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Lineability of Denjoy Integrable Function

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Abstract. Gordon presented an example of a function that is Denjoy integrable but not Lebesgue integrable in the book: The integrals of Lebesgue, Denjoy, Peron and Henstock. That example showed that, the class of Denjoy integrable functions is larger than the class of Lebesgue integrable functions. We are going to show that the set of Denjoy integrable functions that are not Lebesgue integrable, contains an infinite dimensional vector space.

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1. Introduction

Finding infinite dimensional algebraic structures and infinitely generated algebras in different subsets of various spaces is relatively new trend in mathematical analysis.

Recall that a subset S of a vector space V is called lineable if $S \cup \{0\}$ contains an infinite dimensional vector subspace of V. Also if V is a topological vector space then S is called spaceable if $S \cup \{0\}$ contains a closed infinite dimensional vector subspace of V. These notions were first appeared in an unpublished notes of Enflo and Gurariy. Aron and Gurariy published those notes in [2]. We should mention that Enflo's

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and gurariy's unpublished notes were completed in collaboration with Seoane-Sepulvida and finally published in [6].

The origin of lineability is due to Gurariy ([12, 13]) who showed the existence of an infinite dimensional linear space such that every non-zero element of which is a continuous nowhere differentiable function on $\mathcal{C}[0;1]$. Many examples of vector spaces of functions on $\mathbb{R}$ or $\mathbb{C}$ enjoying certain special properties have been constructed in the recent years. More recently, many authors got interested in this subject and gave a wide range of exampls. For more results on lineability we refer the reader to [7, 8, 9].

Our concern in this paper is the set of Denjoy integrable functions that are the extension of Lebesgue measurable functions. To define Denjoy integrable functions we need a variation of the bounded variation functions. The notion of bounded variation and absolute continuity on an interval play a key role in the theory of Lebesgue integral. The extension of these concepts from intervals to arbitrary sets will play a major role in the development of the integrals that generalize the Lebesgue integral. First we bring some definitions and primary results.

Definition 1.1. *Let $F : [a, b] \to \mathbb{R}$ be an arbitrary function. The oscillation of the function F on the interval $[a, b]$ is*

$$\omega(F, [a, b]) = \sup\{|F(y) - F(x)| : a \leqslant x < y \leqslant d\}.$$

Definition 1.2. *Let $F : [a, b] \to \mathbb{R}$ and let $E \subset [a, b]$.*

(a) The weak variation of F on E and the strong variation of F on E are defined by

$$V(F, E) = \sup\{\sum_{i=1}^{n} |F(d_i) - F(c_i)|\};$$

$$V_*(F, E) = \sup\{\sum_{i=1}^{n} \omega(F, [c_i, d_i])\},$$

respectively, where the supremum in each case is taken over all the finite collections $\{[c_i, d_i] : 1 \leqslant i \leqslant n\}$ of non-overlapping intervals that have endpoints in E.

(b) The function F is absolutely continuous on E (F is AC on E) if for each $\varepsilon > 0$ there exits $\delta > 0$ such that $\sum_{i=1}^{n} |F(d_i) - F(c_i)| < \varepsilon$ whenever $\{[c_i, d_i] : 1 \leqslant i \leqslant n\}$ is a finite collection of non-overlapping intervals that have endpoins in E and satisfy $\sum_{i=1}^{n}(d_i - c_i) < \delta$. The function F is absolutely continuous in the restricted sense on E (F is AC_* on E) if for each $\varepsilon > 0$ there exits $\delta > 0$ such that $\sum_{i=1}^{n} \omega(F, [c_i, d_i]) < \varepsilon$ whenever $\{[c_i, d_i] : 1 \leqslant i \leqslant n\}$ is a finite collection of non-overlapping intervals that have endpoins in E and satisfy $\sum_{i=1}^{n}(d_i - c_i) < \delta$.

(c) The function F is generalized absolutely continuous on E (F is ACG on E) if $F|_E$ is continuous on E and E can be written as a countable union of sets on each of which F is AC. The function F is generalized absolutely continuous in restricted sense on E (F is ACG_* on E) if $F|_E$ is continuous on E and E can be written as a countable union of sets on each of which F is AC_*.

It is easy to see that the concept of weak variation and strong variation of a function coincide on an closed interval. In this case f is $AC(ACG)$ if and only if it is $AC_*(ACG_*)$.

The following theorem helps us showing that the Denjoy integral (that will be defined in the sequel)of an ACG function can be uniquely determined up to an additive constant.

Theorem 1.3. *Let $f : [a, b] \to \mathbb{R}$ be ACG on $[a, b]$. If $f' = 0$ almost everywhere on $[a, b]$, then f is constant on $[a, b]$.*

Proof. See page 104 of [10]. □

Now we want to define Denjoy integrable functions. This definition will be achieved by expanding a nice property of Lebesgue measurable functions that is presented in the following theorem.

Theorem 1.4. ([10]) *Let F be a real valued continuous function defined on $[a, b]$. If F is differentiable nearly everywhere on $[a, b]$ and if F' is Lebesgue integrable on $[a, b]$, then $\int_a^x F' = F(x) - F(a)$ for each $x \in [a, b]$.*

The phrase "nearly everywhere" in the above theorem means that the property holds on all points but a countable set. The hypothesis that F'

be Lebesgue measurable on $[a,b]$ is necessary. As an instance take

$$F(x) = \begin{cases} x^2 \sin\left(\frac{\pi}{x^2}\right), & if \quad 0 < x \leqslant 1 \\ 0, & if \quad x = 0 \end{cases}$$

This function has derivative at each point of $[0,1]$, but F is not absolutely continuous on $[0,1]$. Consequently the function F' is not Lebesgue integrable on $[0,1]$. In fact if F' is Lebesgue integrable on $[0,1]$, put $G(x) = \int_0^x F'$ for each $x \in [0,1]$. The function F and G are ACG_* on $[0,1]$ and their derivatives are equal almost everywhere on $[0,1]$. On the other hand $F(0) = G(0)$, thus by Theorem 1.3, the functions F and G are equal on $[0,1]$. But this implies that F is AC on $[0,1]$, a contradiction.

This rose the following question:

Is there an integration process that holds the following property? Let $F : [a,b] \to \mathbb{R}$ be a continuous function. If F is differentiable nearly everywhere on $[a,b]$, then F' is integrable on $[a,b]$ and $\int_a^x F' = F(x) - F(a)$ for each $x \in [a,b]$.

An integral with this property is said to recover a function from its derivative. In addition, any integral that satisfies the above theorem should include the Lebesgue integral. That is, any function that is Lebesgue integrable should be integrable in the new sense and the integrals should be equal. In 1912, A. Denjoy developed an integration process that satisfies the theorem quoted above. He called the process of computing the value of his integral "totalization" and showed that every derivative met the criteria for this process and that the original function was recovered. This totalization is a rather complicated process that involves the use of transfinite numbers. A few months after Denjoy's work, N. Lusin connected the new integral and the notion of generalized absolute continuity. This is the approach that will be followed here.

As it is shown in [10], a function $f : [a,b] \to \mathbb{R}$ is Lebesgue integrable on $[a,b]$ if and only if there exists an AC function $F : [a,b] \to \mathbb{R}$ such that $F' = f$, almost everywhere on $[a,b]$. The Denjoy integral is a simple generalization of this characterization of the Lebesgue integral.

Definition 1.5. *A function $f : [a,b] \to \mathbb{R}$ is Denjoy integrable on $[a,b]$*

if there exists an ACG_* *function* $F : [a, b] \to \mathbb{R}$ *such that* $F' = f$ *almost everywhere on* $[a, b]$*. The function* f *is Denjoy integrable on a measurable set* $E \subseteq [a, b]$ *if* $f\chi_E$ *is Denjoy integrable on* $[a, b]$*.*

By theorem 1.3, the Denjoy integral of a function is uniquely determined up to an additive constant. If we add the condition that $F(a) = 0$, then the function F is unique. We will denote this function by $(D)\int_a^x f$. It is shown in [10] that Denjoy integral recovers a function from its derivative. Actually they proved the following theorem.

Theorem 1.6. *Let* $F : [a, b] \to \mathbb{R}$ *be a continuous function. If* F *is differentiable nearly everywhere on* $[a, b]$*, then* F' *is Denjoy integrable on* $[a, b]$ *and* $(D)\int_a^x F' = F(x) - F(a)$ *for each* $x \in [a, b]$*.*

The Denjoy integral has all of the usual properties of an integral.

2. Main Theorem

In this section we are going to prove that the set of Denjoy integrable functions that are not Lebesgue integrable contains an infinite dimensional vector space. First we need the following lemma.

Lemma 2.1. *Let* $n \in \mathbb{N}$ *be arbitrary and define* $F_n : [0, 1] \to \mathbb{R}$ by

$$F_n(x) = \begin{cases} x^n \sin\left(\frac{\pi}{x^n}\right), & if \ \ 0 < x \leqslant 1 \\ 0, & if \ \ x = 0 \end{cases}$$

Then F_n *is* ACG *on* $[0, 1]$ *but it is not* AC *on* $[0, 1]$*.*

Proof. Since F_n is continuous on each interval $[1/m, 1]$, for all $m \geqslant 1$, so it is ACG on $[0, 1]$. To prove that F is not AC on $[0, 1]$, let $0 < \delta$ be arbitrary. For each $m \geqslant 1$ put $a_m = \left[\frac{2}{4m+1}\right]^{1/n}$ and $b_m = \left[\frac{2}{4m}\right]^{1/n}$. Let $M, N \in \mathbb{N}$ be chosen such that $M < N$ and $\sum_{i=M}^{N} \frac{2}{4i+1} > 1$. Then $\{[a_m, b_m] : M < m < N\}$ is a finite collection of non-overlapping intervals in $[0, 1]$ that satisfy $\sum_{i=M}^{N}(b_i - a_i) < \delta$, but

$$\sum_{i=M}^{N} |F_n(b_i) - F_n(a_i)| = \sum_{i=M}^{N} \frac{2}{4i+1} > 1.$$

This completes the proof. $\square$

Theorem 2.2. *The set of Denjoy integrable functions that are not Lebesgue integrable is lineable.*

Proof. For all $i \geqslant 1$, define $n_i = i!$ and put $G_i = F_{n_i}$. By Lemma 2.1, G_i is ACG but not AC on $[0,1]$. Since the concept of ACG and ACG_* coincide on closed intervals, so G_i is ACG_* on $[0,1]$.

Now for each $i \geqslant 1$ put $g_i = G_i^{'}$. By the Definition 1.5 and the discussion before the definition, each g_i is Denjoy integrable but not Lebesgue integrable. We are going to prove that each finite linear combination of elements of $\{g_i : n \geqslant 1\}$ is still Denjoy integrable and is not Lebesgue integrable. Also we will show that the above set is linearly independent.

To this end let $i_1 < i_2 < \cdots < i_k$ be prime numbers and $\alpha_1, \ldots, \alpha_k$ be arbitrary constants. Let $\alpha_1 g_{i_1} + \ldots + \alpha_k g_{i_k} = 0$. Since Denjoy integral recovers a function from its derivative and also since $g_i(0) = 0$ for all $i \geqslant 1$, so $\alpha_1 G_{i_1} + \ldots + \alpha_k G_{i_k} = 0$. Put $x_1 = \frac{1}{2^{1/(i_1+1)!}}$. For all $2 \leqslant j \leqslant k$, $G_{i_j}(x_1) = 0$ and $G_{i_1}(x_1) \neq 0$. Thus $\alpha_1 = 0$. Put $x_2 = \frac{1}{2^{1/(i_2+1)!}}$. For all $3 \leqslant j \leqslant k$, $G_{i_j}(x_1) = 0$ and $G_{i_2}(x_1) \neq 0$. Thus $\alpha_2 = 0$ ans so on. Therefore the set $\{g_i : i \in \mathbb{N}\}$ is linearly independent.

To prove that each finite linear combination of elements of $\{g_i : i \geqslant 1\}$ is Denjoy integrable and is not Lebesgue integrable, it suffices to show that each finite linear combination of elements of $\{G_i : i \geqslant 1\}$ is ACG but not AC on $[0,1]$. Let $i_1 < i_2 < \cdots < i_k$ be distinct integers and $\alpha_1, \ldots, \alpha_k$ be arbitrary constants and consider the linear combination $\alpha_1 G_{i_1} + \ldots + \alpha_k G_{i_k}$. The proof that this linear combination is ACG, is just similar to the one we brought above to show that a single function is ACG on $[0,1]$.

To show that this combination is not AC on $[0,1]$, we follow the same process that we employed in Lemma 2.1. Let $\delta > 0$ be arbitrary and consider $a_m = \left[\frac{2}{4m+1}\right]^{1/n_1!}_{,}$ $b_m = \left[\frac{2}{4m}\right]^{1/n_1!}_{.}$ Since the series

$$\sum \alpha_1 \frac{2}{4m+1} + \alpha_2 \left[\frac{2}{4m+1}\right]^{(n_2-n_1)!} + \ldots + \alpha_k \left[\frac{2}{4m+1}\right]^{(n_k-n_1)!}_{,}$$

is divergent and the series $\sum(b_m - a_m)$ is convergent, so we can choose M and N such that $M < N$, $\sum_{m=M}^{N}(b_m - a_m) < \delta$ and

$$\sum_{m=M}^{N} \alpha_1 \frac{2}{4m+1} + \alpha_2 \left[\frac{2}{4m+1}\right]^{(n_2-n_1)!} + \ldots + \alpha_k \left[\frac{2}{4m+1}\right]^{(n_k-n_1)!} > 1.$$

Now we have

$$\sum_{m=M}^{N} |(\alpha_1 G_{i_1}(b_m) + \ldots \alpha_k G_{i_k}(b_m)) - (\alpha_1 G_{i_1}(a_m) + \ldots \alpha_k G_{i_k}(a_m))| =$$

$$\sum_{m=M}^{N} \alpha_1 \frac{2}{4m+1} + \alpha_2 \left[\frac{2}{4m+1}\right]^{(n_2-n_1)!} + \ldots + \alpha_k \left[\frac{2}{4m+1}\right]^{(n_k-n_1)!} > 1.$$

This completes the proof. □

Remark 2.3. *Here we proved $\aleph_0$-lineability of the set of Denjoy integrable functions that are not Lebesgue integrable. One can ask whether it is the maximum dimension of the vector space? Or is there any algebra contained in this set of functions?*

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