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A New Constraint Qualification for Vector Semi-Infinite Problem

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Abstract. This paper deals with a class of vector semi-infinite optimization problems with differentiable data and arbitrary index set of inequality constraints. A suitable constraint qualification and a new extension of invexity are introduced, and the weak and strong Karush-Kuhn-Tucker type optimality conditions are investigated.

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1. Introduction

A mathematical programming problem with a finite number of variables and infinitely many constraints is called a semi-infinite programming problem. Problems of this type have been utilized for the modeling and analysis of a wide range of theoretical as well as concrete, real-world, practical problems; see e.g., [8, 19].
Several classes of semi-infinite programming problems have been investigated extensively by many authors, and consequently, numerous optimality conditions, duality relations, sensitively, and numerical solution algorithms are available for these problem in the related literature; see e.g., [3, 8, 12, 13, 14, 19, 20].

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However, a close examination of these and other related sources will readily reveal the fact that so far vector-valued semi-infinite programming have not received much attention in the area of mathematical programming. Indeed, it appears that currently there are few publications dealing with multiobjective (or vector-valued) semi-infinite programming (see [1, 2, 4, 5, 6, 7, 11, 15, 22]) In this paper, we consider the following vector-valued semi-infinite problem:

$$\begin{array}{lll} \text{(P)} & \inf\ f(x) := \big(f_1(x), \dots, f_m(x)\big) & \\ \text{s.t.} & g_j(x) \leqslant 0 \quad i \in J, & \\ & x \in \mathbb{R}^n, & \end{array}$$

where $f : \mathbb{R}^n \to \mathbb{R}^m$ and $g_j : \mathbb{R}^n \to \mathbb{R} \bigcup \{+\infty\}$ for $j \in J$, are continuously differentiable functions. J is assumed to be an arbitrary index set, not necessarily finite (but nonempty).
In Section 2 we introduce a constraint qualification for the problem (P). Then, necessary optimality conditions of Karush-Kuhn-Tucker type are established. In Section 3, sufficient optimality conditions for (P) are obtained under a new invexity assumption.
In the rest of this article, we denotes by $ri(A)$, $conv(A)$, and $cone(A)$, the relative interior of $A \subseteq \mathbb{R}^n$, the convex hull, and the convex cone (containing the origin) generated by A, respectively.

2. Weak and Strong KKT Necessary Conditions

In what follows we shall assume that the feasible set of (P) is nonempty, i.e.,

$$S := \big\{x \in \mathbb{R}^n \mid g_j(x) \leqslant 0, \quad \forall j \in J\big\} \neq \emptyset.$$

For a given $\hat{x} \in S$, let $J(\hat{x})$ denotes the index set of all active constraints at $\hat{x}$,

$$J(\hat{x}) := \big\{j \in J \mid g_j(\hat{x}) = 0\big\}.$$

A feasible point $\hat{x}$ is said to be an efficient solution [resp. weakly efficient solution] to problem (P) if there is no $x \in S$ satisfying $f_i(x) \leqslant f_i(\hat{x}),\ i \in I := \{1, 2, ..., m\}$ and $\big(f_1(x), \dots, f_m(x)\big) \neq \big(f_1(\hat{x}), \dots, f_m(\hat{x})\big)$ [resp. $f_i(x) < f_i(\hat{x}),\ i \in I$]. The set of all efficient solutions and that of all weakly efficient solutions of (P) are denoted by E and W, respectively. Obviously, $E \subseteq W$.

Let $\hat{x} \in S$. On the lines of Ref. [21], for each $i \in I$, define the set

$$Q^i(\hat{x}) := \Big\{x \in S \mid f_l(x) \leqslant f_l(\hat{x}) \quad \forall l \in I \setminus \{i\}\Big\},$$
$$Q^i(\hat{x}) := S, \text{ if } m = 1.$$

For the sake of the simplicity, we denote $Q^i(\hat{x})$ by Q^i in this paper. We also define the following notations for each differentiable function $\varphi : \mathbb{R}^n \to \mathbb{R}$:

$$\nabla^{\odot}\varphi(x_0) := \Big\{z \in \mathbb{R}^n \mid \big\langle \nabla\varphi(x_0), z\big\rangle \leqslant 0\Big\},$$
$$\nabla^{\ominus}\varphi(x_0) := \Big\{z \in \mathbb{R}^n \mid \big\langle \nabla\varphi(x_0), z\big\rangle < 0\Big\}.$$

The aim of this section is to derive the weak (resp. strong) KKT necessary condition at $\hat{x} \in W$ (resp. $\hat{x} \in E$) under the following constraint qualification which is the se mi-infinite analog of the qualification studied by Maeda in [21]:

$$\text{(CQ):} \qquad \Big(\bigcap_{i=1}^{m} \nabla^{\odot} f_i(\hat{x})\Big) \cap \Big(\bigcap_{j\in J(\hat{x})} \nabla^{\odot} g_j(\hat{x})\Big) \subseteq \bigcap_{i=1}^{m} T(Q^i, \hat{x}),$$

where $T(M, x_0)$ denotes the contingent cone of $M \subseteq \mathbb{R}^n$ at $x_0 \in \overline{M}$, i.e.,

$$T(M, x_0) := \Big\{d \in \mathbb{R}^n \mid \exists\{(t_k, d_k) \to (0^+, d), \text{ such that } \hat{x} + t_k d_k \in M \ \ \forall k \in \mathbb{N}\ \}.$$

Owning to the relation $\bigcap_{i=1}^{m} T(Q^i, \hat{x}) \subseteq T(S, \hat{x})$, the following result is a direct consequence of [15, Theorem 3.4(ii)].

Theorem 2.1. (Weak KKT Necessary Condition). *Let $\hat{x}$ be a weakly efficient solution of (P) and cone$\Big(\bigcup_{j\in J(\hat{x})} \nabla g_j(\hat{x})\Big)$ be a closed cone. If in addition, (CQ) holds at $\hat{x}$, then there exist scalars $\alpha_i \geqslant 0$, $i \in I$ with $\sum_{i=1}^{m} \alpha_i = 1$, and an integer $k \geqslant 0$, and a set $\{j_1, j_2, ..., j_k\} \subseteq J(\hat{x})$, and scalars $\beta_{j_r} \geqslant 0$ for $r \in \{1, 2, ..., k\}$, such that*

$$\sum_{i=1}^{m} \alpha_i \nabla f_i(\hat{x}) + \sum_{r=1}^{k} \beta_{j_r} \nabla g_{j_r}(\hat{x}) = 0.$$

In almost all example, we could not obtain positive KKT multipliers associated with the vector-valued objective function, namely, some of the multipliers may be equal to zero. This means that the components of the vector-valued objective function have not role in the necessary conditions for weakly efficiency. In order to avoid the case where some of the KKT multipliers associated with the objective function vanish for a finite vector optimization problem, several approaches have been developed in recent years, and strong KKT necessary optimality conditions have been obtained (see, e.g., [16, 21] for $|J| < \infty$). We say that strong KKT condition holds for a (P), when the KKT multipliers are positive for all components of the objective function.

The following Theorem will be present the strong KKT necessary condition for (P).

Theorem 2.2. (Strong KKT Necessary Condition). *Let $\hat{x}$ be an efficient solution of (P). If in addition, (CQ) and the condition*

$$(\mathfrak{A}): \qquad \Big(\bigcap_{i=1}^{m} \nabla^{\circledcirc} f_i(\hat{x})\Big) \setminus \{0\} \subseteq \bigcup_{i=i}^{m} \nabla^{\ominus} f_i(\hat{x})$$

hold at $\hat{x}$, then there exist scalars $\alpha_i > 0,\ i \in I$, and an integer $k \geqslant 0$, and a set $\{j_1, j_2, ..., j_k\} \subseteq J(\hat{x})$, and scalars $\beta_{j_r} \geqslant 0$ for $r \in \{1, 2, ..., k\}$, such that

$$\sum_{i=1}^{m} \alpha_i \nabla f_i(\hat{x}) + \sum_{r=1}^{k} \beta_{j_r} \nabla g_{j_r}(\hat{x}) = 0.$$

Proof. We present our proof in three steps.

Step 1: We claim that

$$\Big(\bigcup_{i=1}^{m} \nabla^{\ominus} f_i(\hat{x})\Big) \cap \Big(\bigcap_{i=1}^{m} T(Q^i, \hat{x})\Big) = \emptyset. \tag{1}$$

It suffices only to prove that

$$\nabla^{\ominus} f_l(\hat{x}) \cap T(Q^l, \hat{x}) = \emptyset \qquad \forall l \in I.$$

On the contrary, suppose that for some $l \in I$ there is a vector d such that

$$d \in \nabla^{\ominus} f_l(\hat{x}) \cap T(Q^l, \hat{x}). \tag{2}$$

By the definition of contingent cone, there exists sequence $(t_s, d_s) \to (0^+, d)$ such that $\hat{x} + t_s d_s \in Q^l$ for each $s \in \mathbb{N}$. This means for each $s \in \mathbb{N}$ we have

$$f_i(\hat{x} + t_s d_s) \leqslant f_i(\hat{x}) \quad \forall i \in I \setminus \{l\}, \quad \text{and} \quad \hat{x} + t_s d_s \in S. \tag{3}$$

By the mean-value Theorem, for each $s \in \mathbb{N}$, there exist u_s in the open line segment $(\hat{x}\ , \hat{x} + t_s d_s)$ such that

$$f_l(\hat{x} + t_s d_s) - f_l(\hat{x}) = t_s \big\langle \nabla f_l(u_s), d_s \big\rangle. \tag{4}$$

Since $u_s \to \hat{x}$ and $\nabla f_l(.)$ is a continuous function, we deduce

$$\lim_{s\to\infty} \big\langle \nabla f_l(u_s), d_s \big\rangle = \big\langle \nabla f_l(\hat{x}), d \big\rangle.$$

On the other hand, by (2), we have $\langle \nabla f_l(\hat{x}), d \rangle < 0$. Thus, the inequality (4) implies that there is a $N_1 > 0$ such that

$$f_l(\hat{x} + t_s d_s) < f_l(\hat{x}), \qquad \forall\ s > N_1. \tag{5}$$

Therefore, (3) together with (5) contradicts $\hat{x} \in E$, and so (1) is true.

Step 2: Let

$$X := conv\Big(\Big\{\nabla f_i(\hat{x}) \mid i \in I\Big\}\Big),$$
$$Y := cone\Big(\Big\{\nabla g_j(\hat{x}) \mid j \in J(\hat{x})\Big\}\Big).$$

We claim that

$$ri\big(X\big) \cap \big(-Y\big) \neq \emptyset. \tag{6}$$

By contradiction, we suppose that (6) does not hold. Then, by the strong convex separation Theorem ([25, Theorem 11.3]) and noting that $\big(-Y\big)$ is a convex cone, it follows that there is a hyperplane

$$H := \Big\{x \mid \big\langle x, d \big\rangle = 0 \ \text{ for some } d \in \mathbb{R}^n \setminus \{0\}\Big\},$$

separating X and $\big(-Y\big)$ properly. Therefore, there exists $d \in \mathbb{R}^n$ satisfying

$$\begin{aligned} 0 \neq d \in X^0 \cap \big(-Y\big)^0 &= \Big(\bigcup_{i=i}^{m} \nabla f_i(\hat{x})\Big)^0 \cap \Big(\bigcup_{j \in J(\hat{x})} \nabla g_j(\hat{x})\Big)^0 \\ &= \Big(\bigcap_{i=1}^{m} \nabla^{\circledcirc} f_i(\hat{x})\Big) \cap \Big(\bigcap_{j \in J(\hat{x})} \nabla^{\circledcirc} g_j(\hat{x})\Big), \end{aligned}$$

where M^0 denotes the negative polar cone of $M \subseteq \mathbb{R}^n$, i.e.,

$$M^0 := \{z \in \mathbb{R}^n \mid \big\langle z, u \big\rangle \leqslant 0 \quad \text{for all} \bigcup \in M\}.$$

Thus, owning to (CQ) and ($\mathfrak{A}$) we conclude that

$$d \in \Big(\bigcup_{i=1}^{m} \nabla^{\ominus} f_i(\hat{x})\Big) \cap \Big(\bigcap_{i=1}^{m} T(Q^i, \hat{x})\Big),$$

which contradicts (1), and proves (6).

Step 3: Owning to the well-known inclusion (see, e.g., by [25, Theorem 6.9])

$$ri\Big(conv\big(\{\nabla f_i(\hat{x}) \mid i = 1, 2, ..., m\}\big)\Big) \subseteq \Big\{\sum_{i=1}^{m} \alpha_i \nabla f_i(\hat{x}) \mid \ \alpha_i > 0, \ \sum_{i=1}^{m} \alpha_i = 1\Big\},$$

it suffices only to demonstrate

$$0 \in ri\Big(conv\big(\{\nabla f_i(\hat{x}) \mid i = 1, 2, ..., m\}\big)\Big) + cone\big(\{\nabla g_j(\hat{x}) \mid j \in J(\hat{x})\}\big). \quad (7)$$

By contradiction, we suppose that (7) does not hold. Then

$$ri\Big(conv\big(\{\nabla f_i(\hat{x}) \mid i = 1, 2, ..., m\}\big)\Big) \cap \Big(- cone\big(\{\nabla g_j(\hat{x}) \mid j \in J(\hat{x})\}\big)\Big) = \emptyset.$$

Thus, by the strong convex separation Theorem ([25, Theorem 11.3]), it follows that there is a hyperplane $\Big\{x \in \mathbb{R}^n \mid \langle x, d\rangle = 0 \quad \text{for some } d \in \mathbb{R}^n \setminus \{0\}\Big\}$ separating $conv\big(\{\nabla f_i(\hat{x}) \mid i = 1, 2, ..., m\}\big)$ and $\Big(-cone\big(\{\nabla g_j(\hat{x}) \mid j \in J(\hat{x})\}\big)\Big)$ properly. Therefore, there exists $d \in \mathbb{R}^n$ satisfying

$$\begin{aligned} 0 \neq d \quad \in \quad & \Big(conv\big(\{\nabla f_i(\hat{x}) \mid i = 1, 2, ..., m\}\big)\Big)^0 \cap \Big(cone\big\{\nabla g_j(\hat{x}) \mid j \in J(\hat{x})\}\big)\Big)^0 \\ & = \Big(\{\nabla f_i(\hat{x}) \mid i = 1, 2, ..., m\}\Big)^0 \cap \Big(\{\nabla g_j(\hat{x}) \mid j \in J(\hat{x})\}\Big)^0 \\ & = \Big(\bigcap_{i=1}^{m} \nabla^{\circledcirc} f_i(\hat{x})\Big) \cap \Big(\bigcap_{j \in J(\hat{x})} \nabla^{\circledcirc} g_j(\hat{x})\Big). \end{aligned}$$

Thus, owning to (CQ) and ($\mathfrak{A}$) we conclude that

$$d \in \Big(\bigcup_{i=1}^{m} \nabla^{\ominus} f_i(\hat{x})\Big) \cap \Big(\bigcap_{i=1}^{m} T(Q^i, \hat{x})\Big),$$

which contradicts (1). This proves the theorem. □

3. Weak and Strong KKT Sufficient Conditions

In this section, we investigate weak (resp. strong) KKT sufficient conditions for weak efficient (resp. efficient) point of (P). As well as in the classic case, the sufficient results in semi-infinite programming are established under some additional convexity assumptions (see e.g., [3, 7, 11, 20]). On the other hand, the theory of the classical single and multiobjective programming has been considerably extended when the convexity was replaced by weaker invexity like properties.

As this is well-known, the concept of invexity has been introduced in literature in 1981 by Graven [9], after Hanson [10] showed that both weak duality and Karush-Kahn-Tucker sufficiency for optimum in the mathematical programming hold when convexity is replaced by a weaker condition. After the works

of Hanson and Graven, other types of differentiable functions have been introduced with the intent of generalizing invex functions from different points of view; see e.g., [17, 18, 23].
Our first aim in this section is to introduce a new extension of invex function. More specifically, considering a differentiable convex function $\varphi : \mathbb{R}^p \to \mathbb{R}$ at a point $x_0 \in \mathbb{R}^p$. It is easy to see that for each $x \in \mathbb{R}^p$ we have

$$\big\langle \nabla\varphi(x_0), x - x_0 \big\rangle \leqslant \varphi(x) - \varphi(x_0).$$

In definition of invex function, the later $x - x_0$ was replaced by $\eta(x, x_0)$ where the function $\eta : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}$ is called the kernel of φ at x_0, i.e.,

$$\big\langle \nabla\varphi(x_0), \eta(x, x_0) \big\rangle \leqslant \varphi(x) - \varphi(x_0);$$

equivalently,

$$\big\langle \nabla\varphi(x_0), \eta(x, x_0) \big\rangle - \big(\varphi(x) - \varphi(x_0)\big) \leqslant 0.$$

We now extend this idea as below.

Definition 3.1. *Let $\varphi := (\varphi_1, \varphi_2, ..., \varphi_q) : \mathbb{R}^p \to \mathbb{R}^q$ be a differentiable function, and let $x_0 \in \mathbb{R}^p$. We shall say that φ is the extended ν- invex with kernel η at x_0 if there exist functions $\eta : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}^p$ and $\nu_l : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}_+ \backslash \{0\}$ for $l \in \{1, 2, ..., q\}$ such that the condition*

$$\sum_{l=1}^{q} \Big[\big\langle \nabla\varphi_l(x_0), \eta(x, x_0) \big\rangle - \nu_l(x, x_0)\big(\varphi_l(x) - \varphi_l(x_0)\big) \Big] < 0,$$

holds for each $x \in \mathbb{R}^p$.

Theorem 3.2. (Weak KKT Sufficient Condition). *Suppose that there exist a feasible solution $\hat{x} \in S$ and scalars $\alpha_i \geqslant 0$ with $\sum_{i=1}^m \alpha_i = 1$ and a finite set $J^* := \{j_1, j_2, ..., j_k\} \subseteq J(\hat{x})$ and scalars $\beta_{j_l} \geqslant 0$ for $l \in \{1, 2, ..., k\}$ such that*

$$\sum_{i=1}^{m} \alpha_i \nabla f_i(\hat{x}) + \sum_{r=1}^{k} \beta_{j_r} \nabla g_{j_r}(\hat{x}) = 0. \tag{8}$$

Moreover, if the function $(\alpha_1 f_1, \alpha_2 f_2, ..., \alpha_m f_m)$ and $(\beta_{j_1} g_{j_1}, \beta_{j_2} g_{j_2}, ..., \beta_{j_k} g_{j_k})$ are respectively extended ν-invex and extended θ-invex with the same kernel η at $\hat{x}$, then $\hat{x}$ is a weak efficient solution for (P).

Proof. Suppose on the contrary that $\hat{x}$ is not a weak efficient for (P), then there exist $x \in S$ such that $f(x) < f(\hat{x})$. Thus $f_i(x) - f_i(\hat{x}) < 0$ for all

$i \in I$. Since $(\alpha_1, \alpha_2, ..., \alpha_m) \gneqq 0$ and $\nu_i(x, \hat{x}) > 0$ for all $i \in I$, we obtain $\sum_{i=1}^{m} \alpha_i \nu_i(x, \hat{x})\big(f_i(x) - f_i(\hat{x})\big) < 0$, and hence

$$\sum_{i=1}^{m} \nu_i(x, \hat{x})\big(\alpha_i f_i(x) - \alpha_i f_i(\hat{x})\big) < 0.$$

By extended ν-invexity of $(\alpha_1 f_1, \alpha_2 f_2, ..., \alpha_m f_m)$ with kernel η at $\hat{x}$ we get

$$\sum_{i=1}^{m} \big\langle \alpha_i \nabla f_i(\hat{x}), \eta(x, \hat{x}) \big\rangle < 0. \tag{9}$$

On the other hand, since $\{j_1, j_2, ..., j_k\} \subseteq J(\hat{x})$ and $x \in S$, then

$$g_{j_r}(x) \leqslant 0 = g_{j_r}(\hat{x}), \qquad \forall r \in \{1, 2, ..., k\}.$$

Now, Since $\beta_{j_r} \geqslant 0$ and $\theta_{j_r}(x, \hat{x}) > 0$ for all $r \in \{1, 2, ..., k\}$, we obtain

$$\sum_{r=1}^{k} \theta_{j_r}(x, \hat{x})\big(\beta_{j_r} g_{j_r}(x) - \beta_{j_r} g_{j_r}(\hat{x})\big) = \sum_{r=1}^{k} \beta_{j_r} \theta_{j_r}(x, \hat{x})\big(g_{j_r}(x) - g_{j_r}(\hat{x})\big) \leqslant 0.$$

By θ-invexity of $(\beta_{j_1} g_{j_1}, \beta_{j_2} g_{j_2}, ..., \beta_{j_k} g_{j_k})$ with kernel η at $\hat{x}$ we get

$$\sum_{r=1}^{k} \big\langle \beta_{j_r} \nabla g_{j_r}, \eta(x, \hat{x}) \big\rangle \leqslant 0. \tag{10}$$

Adding the inequalities (9) and (10), we get

$$\Big\langle \sum_{i=1}^{m} \alpha_i \nabla f_i(\hat{x}) + \sum_{r=1}^{k} \beta_{j_r} \nabla g_{j_r}, \eta(x, \hat{x}) \Big\rangle < 0,$$

which contradicts (8). This completes the proof. □

Theorem 3.3. (Strong KKT Sufficient Condition). *Suppose that there exist a feasible solution $\hat{x} \in S$ for (P) and scalars $\alpha_i > 0$ and a finite set $J^* := \{j_1, j_2, ..., j_k\} \subseteq J(\hat{x})$ and scalars $\beta_{j_l} \geqslant 0$ for $l \in \{1, 2, ..., k\}$ such that*

$$\sum_{i=1}^{m} \alpha_i \nabla f_i(\hat{x}) + \sum_{r=1}^{k} \beta_{j_r} \nabla g_{j_r}(\hat{x}) = 0$$

Moreover, if the function $(\alpha_1 f_1, \alpha_2 f_2, ..., \alpha_m f_m)$ and $(\beta_{j_1} g_{j_1}, \beta_{j_2} g_{j_2}, ..., \beta_{j_k} g_{j_k})$ are respectively extended ν-invex and extended θ-invex with the same kernel η at $\hat{x}$, then $\hat{x}$ is an efficient solution for (P).

Proof. Suppose on the contrary that $\hat{x}$ is not an efficient solution for (P). Then there exists $x \in S$ such that $f_i(x) \leqslant f_i(\hat{x}), i \in I$, and $f_{i_0}(x) < f_{i_0}(\hat{x})$ for some $i_0 \in I$. The remaining part of the proof is similar to that of Theorem 3.2. $\square$

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