Equitable Coloring of Mycielskian of Some Graphs

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Abstract. In a search for triangle-free graphs with arbitrarily large chromatic numbers, Mycielski developed a graph transformation that transforms a graph $G$ into a new graph $\mu(G)$, we now call the Mycielskian of $G$, which has the same clique number as $G$ and whose chromatic number equals $\chi(G) + 1$. This paper presents exact values of the equitable chromatic number $\chi^e$ for the Mycielski’s graph of complete graphs $\mu(K_n)$, the Mycielski’s graph of cycles $\mu(C_n)$, the Mycielski’s graph of paths $\mu(P_n)$, the Mycielski’s graph of Helm graphs $\mu(H_n)$ and the Mycielski’s graph of Gear graphs $\mu(G_n)$.

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1. Introduction

The notion of equitable coloring was introduced by Meyer in 1973. This model of graph coloring has many applications. Everytime when we have to divide a system with binary conflicting relations into equal or almost
equal conflict-free subsystems we can model this situation by means of equitable graph coloring. This subject is widely discussed in literature [1, 3, 4, 6, 7, 11, 12, 16, 18].

One motivation for equitable coloring suggested by Meyer [13] concerns scheduling problems. In this application, the vertices of a graph represent a collection of tasks to be performed, and an edge connects two tasks that should not be performed at the same time. A coloring of this graph represents a partition of tasks into subsets that may be performed simultaneously. Due to load balancing considerations, it is desirable to perform equal or nearly-equal numbers of tasks in each time step, and this balancing is exactly what an equitable coloring achieves. Furmańczyk [7] mentions a specific application of this type of scheduling problem, namely assigning university courses to time slots in a way that spreads the courses evenly among the available time slots and avoids scheduling incompatible pairs of courses at the same time as each other, since then the usage of additional resources (e.g. rooms) is maximal.

The notion of equitable colorability was introduced by Meyer [13]. However, an earlier work of Hajnal and Szemerédi [9] showed that a graph $G$ with maximal degree $\Delta$ is equitably $k$-colorable if $k \geq \Delta(G) + 1$. Recently, Kierstead et al. [10] have given an $O(\Delta |V(G)|^2)$-time algorithm for equitable $(\Delta + 1)$-coloring of a graph $G$. In 1973, Meyer [13] formulated the following conjecture:

**Conjecture 1.1.** [Equitable Coloring Conjecture (ECC)] For any connected graph $G$, other than complete graph or odd cycle, $\chi_e(G) \leq \Delta(G)$.

This conjecture has been verified for all graphs on six or fewer vertices. Lih and Wu [11] proved that the Equitable Coloring Conjecture is true for all bipartite graphs. Wang and Zhang [16] considered a broader class of graphs, namely $r$-partite graphs. They proved that Meyer’s conjecture is true for complete graphs from this class. Also, the conjecture was confirmed for outerplanar graphs [17] and planar graphs with maximum degree at least 13 [18]. There are very few papers on the complexity of equitable coloring. First of all, a straightforward reduction from graph coloring to equitable coloring by adding sufficiently many isolated vertices to a graph, proves that it is NP-complete to test whether a graph
has an equitable coloring with a given number of colors (greater than two). Secondly, Bodlaender and Fomin [1] showed that equitable coloring can be solved to optimality in polynomial time for trees (previously known due to Chen and Lih [4]) and outerplanar graphs. A polynomial time algorithm is also known for equitable coloring of split graphs [3]. Furmańczyk et al. [8] proved that the problem remains NP-complete for the corona of line graph of cubic graphs (i.e. $L(K_4) \circ K_2$).

2. Preliminaries

If the set of vertices of a graph $G$ can be partitioned into $k$ classes $V_1, V_2, \ldots, V_k$ such that each $V_i$ is an independent set and the condition $|V_i| - |V_j| \leq 1$ holds for every pair $(i, j)$, then $G$ is said to be \textit{equitably $k$-colorable}. The smallest integer $k$ for which $G$ is equitably $k$-colorable is known as the \textit{equitable chromatic number} of $G$ and denoted by $\chi_e(G)$.

For any integer $n \geq 4$, the wheel graph $W_n$ is the $n$-vertex graph obtained by joining a vertex $x_0$ to each of the $n - 1$ vertices $\{x_1, x_2, \ldots, x_{n-1}\}$ of the cycle graph $C_{n-1}$. (see Figure. 1).

![Figure 1. Wheel Graph $W_n$](image)

The Helm graph $H_n$ is the graph obtained from a wheel graph $W_n$ by adjoining a pendant edge to each vertex of the $(n - 1)$-cycle in $W_n$. (see Figure. 2).
The Gear graph $G_n$ is obtained from a wheel graph $W_n$ by adding a vertex to each edge of the $(n - 1)$-cycle in $W_n$. (see Figure 3).

The open neighborhood of a vertex $x$ in a graph $G$, denoted by $N_G(x)$, is the set of all vertices of $G$, which are adjacent to $x$. Also, $N_G[x] = N_G(x) \cup \{x\}$ is called the closed neighborhood of $x$ in the graph $G$. 
In this paper, by $G$ we mean a connected graph. From a graph $G$, by Mycielski’s construction [2, 14, 15], we get the Mycielskian $\mu(G)$ of $G$ with $V(\mu(G)) = V \cup U \cup \{z\}$, where

$$V = V(G) = \{x_1, \ldots, x_n\}, \quad U = \{y_1, \ldots, y_n\}, \quad \text{and}$$

$$E(\mu(G)) = E(G) \cup \{y_i: x \in N_G(x_i) \cup \{z\}, i = 1, \ldots, n\}.$$ 

The paper is organized as follows. We start the next section with a theorem concerning the equitable coloring of Mycielskian of complete graphs. In the subsequent Sections 4, 5, 6 and 7 results concerning the equitable colorability of Mycielskian of cycles, paths, Helm graphs and Gear graphs are discussed. In this way we establish a new class of graphs that can be colored optimally in polynomial time and confirm the ECC conjecture.

3. Equitable Coloring of Mycielskian of Complete Graphs

**Theorem 3.1.** The equitable chromatic number of Mycielskian of complete graph $\chi_=(\mu(K_n)) = n + 1$.

**Proof.** Let $V(K_n) = \{x_i : 1 \leq i \leq n\}$. By Mycielski’s construction,

$$V(\mu(K_n)) = V(K_n) \cup \{y_i : 1 \leq i \leq n\} \cup \{z\} \quad \text{and}$$

$$E(\mu(K_n)) = E(K_n) \cup \{y_i: x \in N_{K_n}(x_i) \cup \{z\}, i = 1, 2, \ldots, n\}.$$ 

Since $z$ is adjacent with each vertex of $\{y_i : 1 \leq i \leq n\}$, in which each vertex is adjacent with $z$. Also $\mu(K_n)$ contains a $n$-clique, $\chi(\mu(K_n)) \geq n$ and hence $\chi_=(\mu(K_n)) \geq n$.

Assume that $\chi_=(\mu(K_n)) = n$. (i.e), There exists a partition $P = \{V_1, V_2, \ldots, V_n\}$, since $|V(\mu(K_n))| = 2n+1$, each $V_i (1 \leq i \leq n)$ contains either 2 or 3 vertices. Since each $x_i$ is adjacent with each $y_j (1 \leq j \leq n)$ where $i \neq j$, $x_i, y_i \in V_i (1 \leq i \leq n)$. Since $z$ is adjacent to each $y_i (1 \leq i \leq n)$, $z \notin V_i$ for every $i, (1 \leq i \leq n)$.
\[ |V_i| = 2 \text{ for all } 1 \leq i \leq n \]

\[ \Rightarrow \sum_{i=1}^{n} |V_i| = 2n \neq 2n + 1 \]

This contradiction shows that \( \chi = (\mu(K_n)) \geq n + 1 \). Now we partition the vertex set \( V(\mu(K_n)) \) as follows,

\[ V_i = \{x_i, y_i : 1 \leq i \leq n\}, \]
\[ V_{n+1} = \{z\}. \]

Clearly \( V_i (1 \leq i \leq n) \) and \( V_{n+1} \) are independent sets, since \( |V_i| = 2(1 \leq i \leq n) \) and \( |V_{n+1}| = 1 \) satisfying the condition \( ||V_i| - |V_j|| \leq 1 \), for any \( i \neq j \), \( \chi = (\mu(K_n)) \leq n + 1 \). Hence \( \chi = (\mu(K_n)) = n + 1 \). \( \square \)

4. Equitable Coloring of Mycielskian of Cycles

**Theorem 4.1.** The equitable chromatic number of Mycielskian of cycle \( \chi = (\mu(C_n)) \) is

\[ \chi = \begin{cases} 
3 & n = 4, 6, 8 \\
4 & \text{if } n \geq 10, n \text{ is even} \\
4 & \text{if } n \text{ is odd}.
\end{cases} \]

**Proof.** Let \( V(C_n) = \{x_i : 1 \leq i \leq n\} \) be the set of vertices of \( C_n \) taken in cyclic order. By Mycielski’s construction,

\[ V(\mu(C_n)) = V(C_n) \cup \{y_i : 1 \leq i \leq n\} \cup \{z\} \text{ and} \]
\[ E(\mu(C_n)) = E(C_n) \cup \{y_ix : x \in N_{C_n}(x_i) \cup \{z\}, i = 1, 2, \ldots, n\}. \]

Now we partition the vertex set of \( V(\mu(C_n)) \) as follows:

**Case (i)a** For \( n = 4 \):

\[ V_1 = \{x_1, x_3, z\}, \]
\[ V_2 = \{x_2, x_4, y_4\}, \]
\[ V_3 = \{y_1, y_2, y_3\}. \]
Case (i)b For $n = 6$:

\[
V_1 = \{x_1, x_3, x_5, z\},
V_2 = \{x_2, x_4, x_6, y_4, y_6\},
V_3 = \{y_1, y_2, y_3, y_5\}.
\]

Case (i)c For $n = 8$:

\[
V_1 = \{x_1, x_3, x_5, x_7, z\},
V_2 = \{x_2, x_4, x_6, x_8, y_4, y_6\},
V_3 = \{y_1, y_2, y_3, y_5, y_7, y_8\}.
\]

It is clear that $V_i \cap V_j = \emptyset$ for $i \neq j$. Since there exists a 5-cycle $x_1 x_2 y_3 y_2 x_1$, $\chi(\mu(C_n)) \geq 3$ and hence $\chi(\mu(C_n)) = 3$.

Case (ii) If $n \geq 10$, $n$ is even and $n$ is odd.

Since there exists a 5-cycle as shown in the previous case, it is clear that $\chi(\mu(C_n)) \geq 3$.

We know that $|V(\mu(C_n))| = 2n + 1$. Out of these $2n + 1$, the vertices of $\{x_{2i} : 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor\} \cup \{z\}$ can be assigned with one color. i.e., $\left\lfloor \frac{n}{2} \right\rfloor + 1$ vertices can be colored with one color. Hence remaining number of vertices to be colored is $(2n + 1) - \left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right) = \left\lfloor \frac{3n}{2} \right\rfloor$.

If $\chi(\mu(C_n)) = 3$, these vertices should be divided into two sets, each containing $\left\lfloor \frac{3n}{4} \right\rfloor$ and $\left\lfloor \frac{3n}{4} \right\rfloor + 1$ vertices.

Since $\left\lfloor \frac{3n}{4} \right\rfloor \neq \left\lfloor \frac{n}{2} \right\rfloor + 1$ and $\left\lfloor \frac{3n}{4} \right\rfloor + 1 \neq \left\lfloor \frac{n}{2} \right\rfloor + 1$ for even $n \geq 10$ and odd $n$, we conclude that the partition of $V(\mu(C_n))$ into three sets satisfying $||V_i| - |V_j|| \leq 1$ for $i \neq j$ is not possible and hence $\chi(\mu(C_n)) \geq 4$, if $n \geq 10$ and $n$ is odd.

Now the following partition gives the equitable coloring of $\mu(C_n)$ for even $n \geq 10$ and odd $n$. 
If $n \geq 10$, $n$ is even:

\[ V_1 = \left\{ x_{2i-1} : 1 \leq i \leq \frac{n}{2} \right\} \cup \{z\}, \]
\[ V_2 = \left\{ x_{2i} : 1 \leq i \leq \frac{n}{2} \right\}, \]
\[ V_3 = \left\{ y_{2i-1} : 1 \leq i \leq \frac{n}{2} \right\}, \]
\[ V_4 = \left\{ y_{2i} : 1 \leq i \leq \frac{n}{2} \right\}. \]

If $n$ is odd:

\[ V_1 = \left\{ x_{2i-1} : 1 \leq i \leq \frac{n-1}{2} \right\} \cup \{z\}, \]
\[ V_2 = \left\{ x_{2i} : 1 \leq i \leq \frac{n-1}{2} \right\} \cup \{y_{n-1}\}, \]
\[ V_3 = \left\{ y_{2i-1} : 2 \leq i \leq \frac{n+1}{2} \right\} \cup \{x_n\}, \]
\[ V_4 = \left\{ y_{2i} : 1 \leq i \leq \frac{n-3}{2} \right\} \cup \{y_1\}. \]

This implies $\chi^e(\mu(C_n)) = 4$. $\square$

5. Equitable Coloring of Mycielskian of Paths

**Theorem 5.1.** The equitable chromatic number of Mycielskian of path

\[ \chi^e(\mu(P_n)) = \begin{cases} 2 & n = 1, \\ 3 & \text{if } n \leq 11, n \neq 10, \\ 4 & \text{if } n \geq 10, n \neq 11. \end{cases} \]

**Proof.** Let $V(P_n) = \{x_i : 1 \leq i \leq n\}$ be the set of vertices of $P_n$. By Mycielski’s construction,

\[ V(\mu(P_n)) = V(P_n) \cup \{y_i : 1 \leq i \leq n\} \cup \{z\} \quad \text{and} \]
\[ E(\mu(P_n)) = E(P_n) \cup \{y_ix : x \in N_{P_n}(x_i) \cup \{z\}, i = 1, 2, \ldots, n\}. \]

Now we partition the vertex set of $V(\mu(P_n))$ as follows:
Case (i) For $n = 1$

\[ V_1 = \{ x_1, z \}, \]
\[ V_2 = \{ y_1 \}. \]

It is obvious.

Case (ii) By this partitions $\chi = (\mu (P_n)) = 3$

1. For $n = 2$:

\[ V_1 = \{ x_1, z \}, \]
\[ V_2 = \{ x_2, y_2 \}, \]
\[ V_3 = \{ y_1 \}. \]

2. For $n = 3$:

\[ V_1 = \{ x_1, x_3, z \}, \]
\[ V_2 = \{ x_2, y_2 \}, \]
\[ V_3 = \{ y_1, y_3 \}. \]

3. For $n = 4$:

\[ V_1 = \{ x_1, x_3, z \}, \]
\[ V_2 = \{ x_2, x_4, y_2 \}, \]
\[ V_3 = \{ y_1, y_3, y_4 \}. \]

4. For $n = 5$:

\[ V_1 = \{ x_1, x_3, x_5, z \}, \]
\[ V_2 = \{ x_2, x_4, y_2, y_4 \}, \]
\[ V_3 = \{ y_1, y_3, y_5 \}. \]

5. For $n = 6$:

\[ V_1 = \{ x_1, x_3, x_5, z \}, \]
\[ V_2 = \{ x_2, x_4, x_6, y_2, y_4 \}, \]
\[ V_3 = \{ y_1, y_3, y_5, y_6 \}. \]
6. For \( n = 7 \):

\[
V_1 = \{x_1, x_3, x_5, x_7, z\},
\]
\[
V_2 = \{x_2, x_4, x_6, y_2, y_4\},
\]
\[
V_3 = \{y_1, y_3, y_5, y_6, y_7\}.
\]

7. For \( n = 8 \):

\[
V_1 = \{x_1, x_3, x_5, x_7, z\},
\]
\[
V_2 = \{x_2, x_4, x_6, x_8, y_2, y_4\},
\]
\[
V_3 = \{y_1, y_3, y_5, y_6, y_7, y_8\}.
\]

8. For \( n = 9 \):

\[
V_1 = \{x_1, x_3, x_5, x_7, x_9, z\},
\]
\[
V_2 = \{x_2, x_4, x_6, x_8, y_2, y_4\},
\]
\[
V_3 = \{y_1, y_3, y_5, y_6, y_7, y_8, y_9\}.
\]

9. For \( n = 11 \):

\[
V_1 = \left\{x_{2i-1} : 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil \right\} \cup \{z\},
\]
\[
V_2 = \left\{x_{2i} : 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil \right\} \cup \{y_2, y_4, y_6\},
\]
\[
V_3 = \{y_i : 7 \leq i \leq n\} \cup \{y_1, y_3, y_5\}.
\]

It is clear that \( V_i \cap V_j = \emptyset \) for \( i \neq j \). Since there exists a 5-cycle \( x_1x_2y_3y_2x_1 \), \( \chi(\mu(P_n)) = 3 \) and hence \( \chi(\mu(P_n)) = 3 \).

**Case (iii)** By this partition \( \chi(\mu(P_n)) = 4 \) for \( n \geq 10, n \neq 11 \).

Since there exists a 5-cycle \( x_1x_2y_3y_2x_1 \), it is clear that \( \chi(\mu(P_n)) = 3 \).

We know that \( |V(\mu(P_n))| = 2n + 1 \). Out of these \( 2n + 1 \), the vertices of \( \{x_{2i} : 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \} \cup \{z\} \) can be assigned with one color. i.e., \( \left\lceil \frac{n}{2} \right\rceil + 1 \) vertices can be colored with one color. Hence remaining number of vertices to be colored is \( (2n + 1) - \left(\left\lceil \frac{n}{2} \right\rceil + 1\right) = \left\lfloor \frac{3n}{2} \right\rfloor \).
If \( \chi_e(\mu(P_n)) = 3 \), these \( \left\lfloor \frac{3n}{2} \right\rfloor \) vertices should be divided into two sets, each containing \( \left\lfloor \frac{3n}{4} \right\rfloor \) and \( \left\lfloor \frac{3n}{4} \right\rfloor + 1 \) vertices.

Since \( \left\lfloor \frac{3n}{4} \right\rfloor \neq \left\lfloor \frac{n}{2} \right\rfloor + 1 \) and \( \left\lfloor \frac{3n}{4} \right\rfloor + 1 \neq \left\lfloor \frac{n}{2} \right\rfloor + 1 \) for \( n \geq 10, n \neq 11 \), we conclude that the partition of \( V(\mu(P_n)) \) into three sets satisfying \( ||V_i| - |V_j|| \leq 1 \) for \( i \neq j \) is not possible and hence \( \chi_e(\mu(P_n)) \geq 4 \), for \( n \geq 10, n \neq 11 \).

Now the following partition gives the equitable coloring of \( \mu(P_n) \) for \( n \geq 10, n \neq 11 \).

\[
\begin{align*}
V_1 &= \{x_{2i} : 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \} \cup \{z\}, \\
V_2 &= \{x_{2i-1} : 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \}, \\
V_3 &= \{y_{2i-1} : 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \}, \\
V_4 &= \{y_{2i} : 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \}.
\end{align*}
\]

This implies \( \chi_e(\mu(P_n)) = 4. \quad \square \)

6. Equitable Coloring of Mycielskian of Helm Graphs

**Theorem 6.1.** The equitable chromatic number of Mycielski’s graph of Helm graph

\[
\chi_e(\mu(H_n)) = \begin{cases} 
4 & \text{if } n \text{ is even} \\
5 & \text{if } n \text{ is odd}.
\end{cases}
\]

**Proof.** Let

\[
\begin{align*}
V(H_n) &= \{x_0\} \cup \{x_i : 1 \leq i \leq n\} \cup \{x'_i : 1 \leq i \leq n\} \\
\text{and } E(H_n) &= \{x_0x_i : 1 \leq i \leq n\} \cup \{x_ix_{i+1} : 1 \leq i \leq n - 1\} \\
&\quad \cup \{x nx_1\} \cup \{x'_ix'_i : 1 \leq i \leq n\}.
\end{align*}
\]
By Mycielski’s construction,

\[ V(\mu(H_n)) = V(H_n) \cup \{y_i : 0 \leq i \leq n\} \cup \{y'_i : 1 \leq i \leq n\} \cup \{z\}. \]

In \(\mu(H_n)\), \(y_0\) is adjacent with each of \(\{x_i : 1 \leq i \leq n\}\), \(y_i\) is adjacent with the neighbour of \(x_i (0 \leq i \leq n)\), \(z\) is adjacent with each of \(\{y_i : 0 \leq i \leq n\}\) and \(\{y'_i : 1 \leq i \leq n\}\). Also each \(y'_i\) is adjacent with \(x_i (1 \leq i \leq n)\).

Since \(x_0\) and \(y_0\) are non adjacent vertices, we use same color to color \(x_0\) and \(y_0\) let it be 1. Since \(y_0\) is adjacent with each vertex of \(\{x_i : 1 \leq i \leq n\}\), the color 1 cannot be assigned to any vertex \(\{x_i : 1 \leq i \leq n\}\).

**Case (i)** If \(n\) is even:

Since the set of vertices \(\{x_i : 1 \leq i \leq n\}\) induce a cycle of length \(n\) in \(\mu(H_n)\), we use two colors 2 and 3 to color all the vertices of this cycle. Also same color cannot be assigned to color the vertices of \(\{y_i : 1 \leq i \leq n\}\), since \(z\) is adjacent with each vertex of \(\{y_i : 0 \leq i \leq n\}\), \(z\) cannot be assigned with any of the three colors and let it be assigned with color 4. Hence \(\chi=(\mu(H_n)) \geq 4\).

We use the following partition to color \(\mu(H_n)\) equitably with 4 given colors.

\[
\begin{align*}
V_1 & = \{x_0, y_0, y'_i : 3 \leq i \leq n\}, \\
V_2 & = \{x_{2i-1}, y_{2i-1} : 1 \leq i \leq \frac{n}{2}\} \cup \{y'_2\}, \\
V_3 & = \{x_{2i}, y_{2i} : 1 \leq i \leq \frac{n}{2}\} \cup \{y'_1\}, \\
V_4 & = \{z, x'_i : 1 \leq i \leq n\}.
\end{align*}
\]

Clearly \(||V_i| - |V_j|| \leq 1\) for \(i \neq j\).

Hence \(\chi=(\mu(H_n)) \leq 4\) and so \(\chi=(\mu(H_n)) = 4\).

**Case (ii)** If \(n\) is odd:

Since the vertices \(\{x_i : 1 \leq i \leq n\}\) induce a cycle of odd length \(n\), in \(\mu(H_n)\), we use three colors (2,3 and 4) to color the vertices of this cycle. Now \(z\) is colored with a new color 5. Hence \(\chi=(\mu(H_n)) \geq 5\).
We use the following partition to color the vertices of $\mu(H_n)$ equitable in the following cases,

**Case (ii)a**

1. For $n = 3$:

   \[ V_1 = \{x_0, y_0, x_3'\}, \]
   \[ V_2 = \{x_1, y_1, y_3'\}, \]
   \[ V_3 = \{x_2, y_2, x_4'\}, \]
   \[ V_4 = \{x_3, y_3, y_4'\}, \]
   \[ V_5 = \{x_1', x_2', z\}. \]

2. For $n = 6k - 3$, $k \geq 2$:

   \[ V_1 = \{x_0, y_0\} \cup \left\{ x_i' : \frac{2n + 6}{3} \leq i \leq n \right\} \cup \left\{ y_i' : \frac{n - 3}{3} \leq i \leq n \right\}, \]
   \[ V_2 = \left\{ x_{3i-2}, y_{3i-2} : 1 \leq i \leq \frac{n}{3} \right\} \cup \left\{ y_{3i} : 1 \leq i \leq \frac{n-3}{3} \right\}, \]
   \[ V_3 = \left\{ x_{3i-1}, y_{3i-1} : 1 \leq i \leq \frac{n}{3} \right\} \cup \left\{ y_{3i-2} : 1 \leq i \leq \frac{n-3}{3} \right\}, \]
   \[ V_4 = \left\{ x_{3i}, y_{3i} : 1 \leq i \leq \frac{n}{3} \right\} \cup \left\{ y_{3i-1} : 1 \leq i \leq \frac{n-3}{3} \right\}, \]
   \[ V_5 = \left\{ x_i' : 1 \leq i \leq \frac{2n + 3}{3} \right\} \cup \{z\}. \]

**Case (ii)b**

1. For $n = 5$:

   \[ V_1 = \{x_0, y_0, x_4', x_5'\}, \]
   \[ V_2 = \{x_1, y_1, x_4, y_4'\}, \]
   \[ V_3 = \{x_2, y_2, x_5, y_5'\}, \]
   \[ V_4 = \{x_3, y_3, y_2', y_4, y_5'\}, \]
   \[ V_5 = \{x_1', x_2', x_3', z\}. \]
2. For \( n = 6k - 1, \ k \geq 2 \):

\[
\begin{align*}
V_1 &= \{x_0, y_0\} \cup \left\{x'_i : \frac{2n + 2}{3} \leq i \leq n \right\} \cup \left\{y'_i : \frac{n + 7}{2} \leq i \leq n \right\}, \\
V_2 &= \left\{x_{3i-2}, y_{3i-2} : 1 \leq i \leq \frac{n+1}{3} \right\} \cup \left\{y'_{3i} : 1 \leq i \leq \frac{n+1}{6} \right\}, \\
V_3 &= \left\{x_{3i-1}, y_{3i-1} : 1 \leq i \leq \frac{n+1}{3} \right\} \cup \left\{y'_{3i-2} : 1 \leq i \leq \frac{n+1}{6} \right\}, \\
V_4 &= \left\{x_{3i}, y_{3i} : 1 \leq i \leq \frac{n-2}{3} \right\} \cup \left\{y'_{3i-1} : 1 \leq i \leq \frac{n+1}{6} \right\}, \\
V_5 &= \left\{x'_i : 1 \leq i \leq \frac{2n-1}{3} \right\} \cup \{z\}.
\end{align*}
\]

**Case (ii)c**

1. For \( n = 7 \):

\[
\begin{align*}
V_1 &= \{x_0, y_0, x_6, x_7, y_1, y_4\}, \\
V_2 &= \{x_1, y_1, x_4, y_4, y'_3, y'_7\}, \\
V_3 &= \{x_2, y_2, x_5, y_5, x_7, y_7\}, \\
V_4 &= \{x_3, y_3, x_6, y_6, y_2, y'_5\}, \\
V_5 &= \{x'_1, x'_2, x'_3, x'_4, x'_5, z\}.
\end{align*}
\]

2. For \( n = 6k + 1, \ k \geq 2 \):

\[
\begin{align*}
V_1 &= \{x_0, y_0\} \cup \left\{x'_i : \frac{5n + 1}{6} \leq i \leq n \right\} \cup \left\{y'_i : \frac{n + 11}{3} \leq i \leq n \right\}, \\
V_2 &= \left\{x_{3i-2}, y_{3i-2} : 1 \leq i \leq \frac{n+2}{3} \right\} \cup \left\{y'_{3i+2} : 1 \leq i \leq \frac{n-7}{6} \right\}, \\
V_3 &= \left\{x_{3i-1}, y_{3i-1} : 1 \leq i \leq \frac{n+2}{3} \right\} \cup \left\{y'_{3i+3} : 1 \leq i \leq \frac{n-7}{6} \right\}, \\
&\quad \cup \left\{y'_1, y'_3\right\}, \\
V_4 &= \left\{x_{3i}, y_{3i} : 1 \leq i \leq \frac{n-1}{3} \right\} \cup \left\{y'_{3i+4} : 1 \leq i \leq \frac{n-7}{6} \right\}, \\
&\quad \cup \left\{y'_2, y'_4\right\}, \\
V_5 &= \left\{x'_i : 1 \leq i \leq \frac{5n-5}{6} \right\} \cup \{z\}.
\end{align*}
\]
Clearly in all the cases \(||V_i| - |V_j|| \leq 1\) for \(i \neq j\).
Hence \(\chi_\pm(\mu(H_n)) \leq 5\) and so \(\chi_\pm(\mu(H_n)) = 5\) for \(n\) is odd. □

7. Equitable Coloring of Mycielskian of Gear Graphs

Theorem 7.1. The equitable chromatic number of Mycielski’s graph of Gear graph

\[
\chi_\pm(\mu(G_n)) = \begin{cases} 
3 & \text{if } 3 \leq n \leq 5 \\
4 & \text{if } n \geq 6 
\end{cases}
\]

Proof. Let

\[
V(G_n) = \{x_0\} \cup \{x_i : 1 \leq i \leq n\} \cup \{x'_i : 1 \leq i \leq n\}
\]
and \(E(G_n) = \{x_0x_i : 1 \leq i \leq n\} \cup \{x_ix'_i : 1 \leq i \leq n\}
\]
\[
\cup \{x'_ix_{i+1} : 1 \leq i \leq n - 1\} \cup \{x'_nx_1\}.
\]

By Mycielski’s construction,

\[
V(\mu(G_n)) = V(G_n) \cup \{y_i : 1 \leq i \leq n\} \cup \{y'_i : 1 \leq i \leq n\} \cup \{z\}.
\]

In \(\mu(H_n)\), \(y_0\) is adjacent with each vertex of \(\{x_i : 1 \leq i \leq n\}\), each \(y_i\) is adjacent with the neighbour of \(x_i (1 \leq i \leq n)\) and each \(y'_i\) is adjacent with the neighbours of \(x'_i (1 \leq i \leq n)\), \(z\) is adjacent with each vertex of \(\{y_i : 1 \leq i \leq n\} \cup \{y'_i : 1 \leq i \leq n\} \cup \{y_0\}\).

Case (i) For \(n = 3, 4, 5\):

Since the set of vertices \(\{x_i : 1 \leq i \leq n\} \cup \{x'_i : 1 \leq i \leq n\}\) induce an even cycle in \(\mu(G_n)\), it requires atleast two colors to color the vertices of this cycle (say 1 and 2). If we assign the same color 1 and 2 to color the vertices of \(\{y_i : 1 \leq i \leq n\} \cup \{y'_i : 1 \leq i \leq n\}\) then \(z\) will receive a new color other than 1 and 2. Hence \(\chi_\pm(\mu(G_n)) \geq 3\).
1. For \( n = 3 \):

\[
V_1 = \{ x_0', x_1', x_2', x_3', z \}, \\
V_2 = \{ x_1, x_2, x_3, y_1, y_3 \}, \\
V_3 = \{ y_0, y_2, y'_1, y'_2, y'_3 \}.
\]

2. For \( n = 4 \):

\[
V_1 = \{ x_0, x_1', x_2', x_3', x_4', z \}, \\
V_2 = \{ x_1, x_2, x_3, x_4, y_1, y_2 \}, \\
V_3 = \{ y_0, y_3, y_4, y'_1, y'_2, y'_3, y'_4 \}.
\]

3. For \( n = 5 \):

\[
V_1 = \{ x_0, x_1', x_2', x_3', x_4', x_5', z \}, \\
V_2 = \{ x_1, x_2, x_3, x_4, x_5, y_1, y_2, y_3 \}, \\
V_3 = \{ y_0, y_4, y_5, y'_1, y'_2, y'_3, y'_4, y'_5 \}.
\]

Clearly \( ||V_i| - |V_j|| \leq 1 \) for \( i \neq j \).

Hence \( \chi_\leq(\mu(G_n)) \leq 3 \) and hence \( \chi_\leq(\mu (G_n)) = 3 \).

**Case (ii)** For \( n \geq 6 \):

We know that \( |V(\mu(G_n))| = 4n + 3 \). Out of these \( 4n + 3 \), the vertices of \( \{x'_i : 1 \leq i \leq n\} \cup \{x_0\} \cup \{z\} \) can be assigned with one color. i.e., \( n + 2 \) vertices can be colored with one color. Hence remaining number of vertices to be colored is \( (4n + 3) - (n + 2) = 3n + 1 \).

If \( \chi_\leq(\mu(G_n)) = 3 \), these \( 3n + 1 \) vertices should be divided into two sets, each containing \( \frac{3n}{2} \) and \( \frac{3n + 1}{2} \) vertices.

Since \( \frac{3n}{2} \neq n + 2 \) and \( \frac{3n + 1}{2} \neq n + 2 \) for \( n \geq 6 \), we conclude that the partition of \( V(\mu(G_n)) \) into three sets satisfying \( ||V_i| - |V_j|| \leq 1 \) for
$i \neq j$ is not possible and hence $\chi(\mu(G_n)) \geq 4$.

$$
\begin{align*}
V_1 &= \{x_i' : 1 \leq i \leq n\} \cup \{x_0\}, \\
V_2 &= \{x_i : 1 \leq i \leq n\} \cup \{z\}, \\
V_3 &= \{y_i' : 1 \leq i \leq n\} \cup \{y_0\}, \\
V_4 &= \{y_i : 1 \leq i \leq n\}.
\end{align*}
$$

Clearly $|V_i| - |V_j| \leq 1$ for $i \neq j$.

Hence $\chi(\mu(G_n)) \leq 4$ and so $\chi(\mu(G_n)) = 4$. \hfill $\square$

References


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