^* B^\dagger. \]
\[ \Rightarrow U^* |A|^t \Lambda = \Lambda V^* B^\dagger. \]
Therefore, $A \in \text{Com}(A^\dagger, B^\dagger)$. Also by the Lemma 1.1, and hypothesis, $\mathcal{R}(\Lambda)$ reduces $A^\dagger$, $(\ker \Lambda)^\perp$ reduces $B^\dagger$ and $A^{\dagger}_{\mathcal{R}(\Lambda)}, B^\dagger_{(\ker \Lambda)^\perp}$ are unitarily equivalent normal operators. Thus,

$$A^\dagger = A_1^\dagger \oplus A_2^\dagger \quad \text{on } \mathcal{R}(\Lambda) \oplus R(\Lambda)^\perp,$$

and

$$B^\dagger = B_1^\dagger \oplus B_2^\dagger \quad \text{on } N(\Lambda)^\perp \oplus N(\Lambda),$$

where $A_1^\dagger, B_1^\dagger$ are unitarily equivalent normal operators. Since $A$ is invertible and $A_1^\dagger, B_1^\dagger$ are unitarily equivalent then, $B_1^\dagger$ is invertible. Let,

$$X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}$$

and

$$\Lambda = \begin{pmatrix} \Lambda_1 & 0 \\ 0 & 0 \end{pmatrix}.$$ 

It follows from $\Lambda = (|A|^\dagger)^{-\frac{1}{2}} X (|B|^\dagger)^{\frac{1}{2}}$, that,

$$\Lambda = \begin{pmatrix} \Lambda_1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} (|A_1|^\dagger)^{-\frac{1}{2}} X_1 (|B_1|^\dagger)^{\frac{1}{2}} & (|A_1|^\dagger)^{-\frac{1}{2}} X_2 (|B_1|^\dagger)^{\frac{1}{2}} \\ (|A_2|^\dagger)^{-\frac{1}{2}} X_3 (|B_1|^\dagger)^{\frac{1}{2}} & (|A_2|^\dagger)^{-\frac{1}{2}} X_4 (|B_2|^\dagger)^{\frac{1}{2}} \end{pmatrix}.$$ 

Therefore, $X_2 (|B_1|^\dagger)^{\frac{1}{2}} = 0$, $X_3 = X_4 (|B_2|^\dagger)^{\frac{1}{2}} = 0$, then as the result, $X_2 (B_2)^\dagger = 0$, $X_4 (B_2)^\dagger = 0$. Thus $(\tilde{A})^\dagger X = X(\tilde{B})^\dagger$, implies that,

$$\begin{pmatrix} A_1^\dagger X_1 & A_1^\dagger X_2 \\ 0 & A_2^\dagger X_4 \end{pmatrix} = \begin{pmatrix} X_1 B_1^\dagger & 0 \\ 0 & 0 \end{pmatrix}.$$ 

Then, $X_2 = 0$, $X_4 = 0$, $A_1^\dagger X_1 = X_1 B_1^\dagger$, and since

$$X = \begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix} = X_1 \oplus 0.$$

Therefore, $A^\dagger X = XB^\dagger$ and the proof is completed. □
Corollary 2.11. Let $A^*, B^*$ be quasinormal. If $(A, B)$ have the FP-property. Then $\text{Com}(A, B) = \text{Com}(\tilde{A}^{(1)}, \tilde{B}^{(1)})$.

3. Moore-Penrose Inverse of Lambert Multiplication Operators

Let $(X, \Sigma, \mu)$ be a $\sigma$-finite measure space. For any $\sigma$-finite subalgebra $\mathcal{A} \subseteq \Sigma$ the Hilbert space $L^2(X, \mathcal{A}, \mu|_{\mathcal{A}})$ is abbreviated to $L^2(\mathcal{A})$. The support of a measurable function $f$ is defined by $\sigma(f) = \{ x \in X : f(x) \neq 0 \}$. All sets and functions statements are to be interpreted as being valid almost everywhere with respect to $\mu$. For each $f \in L^2(\Sigma)$, by the Radon-Nikodym theorem, there exists a unique $\mathcal{A}$-measurable function $E^A(f)$ such that $\int_A f d\mu = \int_A E^A(f) d\mu$, where $A$ is any $\mathcal{A}$-measurable set for which $\int_A f d\mu$ exists. Put $E = E^A$. The mapping $E$ is a linear orthogonal projection. For more details on the properties of $E$ see [9, 17].

Let $w, u \in L^0(\Sigma)$, the linear space of all complex-valued $\Sigma$-measurable functions on $X$. The mapping $T : L^2(\Sigma) \to L^0(\Sigma)$ defined by $T(f) = M_w E M_u(f) = wE(uf)$ is called Lambert multiplication operator. It is easy to check that for each $f \in L^2(\Sigma)$, $\|Tf\| = \|E M_u f\|$, where $v := u(E(|w|^2))^{1/2}$. Thus, $M_w E M_u$ is bounded (has closed range) if and only $E M_u$ is bounded (has closed range). Interesting articles related to this topic are [5, 6, 9]. A combination of conditional expectation operators and multiplication operators appears more often in the service of the study of other operators, such as operators generated by random measures and Markov operators. A good article related to the conditional type operators is [8].

Here we recall some results of [11] that state our results is valid for $T = M_w E M_u$.

Proposition 3.1. Let $T : L^2(\Sigma) \to L^0(\Sigma)$ defined by $T = M_w E M_u$ is a Lambert multiplication operator.

(i) $T \in B(L^2(\Sigma))$ if and only if $E(|w|^2)E(|u|^2) \in L^\infty(\mathcal{A})$, and in this case $\|T\| = \|E(|w|^2)E(|u|^2)\|^{1/2}_{\infty}$.

(ii) Let $T \in B(L^2(\Sigma))$, $0 \leq u \in L^0(\Sigma)$ and $v = u(E(|w|^2))^{1/2}$. If $E(v)$ is
δ on σ(ν), then T has closed range.

Put
\[ A(f) = \frac{u\chi_{G}}{E(u^2)E(w^2)}E(wf), \quad f \in L^2(\Sigma), \quad G = \sigma(E(w)). \quad (3) \]

Then by Proposition 3.1, \( A \in B(L^2(\Sigma)) \). Also, it is easy to check that
\[ TAT = T, \quad ATA = A, \quad (TA)^* = TA, \quad (AT)^* = AT. \]

Thus, \( A = T^\dagger \). We now turn to the computation of \( T^\dagger \), \( \widetilde{T} \), \( (\widetilde{T})^\dagger \) and \( \widetilde{T}^\dagger \). Direct computations give the following proposition.

**Proposition 3.2.** Let \( T, \widetilde{T} \in CR(L^2(\Sigma)) \) with \( u, w \geq 0 \). Then

\[ (a) \quad T^\dagger = M \frac{u\chi_{\sigma(E(w))}}{E(u^2)E(w^2)} EM_w. \]
\[ (b) \quad \widetilde{T} = M \frac{uE(uw)}{E(u^2)} EM_u. \]
\[ (c) \quad (\widetilde{T})^\dagger = M \frac{u\chi_{\sigma(E(uw))}}{E(u^2)E(uw)} EM_u. \]
\[ (d) \quad \widetilde{T}^\dagger = M \frac{\chi_{G(wu)E(w^2)}}{E(u^2)E(w^2)^2} EM_w. \]

Note that, if \( w \) or \( u \) is not \( A \)-measurable, then \( (\widetilde{T})^\dagger \neq \widetilde{T}^\dagger \). Moreover, if \( T \in CR(L^2(\Sigma)) \), then by Proposition 3.1(ii), \( \widetilde{T} \in CR(L^2(\Sigma)) \) whenever \( (E(uw))^2 \geq E(u^2)E(w^2) \).

**Proposition 3.3.** Let for \( i = 1, 2 \), \( T_i = M_{w_i}EM_{u_i} \in B(L^2(\Sigma)) \) and \( (E(w_i^2))^{1/2}w_i = u_i(E(u_i^2))^{1/2} \). Then \( (T_1, T_2) \) has the FP-property.

**Proof.** We know that if \( T_1 \) and \( T_2 \) are normal, then \( (T_1, T_2) \) has the FP-property. Thus it is sufficient to prove that \( T_1 \) and \( T_2 \) are normal. Since \( T_i^* = M_{u_i}EM_{w_i} \), it is easy to check that \( T_i^*T_i = M_{u_iE(w_i^2)}EM_{u_i} \) and \( T_iT_i^* = M_{w_iE(u_i^2)}EM_{w_i} \). So \( T_i^*T_i - T_iT_i^* = M_{u_iE(w_i^2)}EM_{u_i} - M_{w_iE(u_i^2)}EM_{w_i} \). Then by hypothesis we obtain
\[ \langle (T_i^*T_i - T_iT_i^*)f, f \rangle = \int_X \{ (E(w_i^2))E(u_if)uf - E(u_i^2)E(w_if)wf \}d\mu \]
\[ = \int_X \{ (E(u_i(E(w_i^2))^{1/2}f))^2 - (E((E(u_i^2))^{1/2}w_if))^2 \}d\mu = 0, \]
for each \( f \in L^2(\Sigma) \). This implies that \( T_i \) are normal. \qed
In [5], Estaremi show that the Aluthge transform of $M_u EM_u$ is always normal. So we have the following corollary.

**Corollary 3.4.** Let $T, \check{T} \in CR(L^2(\Sigma))$ with $u, w \geq 0$. Then

(a) $(\check{T}_1, \check{T}_2)$ has the FP-property.

(b) $((\check{T}_1)\dagger, (\check{T}_2)\dagger)$ has the FP-property.

(b) $(T_1\dagger, T_2\dagger)$ has the FP-property.

Let $A = \varphi^{-1}(\Sigma), 0 \leq u \in L^0(\Sigma)$ and $\varphi : X \to X$ be a measurable transformation such that $\mu \circ \varphi^{-1}$ is absolutely continuous with respect to $\mu$. The weighted composition operator $W$ on $L^2(\Sigma)$ induced by the pair $(u, \varphi)$ is given by $W = M_u \circ C_\varphi$, where $C_\varphi$ is the composition operator defined by $C_\varphi f = f \circ \varphi$. It is a classical fact that $W$ is a bounded linear operator on $L^2(\Sigma)$, if and only if $J := h E(u^2) \circ \varphi^{-1} \in L^\infty(\Sigma)$. Also, $W \in CR(L^2(\Sigma))$ if and only if $J$ is bounded away from zero on $\sigma(J)$ (see [10]). From now on, we assume that $W$ has closed range. It is easy to check that $W\dagger = M x_{\sigma(J)} W^*$ and $(W\dagger)^* = M x_{\sigma(J\circ \varphi)} W$.

Now, we can compute the polar decomposition and Aluthge transformations of $W = U|W|$ and $W\dagger = U^*|W\dagger|$ as follows:

$$|W| = M \sqrt{J};$$

$$U = M x_{\sigma(J \circ \varphi)} W;$$

$$U^* = M x_{\sigma(J)} W^*;$$

$$|W\dagger| = M \frac{u x_{\sigma(J \circ \varphi)}}{\sqrt{h E(u^2)}} EM_u;$$

$$|W\dagger|^\frac{1}{2} = M \frac{h \sqrt{J} C_{\varphi^{-1}}}{\frac{1}{2} (E(u^2))^\frac{1}{2}} E(u f);$$

Consequently, for each $f \in L^2(\Sigma)$ we get that

$$\tilde{W} (f) = u \left( \frac{J x_{\sigma(E(u))}}{(h \circ \varphi) E(u^2)} \right)^\frac{1}{2} (f \circ \varphi);$$

$$\tilde{W}\dagger (f) = \left( \frac{1}{(h \circ \varphi)(E(u^2))^\frac{1}{2}} \right)^\frac{1}{2} u E \left( \frac{u \sqrt{h} C_{\varphi^{-1}}}{(E(u^2) \circ \varphi^{-1})^\frac{1}{2}} E(u f) \circ \varphi^{-1} \right);$$

$$\check{W} (f) = \left( \frac{x_{\sigma(J)}}{\sqrt{h} E(u^2) \circ \varphi^{-1}} \right)^\frac{1}{2} E(u \sqrt{J} f) \circ \varphi^{-1};$$

$$\check{W}\dagger (f) = \left( \frac{x_{\sigma(J)}}{\sqrt{h} E(u^2) \circ \varphi^{-1}} \right)^\frac{1}{2} E(u f) \circ \varphi^{-1},$$

$$\check{W}\dagger (f) = \left( \frac{x_{\sigma(J)}}{\sqrt{h} E(u^2) \circ \varphi^{-1}} \right)^\frac{1}{2} E(u f) \circ \varphi^{-1},$$
where $C = \chi_{\sigma(E(u^2) \circ \varphi^{-1})}$.

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