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Fuglede-Putnam Type Theorems Via the Moore-Penrose Inverse and Aluthge Transform

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Abstract. Let $A, B \in B(H)$, where H is a Hilbert space. Let $\widetilde{T}$ and $T^{\dagger}$ denote the Aluthge transform and the Moore-Penrose inverse of T, respectively. We show that (i) if A^* is quasinormal, then $((\widetilde{A})^{\dagger}, (\widetilde{B})^{\dagger})$ has the FP-property; (ii) if $(A^{\dagger}, B^{\dagger})$ has the FP-property, then so has $((\widetilde{A})^{\dagger}, (\widetilde{B})^{\dagger})$. In general, $(\widetilde{T})^{\dagger} \neq \widetilde{T^{\dagger}}$. Finally, we give some applications to the Lambert multiplication operator $M_w E M_u$ on $L^2(\Sigma)$, where E is the conditional expectation operator.

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1. Introduction

In this section our purpose is to investigate some Fuglede-Putnam properties (shortened to FP-properties) for operators acting on Hilbert spaces.

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The classical Fuglede-Putnam commutativity theorem says that if A, B are normal (see, e.g., [14, p. 84]), then the pair (A, B) has the FP-property.
Given a complex separable Hilbert space H, let $B(H)$ denotes the linear space of all bounded linear operators on H. Let $T = U|T|$ be the polar decomposition of T. An operator T is said to be binormal, if $[|T|, |T^*|] = 0$, where $[A, B] = AB - BA$ for operators A and B. T is said to be quasinormal, if $(T^*T)T = T(T^*T)$. Associated with $T \in B(H)$, there is a useful related operator $\widetilde{T} = |T|^{1/2}U|T|^{1/2}$, called the Aluthge transform of T. Let $CR(H)$ be the set of all bounded linear operators on H with closed range. For $T \in CR(H)$, the Moore-Penrose inverse of T, denoted by $T^\dagger$, is the unique operator $T^\dagger \in CR(H)$ which satisfies

$$TT^\dagger T = T, \quad T^\dagger TT^\dagger = T^\dagger, \quad (TT^\dagger)^* = TT^\dagger, \quad (T^\dagger T)^* = T^\dagger T. \tag{1}$$

We recall that $T^\dagger$ exists if and only if $T \in CR(H)$. The Moore-Penrose inverse is designed as a measure for the invertibility of an operator. If $T = U|T|$ is invertible, then U is unitary and so $|T| = (T^*T)^{1/2}$ is invertible. It is a classical fact that the polar decomposition of T^* is $U^*|T^*|$. It is easy to check that $U^*|T^*|^\dagger$ and $|T^\dagger|^{\frac{1}{2}}U^*|T^\dagger|^{\frac{1}{2}}$ are the polar decomposition and Aluthge transform of $T^\dagger$, respectively. It is sufficient to show that $(T^*)^\dagger = |T^*|^\dagger U$, because $T^\dagger = ((T^*)^\dagger)^*$.

$$\begin{aligned} T^*(T^*)^\dagger T^* &= U^*|T^*|(|T^*|^\dagger U)U^*|T^*| \\ &= U^*|T^*||T^*|^\dagger |T^*| \\ &= U^*|T^*| = T^*, \end{aligned}$$

$$\begin{aligned} (T^*)^\dagger T^*(T^*)^\dagger &= |T^*|^\dagger UU^*|T^*||T^*|^\dagger U \\ &= |T^*|^\dagger |T^*||T^*|^\dagger U \\ &= |T^*|^\dagger U = (T^*)^\dagger. \end{aligned}$$

Since $T^* = U^*|T^*|$ is polar decomposition for T^*, so $N(T^*) = N(U^*)$. But $N(T^*) = N(T^\dagger)$. Hence$N(U^*) = N(T^\dagger)$. Also it is easy to check that $T^*(T^*)^\dagger$ and $(T^*)^\dagger T^*$ are self-adjoint operators. Therefore, we have $T^\dagger = U^*|T^*|^\dagger$ is polar decomposition. Similarly it can be shown $|T^\dagger|^{\frac{1}{2}}U^*|T^\dagger|^{\frac{1}{2}}$

is Aluthge transform of $T^\dagger$. We shall make use of the following general properties of T^*, $\widetilde{T}$, $T^\dagger$, parts of their projections and polar decompositions. For proofs and discussions of some of these facts see [1, 2, 3, 7, 12, 13, 15, 19].

P(1) $\widetilde{T^\dagger} = |T^\dagger|^{\frac{1}{2}}U^*|T^\dagger|^{\frac{1}{2}}$, $(\widetilde{T^\dagger})^\dagger = \widetilde{T}^{(\dagger)}$;
P(2) If T is binormal then $\widetilde{T^\dagger} = (|T|^\dagger)^{\frac{1}{2}}U^*(|T|^\dagger)^{\frac{1}{2}}$;
P(3) If T is binormal, then so is $T^\dagger$ and $(\widetilde{T^*})^* = (\widetilde{T^\dagger})^\dagger$;
P(4) $|T^\dagger| = |T^*|^\dagger$ and $|T^\dagger|^{\frac{1}{2}} = (|T^*|^{\frac{1}{2}})^\dagger$;
P(5) $U^*(|T^*|^\dagger)^{\frac{1}{2}} = (|T|^\dagger)^{\frac{1}{2}}U^*$;
P(6) $|(T^*)^\dagger| = |T|^\dagger$;
P(7) $U^*|T^*|$ and $U^*|T^*|^\dagger$ are the polar decompositions of T^* and $T^\dagger$, respectively;
P(8) $UU^*|T^*|^\dagger = |T^*|^\dagger$;
P(9) $(T^\dagger)^* = |T^*|^\dagger U$.

The next lemmas are concerned with the Fuglede-Putnam theorem and we need them in the future.

Lemma 1.1. [18, 16] *Let $A, B \in B(H)$. Then the following assertions are equivalent:*
(i) The pair (A, B) has the FP-property.
(ii) If $X \in Com(A, B)$, then $\overline{R(X)}$ reduces A , $(kerX)^\perp$ reduces B and $A|_{\overline{R(X)}}$, $B|_{(kerX)^\perp}$ are unitarily equivalent normal operators.

Lemma 1.2. *If $A, B \in B(H)$ are invertible, then*
(i) $X \in Com(A^\dagger, B^\dagger) \Leftrightarrow |A^*|^\dagger X(|B^*|^\dagger)^{-1} = UXV^*$.
(ii) $X \in Com(A^\dagger, B^\dagger) \cap Com((A^\dagger)^*, (B^\dagger)^*) \Leftrightarrow |A^*|^\dagger X(|B^*|^\dagger)^{-1} = UXV^* = X$.

Proof.
(i). It is clear by definition.
(ii) Let $X \in \mathrm{Com}(A^\dagger, B^\dagger) \cap \mathrm{Com}((A^\dagger)^*, (B^\dagger)^*)$. Then $A^\dagger X = XB^\dagger, (A^\dagger)^*X = X(B^\dagger)^*$, thus we get that $U^*|A^*|^\dagger X = XV^*|B^*|^\dagger, |A^*|^\dagger VX = X|B^*|^\dagger V$. Hence, $UXV^* = |A^*|^\dagger X(|B^*|^\dagger)^{-1}$, $(|A^*|^\dagger)^2X = (A^\dagger)^*A^\dagger X = (A^\dagger)^*(XB^\dagger) = X(B^\dagger)^*B^\dagger = X(|B^*|^\dagger)^2$. Then $(|A^*|^\dagger)^2X = X(|B^*|^\dagger)^2$. Utilizing a sequence of polynomials uniformly converging to $f(t) = \sqrt{t}$ on $SP((|A^*|^\dagger)^2) \cup SP((|B^*|^\dagger)^2)$ and the functional calculus we get $|A^*|^\dagger X = X|B^*|^\dagger$, that

is $|A^*|^\dagger X(|B^*|^\dagger)^{-1} = X$. Hence from (i) we have $UXV^* = X$. The reverse direction is trivial. □

In the following, we try to provide some results concerning this problem that we call it the Fuglede-Putnam-Moore-Penrose problem. More precisely, we prove that if $(A^\dagger, B^\dagger)$ has the FP-property, then $\mathrm{Com}(A, B) \subseteq \mathrm{Com}(A^\dagger, B^\dagger)$ and if, moreover, A is invertible operator then, $\mathrm{Com}(A^\dagger, B^\dagger) = \mathrm{Com}((\widetilde{A})^\dagger, (\widetilde{B})^\dagger)$. Note that if $A = U|A|$ is invertible then U is unitary and $|A|$ is also invertible.

2. Fuglede-Putnam Theorem for Moore-Penrose Inverse and Aluthge Transforms

As mentioned above, in this section we present some results concerning the Fuglede-Putnam-Moore-Penrose problem.

Lemma 2.1. *If $A, B \in CR(H)$ are binormal, then*
(i) $X \in Com(A^\dagger, B^\dagger) \Leftrightarrow (|A|^\dagger)^{\frac{1}{2}} U^* X (|B|^\dagger)^{\frac{1}{2}} \in Com((\widetilde{A})^\dagger, (\widetilde{B})^\dagger)$.
(ii) $X \in Com((A^\dagger)^*, (B^\dagger)^*) \Leftrightarrow (|A|^\dagger)^{\frac{1}{2}} X V (|B|^\dagger)^{\frac{1}{2}} \in Com(((\widetilde{A})^\dagger)^*, ((\widetilde{B})^\dagger)^*)$.

Proof. (i) If $A^\dagger = U^*|A^*|^\dagger$,$B^\dagger = V^*|B^*|^\dagger$ be polar decompositions, since $U^*(|T^*|^\dagger) = (|T|^\dagger)U^*$ then we have,

$$\begin{aligned} X \in \mathrm{Com}(A^\dagger, B^\dagger) &\Rightarrow U^*|A^*|^\dagger X = XV^*|B^*|^\dagger \\ &\Rightarrow |A|^\dagger U^* X = X|B|^\dagger V^*. \end{aligned}$$

Hence,

$$\begin{aligned} (\widetilde{A})^\dagger(|A|^{\dagger\frac{1}{2}} U^* X |B|^{\dagger\frac{1}{2}}) &= |A|^{\dagger\frac{1}{2}} U^* (|A|^\dagger U^* X |B|^{\dagger\frac{1}{2}}) \\ &= |A|^{\dagger\frac{1}{2}} U^* (X|B|^\dagger V^*) |B|^{\dagger\frac{1}{2}} \\ &= (|A|^{\dagger\frac{1}{2}} U^* X |B|^{\dagger\frac{1}{2}})(\widetilde{B})^\dagger. \end{aligned}$$

The converse obviously holds.
(ii) It could be proved in a similar way (i). □

Corollary 2.2. *If $A, B, \widetilde{A}, \widetilde{B} \in CR(H)$ and A, B are binormal. Then*

$(|A^{\dagger}|)^{\frac{1}{2}}XV(|B^{\dagger}|)^{\frac{1}{2}} \in Com(\widetilde{A^{\dagger}}, \widetilde{B^{\dagger}})$ *if and only if* $(|A|^{\dagger})^{\frac{1}{2}}U^*X(|B|^{\dagger})^{\frac{1}{2}} \in Com((\widetilde{A})^{\dagger}, (\widetilde{B})^{\dagger})$.

Proof. It follows from P(3) and Lemma 2.1. $\square$

Theorem 2.3. *Let $A, B, \widetilde{A}, \widetilde{B} \in CR(H)$ and A, B are binormal, A^* is quasinormal. Then the pair $((\widetilde{A})^{\dagger}, (\widetilde{B})^{\dagger})$ has the FP-property.*

Proof. Let $A = U|A|$ and $B = V|B|$ be the polar decompositions. We show that $\mathrm{Com}((\widetilde{A})^{\dagger}, (\widetilde{B})^{\dagger}) \subseteq \mathrm{Com}(((\widetilde{A})^{\dagger})^*, ((\widetilde{B})^{\dagger})^*)$ if and only if $U^2X = XV^2$, for any $X \in \mathrm{Com}(A^{\dagger}, B^{\dagger})$. First, we prove that the FP-property for $\mathrm{Com}((\widetilde{A})^{\dagger}, (\widetilde{B})^{\dagger})$ is equivalent to following requirement,

$$U^*XV^* \in \mathrm{Com}((A^{\dagger})^*, (B^{\dagger})^*), \qquad (X \in \mathrm{Com}(A^{\dagger}, B^{\dagger})). \tag{2}$$

Let $((\widetilde{A})^{\dagger}, (\widetilde{B})^{\dagger})$ have the FP-property and $(X \in \mathrm{Com}(A^{\dagger}, B^{\dagger}))$. By Lemma 2.1(i), $(|A^*|^{\dagger})^{\frac{1}{2}}U^*X(|B|^{\dagger})^{\frac{1}{2}} \in \mathrm{Com}(((\widetilde{A})^{\dagger}), ((\widetilde{B})^{\dagger}))$.
Since $\mathrm{Com}((\widetilde{A})^{\dagger}, (\widetilde{B})^{\dagger})$ has the FP-property then, $(|A^*|^{\dagger})^{\frac{1}{2}}U^*X(|B|^{\dagger})^{\frac{1}{2}} \in \mathrm{Com}(((\widetilde{A})^{\dagger}), ((\widetilde{B})^{\dagger}))$. By Lemma 2.1(ii),

$$(|A|^{\dagger})^{\frac{-1}{2}}(|A|^{\dagger})^{\frac{1}{2}}U^*X(|B|^{\dagger})^{\frac{1}{2}}(|B|^{\dagger})^{\frac{1}{2}}(|B|^{\dagger})^{\frac{-1}{2}}V^* \in \mathrm{Com}((A^{\dagger})^*, (B^{\dagger})^*).$$

Then, $U^*XV^* \in \mathrm{Com}((A^{\dagger})^*, (B^{\dagger})^*)$.
To prove the revers, assume that (2.1) holds and let $X \in \mathrm{Com}((\widetilde{A})^{\dagger}, (\widetilde{B})^{\dagger})$. It follows from Lemma 2.1(i), that $U(|A|^{\dagger})^{\frac{-1}{2}}X(|B|^{\dagger})^{\frac{1}{2}} \in \mathrm{Com}(A^{\dagger}, B^{\dagger})$. Then by (2.1)we have $U^*U(|A|^{\dagger})^{\frac{-1}{2}}X(|B|^{\dagger})^{\frac{1}{2}}V^* \in \mathrm{Com}((A^{\dagger})^*, (B^{\dagger})^*)$ and by Lemma 2.1(ii), implies that $X \in \mathrm{Com}((\widetilde{A^{\dagger}})^*, (\widetilde{B^{\dagger}})^*)$, therefor $\mathrm{Com}(A^{\dagger}, B^{\dagger})$ has the FP-property.
Let (2.1) hold, then by Lemma 2.1(i) for any $X \in \mathrm{Com}(A^{\dagger}, B^{\dagger})$, it follows that the $|A^*|^{\dagger}X(|B^*|^{\dagger})^{-1} = UXV^*$. By using (2.1), we obtain

$$(A^{\dagger})^*U^*XV^* = U^*XV^*(B^{\dagger})^* \Rightarrow |A^*|^{\dagger}XV^* = U^*XV^*|B^*|^{\dagger}V =$$

$$(U^*)^2UXV^*|B^*|^{\dagger}V.$$

Then by lemma 2.1(i) we get that, $|A^*|^{\dagger}X(V^*)^2 = (U^*)^2|A^*|^{\dagger}X$. Since A^* is quasinormal then, $X(V^*)^2 = (U^*)^2X$. Thus $XV^2 = U^2X$. The converse can be proved in the same way. $\square$

Corollary 2.4. *If* $(A^\dagger, B^\dagger)$ *has the* FP*-property, then so has* $((\widetilde{A})^\dagger, (\widetilde{B})^\dagger)$.

Proof. If $(A^\dagger, B^\dagger)$ has the FP-property, then by Lemma 2.1(ii), $UX = XV$ for any $X \in (A^\dagger, B^\dagger)$. Thus $U^2X = U(XV) = (UX)V = (XV)V = XV^2$. Therefore by the theorem 2.3, $((\widetilde{A})^\dagger, (\widetilde{B})^\dagger)$ has the FP-property. $\square$

Corollary 2.5. *If* (A, B) *has the* FP*-property, then so has* $(\widetilde{A}^{(\dagger)}, \widetilde{B}^{(\dagger)})$.

Proof. By corollary 2.4 and P(1) the proof is completes. $\square$

Corollary 2.6. *If* A, B *be binormal,*(A, B) *has the* FP*-property, then so has* $(\widetilde{A}^{(*)}, \widetilde{B}^{(*)})$.

Proof. By corollary 2.4 and P(3) the proof is completes. $\square$

Theorem 2.7. *Let* $A, B \in CR(H)$ *and let* $(A^\dagger, B^\dagger)$ *have the* FP*-property. Then* $Com(A^\dagger, B^\dagger) \subseteq Com((\widetilde{A})^\dagger, (\widetilde{B})^\dagger)$.

Proof. Let $A^\dagger = U^*|A^*|^\dagger$,$B^\dagger = V^*|B^*|^\dagger$ be the polar decomposition and let $\{f_n\}$ be a sequence of polynomials with no constant term such that $\{f_n(t)\} \to t^{\frac{1}{2}}$ as $t \to \infty$. Now let $X \in \mathrm{Com}(A^\dagger, B^\dagger)$, then by the hypothesis, $(A^\dagger)X = X(B^\dagger)$,$(A^\dagger)^*X = X(B^\dagger)^*$, thus $(|A|^\dagger)^2X = X(|B|^\dagger)^2$ then $f_n(|A|^\dagger)^2X = Xf_n(|B|^\dagger)^2$. Using the same argument we get $(|A|^\dagger)^{\frac{1}{2}}X = X(|B|^\dagger)^{\frac{1}{2}}$. Also $U^*|A^*|^\dagger X = XV^*|B^*|^\dagger$, then $|A|^\dagger U^*X = X|B|^\dagger V^*$, and the same argument above show that, $(|A|^\dagger)^{\frac{1}{2}}U^*X = X(|B|^\dagger)^{\frac{1}{2}}V^*$. Therefore,

$$\begin{aligned}(\widetilde{A})^\dagger X &= (|A|^\dagger)^{\frac{1}{2}}U^*(|A|^\dagger)^{\frac{1}{2}}X = (|A|^\dagger)^{\frac{1}{2}}U^*X(|B|^\dagger)^{\frac{1}{2}} \\ &= X(|B|^\dagger)^{\frac{1}{2}}V^*(|B|^\dagger)^{\frac{1}{2}} = X(\widetilde{B})^\dagger. \quad \square\end{aligned}$$

Lemma 2.8. *Let* $A \in CR(H)$*, then*
(i) $(|A|^\dagger)^q = U^*(|A^*|^\dagger)^qU$.
(ii) $A^\dagger$ *is quasinormal if and only if* $U^*|A^*|^\dagger = |A^*|^\dagger U^*$.

Proof. (i) By P(4), P(7), P(8) and P(9) we have

$$\begin{aligned}(|A|^\dagger)^2 &= |(A^*)^\dagger|^2 = |(A^\dagger)^*|^2 = A^\dagger(A^\dagger)^* = U^*|A^*|^\dagger|A^*|^\dagger U \\ &= U^*|A^*|^\dagger UU^*|A^*|^\dagger U = (U^*|A^*|^\dagger U)^2.\end{aligned}$$

Let $\{p_n\}$ be a sequence of polynomials with no constant term such that $p_n(t) \to \sqrt{t}$ uniformly on a certain compact subset of $\mathbb{R}^+$ as $n \to \infty$. It follows that $p_n((|A|^\dagger)^2) = p_n((U^*|A^*|^\dagger U)^2)$, and so $|A|^\dagger = U^*|A^*|^\dagger U$. By induction, $(|A|^\dagger)^{\frac{m}{n}} = U^*(|A^*|^\dagger)^{\frac{m}{n}} U$ holds for each $m, n \in \mathbb{N}$. Now, by using of the functional calculus, $(|A|^\dagger)^q = U^*(|A^*|^\dagger)^q U$.
(ii) It is a classical fact that A is quasinormal if and only if $U|A| = |A|U$ (see [7, Theorem 3]. Now, the desired conclusion follows from this and P(7). □

Theorem 2.9. *Let $A \in B(H)$ be onto. Then the following statements are equivalent:*
(i) A^* is quasinormal.
(ii) $A^\dagger$ is quasinormal.

Proof. (i)⇔(ii) By (1.1) we have $|A^*||A^*|^\dagger|A^*| = |A^*|$, then we get that

$$\begin{aligned} A^* \text{is quasinormal} &\Longleftrightarrow U^*|A^*| = |A^*|U^* \\ &\Longleftrightarrow U^*|A^*||A^*|^\dagger|A^*| = |A^*||A^*|^\dagger|A^*|U^* \\ &\Longleftrightarrow |A^*|U^*|A^*|^\dagger|A^*| = |A^*||A^*|^\dagger U^*|A^*| \\ &\Longleftrightarrow |A^*|(U^*|A^*|^\dagger - |A^*|^\dagger U^*)|A^*| = 0. \end{aligned}$$

By hypothesis, $\mathcal{N}(|A^*|) = \mathcal{N}(A^*) = \{0\}$. Hence $(U^*|A^*|^\dagger - |A^*|^\dagger U^*)|A^*| = 0$, and so $U^*|A^*|^\dagger = |A^*|^\dagger U^*$ on $\overline{R(|A^*|^\dagger)}$. On the other hand, $U^*|A^*|^\dagger = |A^*|^\dagger U^*$ on $N(|T^*|^\dagger) = N(U^*)$. Thus, $U^*|A^*|^\dagger = |A^*|^\dagger U^*$ on H. Consequently, by Lemma 2.8, (i)⇔(ii) holds. □

Theorem 2.10. *Let $A, B \in CR(H)$ be onto and A^*, B^* be quasinormal. If $(A^\dagger, B^\dagger)$ has the FP-property then $Com(A^\dagger, B^\dagger) = Com((\widetilde{A})^\dagger, (\widetilde{B})^\dagger)$.*

Proof. According to Theorem 2.7, it is sufficient to prove $Com(A^\dagger, B^\dagger) \supseteq Com((\widetilde{A})^\dagger, (\widetilde{B})^\dagger)$. Let $X \in Com((\widetilde{A})^\dagger, (\widetilde{B})^\dagger)$, $\Lambda = (|A|^\dagger)^{\frac{-1}{2}} X (|B|^\dagger)^{\frac{1}{2}}$. Since $(\widetilde{A})^\dagger X = X(\widetilde{B})^\dagger$ Then,

$(|A|^\dagger)^{\frac{-1}{2}}(\widetilde{A})^\dagger X(|B|^\dagger)^{\frac{1}{2}} = (|A|^\dagger)^{\frac{-1}{2}} X(\widetilde{B})^\dagger(|B|^\dagger)^{\frac{1}{2}}$.
$\Rightarrow U^*(|A|^\dagger)^{\frac{1}{2}} X(|B|^\dagger)^{\frac{1}{2}} = (|A|^\dagger)^{\frac{-1}{2}} X(|B|^\dagger)^{\frac{1}{2}} V^*|B|^\dagger$.
$\Rightarrow U^*(|A|^\dagger)(|A|^\dagger)^{\frac{-1}{2}} X(|B|^\dagger)^{\frac{1}{2}} = \Lambda V^*|B|^\dagger$.
$\Rightarrow U^*|A|^\dagger\Lambda = \Lambda V^*|B|^\dagger$.

$\Rightarrow U^*(U^*|A^*|^\dagger U)\Lambda = \Lambda V^*(V^*|B^*|^\dagger V)$. by Lemma 2.8(i)
$\Rightarrow U^*(|A^*|^\dagger U^*U)\Lambda = \Lambda V^*(|B^*|^\dagger V^*V)$. by Lemma 2.8(ii) and Theorem 2.9
$\Rightarrow U^*|A^*|^\dagger\Lambda = \Lambda V^*|B^*|^\dagger \Rightarrow A^\dagger\Lambda = \Lambda B^\dagger$.

Therefor $\Lambda \in Com(A^\dagger, B^\dagger)$. Also by the Lemma 1.1, and hypothesis, $\overline{R(\Lambda)}$ reduces $A^\dagger$, $(ker\Lambda)^\perp$ reduces $B^\dagger$ and $A^\dagger_{\overline{R(\Lambda)}}$, $B\ |^\dagger_{(ker\Lambda)^\perp}$ are unitarily equivalent normal operators. Thus,

$$A^\dagger = A_1^\dagger \oplus A_2^\dagger \qquad\qquad on\ \overline{R(\Lambda)} \oplus R(\Lambda)^\perp,$$

and

$$B^\dagger = B_1^\dagger \oplus B_2^\dagger \qquad\qquad on\ N(\Lambda)^\perp \oplus N(\Lambda),$$

where $A_1^\dagger$,$B_1^\dagger$ are unitarily equivalent normal operators. Since A is invertible and $A_1^\dagger$,$B_1^\dagger$ are unitarily equivalent then, $B_1^\dagger$ is invertible. Let,

$$X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}$$

and

$$\Lambda = \begin{pmatrix} \Lambda_1 & 0 \\ 0 & 0 \end{pmatrix}.$$

It follows from $\Lambda = (|A|^\dagger)^{\frac{-1}{2}} X (|B|^\dagger)^{\frac{1}{2}}$, that,

$$\Lambda = \begin{pmatrix} \Lambda_1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} (|A_1|^\dagger)^{\frac{-1}{2}} X_1 (|B_1|^\dagger)^{\frac{1}{2}} & (|A_1|^\dagger)^{\frac{-1}{2}} X_2 (|B_1|^\dagger)^{\frac{1}{2}} \\ (|A_2|^\dagger)^{\frac{-1}{2}} X_3 (|B_1|^\dagger)^{\frac{1}{2}} & (|A_2|^\dagger)^{\frac{-1}{2}} X_4 (|B_2|^\dagger)^{\frac{1}{2}} \end{pmatrix}.$$

Therefore, $X_2(|B_1|^\dagger)^{\frac{1}{2}} = 0$, $X_3 = 0$, $X_4(|B_2|^\dagger)^{\frac{1}{2}} = 0$, then as the result, $X_2(\widetilde{B_2})^\dagger = 0$, $X_4(\widetilde{B_2})^\dagger = 0$. Thus $(\widetilde{A})^\dagger X = X(\widetilde{B})^\dagger$, implies that,

$$\begin{pmatrix} A_1^\dagger X_1 & A_1^\dagger X_2 \\ 0 & A_2^\dagger X_4 \end{pmatrix} = \begin{pmatrix} X_1 B_1^\dagger & 0 \\ 0 & 0 \end{pmatrix}.$$

Then, $X_2 = 0$, $X_4 = 0$, $A_1^\dagger X_1 = X_1 B_1^\dagger$, and since

$$X = \begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix} = X_1 \oplus 0.$$

Therefore, $A^\dagger X = XB^\dagger$ and the proof is completed. $\square$

Corollary 2.11. *Let A^*, B^* be quasinormal. If (A, B) have the FP-property. Then $Com(A, B) = Com(\widetilde{A}^{(\dagger)}, \widetilde{B}^{(\dagger)})$.*

3. Moore-Penrose Inverse of Lambert Multiplication Operators

Let (X, Σ, μ) be a σ-finite measure space. For any σ-finite subalgebra $\mathcal{A} \subseteq \Sigma$ the Hilbert space $L^2(X, \mathcal{A}, \mu_{|\mathcal{A}})$ is abbreviated to $L^2(\mathcal{A})$. The support of a measurable function f is defined by $\sigma(f) = \{x \in X : f(x) \neq 0\}$. All sets and functions statements are to be interpreted as being valid almost everywhere with respect to μ. For each $f \in L^2(\Sigma)$, by the Radon-Nikodym theorem, there exists a unique $\mathcal{A}$-measurable function $E^{\mathcal{A}}(f)$ such that $\int_A f d\mu = \int_A E^{\mathcal{A}}(f) d\mu$, where A is any $\mathcal{A}$-measurable set for which $\int_A f d\mu$ exists. Put $E = E^{\mathcal{A}}$. The mapping E is a linear orthogonal projection. For more details on the properties of E see [9, 17].

Let $w, u \in L^0(\Sigma)$, the linear space of all complex-valued Σ-measurable functions on X. The mapping $T : L^2(\Sigma) \to L^2(\Sigma)$ defined by $T(f) = M_w E M_u(f) = wE(uf)$ is called Lambert multiplication operator. It is easy to check that for each $f \in L^2(\Sigma)$, $\|Tf\| = \|EM_\upsilon f\|$, where $\upsilon := u(E(|w|^2))^{\frac{1}{2}}$. Thus, $M_w E M_u$ is bounded (has closed range) if and only EM_υ is bounded (has closed range). Interesting articles related to this topic are [5, 6, 9]. A combination of conditional expectation operators and multiplication operators appears more often in the service of the study of other operators, such as operators generated by random measures and Markov operators. A good article related to the conditional type operators is [8].

Here we recall some results of [11] that state our results is valid for $T = M_w E M_u$.

Proposition 3.1. *Let $T : L^2(\Sigma) \to L^0(\Sigma)$ defined by $T = M_w E M_u$ is a Lambert multiplication operator.*
(i) $T \in B(L^2(\Sigma))$ if and only if $E(|w|^2)E(|u|^2) \in L^\infty(\mathcal{A})$, and in this case $\|T\| = \|E(|w|^2)E(|u|^2)\|_\infty^{1/2}$.
(ii) Let $T \in B(L^2(\Sigma))$, $0 \leqslant u \in L^0(\Sigma)$ and $\upsilon = u(E(|w|^2))^{\frac{1}{2}}$. If $E(\upsilon) \geqslant$

δ on $\sigma(v)$, then T has closed range.

Put

$$A(f) = \frac{u\chi_G}{E(u^2)E(w^2)}E(wf), \qquad f \in L^2(\Sigma),\ G = \sigma(E(w)). \tag{3}$$

Then by Proposition 3.1, $A \in B(L^2(\Sigma))$. Also, it is easy to check that

$$TAT = T, \quad ATA = A, \quad (TA)^* = TA, \quad (AT)^* = AT.$$

Thus, $A = T^\dagger$. We now turn to the computation of $T^\dagger$, $\widetilde{T}$, $(\widetilde{T})^\dagger$ and $\widetilde{T^\dagger}$. Direct computations give the following proposition.

Proposition 3.2. *Let $T, \widetilde{T} \in CR(L^2(\Sigma))$ with $u, w \geqslant 0$. Then*

(a) $T^\dagger = M_{\frac{u\chi_{\sigma(E(w))}}{E(u^2)E(w^2)}} EM_w$.

(b) $\widetilde{T} = M_{\frac{uE(uw)}{E(u^2)}} EM_u$.

(c) $(\widetilde{T})^\dagger = M_{\frac{u\chi_{\sigma(E(uw))}}{E(u^2)E(uw)}} EM_u$.

(d) $\widetilde{T^\dagger} = M_{\frac{\chi_S wE(uw)}{E(u^2)(E(w^2))^2}} EM_w$.

Note that, if w or u is not $\mathcal{A}$-measurable, then $(\widetilde{T})^\dagger \neq \widetilde{T^\dagger}$. Moreover, if $T \in CR(L^2(\Sigma))$, then by Proposition 3.1(ii), $\widetilde{T} \in CR(L^2(\Sigma))$ whenever $(E(uw))^2 \geqslant E(u^2)E(w^2)$.

Proposition 3.3. *Let for $i = 1, 2$, $T_i = M_{w_i}EM_{u_i} \in B(L^2(\Sigma))$ and $(E(w_i^2))^{\frac{1}{2}} w_i = u_i(E(u_i^2))^{\frac{1}{2}}$. Then (T_1, T_2) has the FP-property.*

Proof. We know that if T_1 and T_2 are normal, then (T_1, T_2) has the FP-property. Thus it is sufficient to prove that T_1 and T_2 are normal. Since $T_i^* = M_{u_i}EM_{w_i}$, it is easy to check that $T_i^*T_i = M_{u_iE(w_i^2)}EM_{u_i}$ and $T_iT_i^* = M_{w_iE(u_i^2)}EM_{w_i}$. So $T_i^*T_i - T_iT_i^* = M_{u_iE(w_i^2)}EM_{u_i} - M_{w_iE(u_i^2)}EM_{w_i}$. Then by hypothesis we obtain

$$\begin{aligned}\langle (T_i^*T_i - T_iT_i^*)f, f\rangle &= \int_X \{(E(w_i^2)E(u_if)uf - E(u_i^2)E(w_if)wf\}d\mu \\ &= \int_X \{(E(u_i(E(w_i^2))^{\frac{1}{2}}f))^2 - (E((E(u_i^2))^{\frac{1}{2}}w_if))^2\}d\mu = 0,\end{aligned}$$

for each $f \in L^2(\Sigma)$. This implies that T_i are normal. □

In [5], Estaremi show that the Aluthge transform of $M_w E M_u$ is always normal. So we have the following corollary.

Corollary 3.4. *Let* $T, \widetilde{T} \in CR(L^2(\Sigma))$ *with* $u, w \geqslant 0$. *Then*

(a) $(\widetilde{T_1}, \widetilde{T_2})$ *has the FP-property.*

(b) $((\widetilde{T_1})^\dagger, (\widetilde{T_2})^\dagger)$ *has the FP-property.*

(b) $(\widetilde{T_1^\dagger}, \widetilde{T_2^\dagger})$ *has the FP-property.*

Let $\mathcal{A} = \varphi^{-1}(\Sigma)$, $0 \leqslant u \in L^0(\Sigma)$ and $\varphi : X \to X$ be a measurable transformation such that $\mu \circ \varphi^{-1}$ is absolutely continuous with respect to μ. The weighted composition operator W on $L^2(\Sigma)$ induced by the pair (u, φ) is given by $W = M_u \circ C_\varphi$, where C_φ is the composition operator defined by $C_\varphi f = f \circ \varphi$. It is a classical fact that W is a bounded linear operator on $L^2(\Sigma)$, if and only if $J := hE(u^2) \circ \varphi^{-1} \in L^\infty(\Sigma)$. Also, $W \in CR(L^2(\Sigma))$ if and only if J is bounded away from zero on $\sigma(J)$ (see [10]). From now on, we assume that W has closed range. It is easy to check that $W^\dagger = M_{\frac{\chi_{\sigma(J)}}{J}} W^*$ and $(W^\dagger)^* = M_{\frac{\chi_{\sigma(J\circ\varphi)}}{J\circ\varphi}} W$.

Now, we can compute the polar decomposition and Aluthge transformations of $W = U|W|$ and $W^\dagger = U^*|W^\dagger|$ as follows:

$$
\begin{aligned}
|W| &= M_{\sqrt{J}};\\
U &= M_{\frac{\chi_{\sigma(J\circ\varphi)}}{\sqrt{J\circ\varphi}}} W;\\
U^* &= M_{\frac{\chi_{\sigma(J)}}{\sqrt{J}}} W^*;\\
|W^\dagger| &= M_{\frac{u\chi_{J\circ\varphi}}{\sqrt{J\circ\varphi}(E(u^2))}} E M_u;\\
|W^\dagger|^{\frac{1}{2}} &= M_{\frac{u}{(h\circ\varphi)^{\frac{1}{4}}(E(u^2))^{\frac{5}{4}}}} E M_u.
\end{aligned}
$$

Consequently, for each $f \in L^2(\Sigma)$ we get that

$$
\begin{aligned}
\widetilde{W}\ (f) &= u\left(\frac{J\chi_{\sigma(E(u))}}{(h\circ\varphi)E(u^2)}\right)^{\frac{1}{4}}(f\circ\varphi);\\
\widetilde{W^\dagger}\ (f) &= \left(\frac{1}{(h\circ\varphi)(E(u^2))^5}\right)^{\frac{1}{4}} uE\left(\frac{u\sqrt[4]{h}\chi_C}{(E(u^2)\circ\varphi^{-1})^{\frac{3}{4}}}E(uf)\circ\varphi^{-1}\right);\\
(\widetilde{W})^\dagger(f) &= \left(\frac{\chi_{\sigma(J)}}{\sqrt[5]{h}E(u^2)\circ\varphi^{-1}}\right)^{\frac{5}{4}} E(u\sqrt[4]{J}f)\circ\varphi^{-1};\\
(\widetilde{W^*})^\dagger(f) &= \left(\frac{\chi_C}{E(u^2)\circ\varphi^{-1}}\right)^{\frac{5}{2}} E(uf)\circ\varphi^{-1},
\end{aligned}
$$

where $C = \chi_{\sigma(E(u^2)\circ\varphi^{-1})}$.

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References

[1] A. Ben-Israel and T. N. E. *Greville, Generalized Inverses: Theory and Applications,* Wiley-Interscience, New York, 1974.

[2] D. S. Djordjević and N. C. Dinčić, Reverse order law for the MoorePenrose inverse, *J. Math. Anal. Appl.*, 361 (2010), 252-261.

[3] D. S. Djordjević, Further results on the reverse order law for generalized inverses, *SIAM J. Matrix Anal. Appl.*, 29 (2007), 1242-1246.

[4] Y. Estaremi and M. R. Jabbarzadeh, Weighted lambert type operators on Lp-spaces, *Oper. Matrices.,* 7 (2013), 101116.

[5] Y. Estaremi, On a class of operators with normal Aluthge transformations, *Filomat,* 29 (2015), 969975.

[6] Y. Estaremi, Some classes of weighted conditional type operators and their spectra, *Positivity,* 19 (2015), 83-93.

[7] T. Furuta, On the polar decomposition of an operator, *Acta Sci. Math. (Szeged),* 46 (1983), 261-268.

[8] J. J. Grobler and B. De Pagter, Operators representable as multiplicationconditional expectation operators, *J. Oper. Theory*, 48 (2002), 15-40.

[9] J. Herron, Weighted conditional expectation operators, *Oper. Matrices,* 5 (2011), 107-118.

[10] T. Hoover, A. Lambert, and J. Queen, The Markov process determined by a weighted composition operator, *Studia Math.*, (Poland) (1982), 225-235.

[11] M. R. Jabbarzadeh, A conditional expectation type operator on L^p spaces, *Oper. Matrices,* 4 (2010), 445-453.

[12] M. R. Jabbarzadeh and M. Jafari Bakhshkandi, *Centered operators via Moore-Penrose inverse and Aluthge transformations,* submitted.

[13] F. Kimura, Analysis of non-normal operators via Aluthge transformation, *Integral Equations Operator Theory,* 50 (2004), 375-384.

[14] C. S. Kubrusly, *Hilbert Space Operators: A Problem Solving Approach,* Birkhuser, Boston, 2003.

[15] S. H. Kulkarni, M. T. Nair, and G. Ramesh, Some properties of unbounded operators with closed range. Proc. *Indian Acad. Sci. Math. Sci.,* 118 (4) (2008), 613-625.

[16] M. H. M. Rashid, Fuglede-Putnam type theorems via the generalized Aluthge transform, Rev. R. Acad. Cienc. *Exactas Fís. Nat. Ser. A Math. RACSAM,* 108 (2014), 1021-1034.

[17] M. M. Rao, *Conditional Measure and Applications,* Marcel Dekker, New York, 1993.

[18] K. Takahashi, On the converse of Fuglede-Putnam theorem, *Acta Sci. Math.,* 43 (1981), 123-125.

[19] T. Yamazaki, Parallelisms between Aluthge transformation and powers of operators, *Acta Sci. Math.,* 67 (2001), 809-820.

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