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Some Results on Weakly Compact Positive Left Multipliers of a Certain Group Algebra

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Abstract. Let T be a weakly compact left multiplier on $L_0^\infty(\mathcal{G})^*$. In this paper, we prove that if T is positive, then T maps $L_0^\infty(\mathcal{G})^*$ into $L^1(\mathcal{G})$ and T is of the form ${}_\phi T$ for some positive function $\phi \in L^1(G)$. Using this result, we show that $T = T^+ - T^-$ for some positive weakly compact left multipliers T^+, T^- on $L_0^\infty(\mathcal{G})^*$ if and only if T maps $L_0^\infty(\mathcal{G})^*$ into $L^1(\mathcal{G})$.

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1. Introduction

We always denote by $\mathcal{G}$ a locally compact group with a fixed left Haar measure λ. The Banach spaces $L^1(\mathcal{G})$ and $L^\infty(\mathcal{G})$ are as defined in [7]. We say that a function $f \in L^\infty(\mathcal{G})$ *vanishes at infinity* if for each $\varepsilon > 0$, there is a compact subset $\mathcal{C}$ of $\mathcal{G}$ for which $\|f\ \chi_{\mathcal{G}\setminus\mathcal{C}}\|_\infty < \varepsilon$, where $\chi_{\mathcal{G}\setminus\mathcal{C}}$ denotes characteristic function of $\mathcal{G}\setminus\mathcal{C}$ on $\mathcal{G}$. We denote by $L_0^\infty(\mathcal{G})$ the subspace of $L^\infty(\mathcal{G})$ consisting of all functions $f \in L^\infty(\mathcal{G})$ vanishing at infinity. This space is a left introverted subspace of $L^\infty(\mathcal{G})$; that is, for each $n \in L_0^\infty(\mathcal{G})^*$, $f \in L_0^\infty(\mathcal{G})$ and $\phi \in L^1(\mathcal{G})$, the functional nf defined by

$$\langle nf, \phi\rangle = \langle n, f\phi\rangle,$$

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is also an element in $L_0^\infty(\mathcal{G})$, where

$$\langle f\phi, \psi\rangle = \langle f, \phi * \psi\rangle,$$

and

$$\phi * \psi(x) = \int_{\mathcal{G}} \phi(y)\psi(y^{-1}x)\, d\lambda(y),$$

for all $\psi \in L^1(\mathcal{G})$ and $x \in \mathcal{G}$. This lets us to endow $L_0^\infty(\mathcal{G})^*$ with the *first Arens product* "$\diamond$" defined by

$$\langle m \diamond n, f\rangle = \langle m, nf\rangle,$$

for all $m, n \in L_0^\infty(\mathcal{G})^*$ and $f \in L_0^\infty(\mathcal{G})$. Then $L_0^\infty(\mathcal{G})^*$ with this product is a Banach algebra and the group algebra $L^1(\mathcal{G})$ can be isometrically embedded into $L_0^\infty(\mathcal{G})^*$ as a closed ideal via

$$\langle \phi, f\rangle = \int_{\mathcal{G}} \phi(x) f(x)\, d\lambda(x),$$

for all $\phi \in L^1(\mathcal{G})$ and $f \in L_0^\infty(\mathcal{G})$; see [9]. Let $\Lambda(L_0^\infty(\mathcal{G})^*)$ denote the set of all weak*-cluster points of an approximate identity in $L^1(\mathcal{G})$ bounded by one. It is easy to see that if $u \in \Lambda(L_0^\infty(\mathcal{G})^*)$, then for every $m \in L_0^\infty(\mathcal{G})^*$ and $\phi \in L^1(\mathcal{G})$

$$m \diamond u = m \quad \text{and} \quad u \diamond \phi = \phi.$$

Let $P(L_0^\infty(\mathcal{G})^*)$ be the set of all positive functionals on $L_0^\infty(\mathcal{G})$ and $P(L^1(\mathcal{G}))$ be the set of all positive functions in $L^1(\mathcal{G})$. Note that

$$P(L^1(\mathcal{G})) = P(L_0^\infty(\mathcal{G})^*) \cap L^1(\mathcal{G}) \quad \text{and} \quad \Lambda(L_0^\infty(\mathcal{G})^*) \subseteq P(L_0^\infty(\mathcal{G})^*);$$

see [3]. By $\mathrm{ran}(L_0^\infty(\mathcal{G})^*)$ will be understood the set of all $r \in L_0^\infty(\mathcal{G})^*$ such that

$$L_0^\infty(\mathcal{G})^* \diamond r = \{0\}.$$

Let us remark from [9] that if $r \in \mathrm{ran}(L_0^\infty(\mathcal{G})^*)$, then $\langle r, f\rangle = 0$ for all $f \in C_0(\mathcal{G})$, the space of all complex-valued continuous functions on $\mathcal{G}$ vanishing at infinity.

A bounded operator T on a Banach algebra A is called a *left multiplier* provided that

$$T(ab) = T(a)b,$$

for all $a, b \in A$. For any $a \in A$, the left multiplier $b \mapsto ab$ on A is denoted by ${}_aT$; also a is said to be a *left weakly completely continuous element* of A if ${}_aT$

is a weakly compact operator. We denote the set of all left weakly completely continuous elements of A by $\mathcal{L}_{wcc}(A)$.
Weakly compact left multipliers have been studied by several authors [1, 4, 5, 6, 10, 13]. For example, Ghahramani and Lau [4, 6] have obtained some results on the question of existence of non-zero weakly compact left multipliers on $L^\infty(\mathcal{G})^*$. Losert [10], among other things, has shown that if $\mathcal{G}$ is a locally compact non-compact group, then zero is the only weakly compact left multiplier on $L^\infty(\mathcal{G})^*$. The author and Nasr-Isfahani [12] have proved that the existence of a non-zero weakly compact left multiplier on $L_0^\infty(\mathcal{G})^*$ is equivalent to the compactness of $\mathcal{G}$; see also [11].

Note that the space of all bounded operators on a Banach space coincides with the vector space generated by the positive operators on it; see for example [2]. Hence it is interesting that we investigate the following question. Is the Banach space weakly compact left multiplier on $L_0^\infty(\mathcal{G})^*$ equal to the vector space generated by the positive weakly compact left multipliers of it? For this end, we first determine the range of positive weakly compact left multipliers on $L_0^\infty(\mathcal{G})^*$. We prove that if T is a positive weakly compact left multiplier on $L_0^\infty(\mathcal{G})^*$, then T maps $L_0^\infty(\mathcal{G})^*$ into $L^1(\mathcal{G})$ and $T = {}_\phi T$ for some $\phi \in P(L^1(\mathcal{G}))$. Then, for a weakly compact left multiplier T on $L_0^\infty(\mathcal{G})^*$, we show that

$$T = T^+ - T^-,$$

for some positive weakly compact left multipliers T^+, T^- on $L_0^\infty(\mathcal{G})^*$ if and only if T maps $L_0^\infty(\mathcal{G})^*$ into $L^1(\mathcal{G})$.

2. The Results

The main result of the paper is the following.

Theorem 2.1. *Let T be a positive weakly compact left multiplier on $L_0^\infty(\mathcal{G})^*$. Then T maps $L_0^\infty(\mathcal{G})^*$ into $L^1(\mathcal{G})$ and $T = {}_\phi T$ for some $\phi \in P(L^1(\mathcal{G}))$. In this case ϕ is unique.*

Proof. Choose $u \in \Lambda(L_0^\infty(\mathcal{G})^*)$. Define the function

$$T_1 : L_0^\infty(\mathcal{G})^* \to L_0^\infty(\mathcal{G})^*,$$

by

$$T_1(m) = u \diamond T(m),$$

for all $m \in L_0^\infty(\mathcal{G})^*$. It is easy to see that T_1 is a weakly compact left multiplier from $L_0^\infty(\mathcal{G})^*$ into $u \diamond L_0^\infty(\mathcal{G})^*$. Let $m \in L_0^\infty(\mathcal{G})^*$ and $r = m - u \diamond m$. For any

$f \in L_0^\infty(\mathcal{G})$ and $\phi \in L^1(\mathcal{G})$, we have

$$\langle T_1(r) \diamond \phi, f\rangle = \langle T_1(r \diamond \phi), f\rangle = 0.$$

This together with the fact that

$$L^1(\mathcal{G})L_0^\infty(\mathcal{G}) = C_0(\mathcal{G}),$$

shows that

$$T_1(r) \in \text{ran}(L_0^\infty(\mathcal{G})^*) \cap u \diamond L_0^\infty(\mathcal{G})^*.$$

Thus $T_1(r) = 0$ and so

$$T_1(m) = T_1(u \diamond m) = T_1(u) \diamond m.$$

This implies that $T_1 = {}_{T_1(u)}T$ on $L_0^\infty(\mathcal{G})^*$. Set $\phi = T_1(u)$. Since ${}_\phi T|_{L^1(\mathcal{G})}$ is a weakly compact left multiplier on $L^1(\mathcal{G})$, there exists $\xi \in L^1(\mathcal{G})$ such that ${}_\phi T = {}_\xi T$ on $L^1(\mathcal{G})$; see [1]. Hence for every $\psi \in L^1(\mathcal{G})$ and $f \in L_0^\infty(\mathcal{G})$, we have

$$\langle \phi - \xi, \psi f\rangle = \langle (\phi - \xi) \diamond \psi, f\rangle = 0.$$

Thus

$$\phi - \xi \in \text{ran}(L_0^\infty(\mathcal{G})^*) \cap u \diamond L_0^\infty(\mathcal{G})^*$$

and hence

$$\phi = \xi \in L^1(\mathcal{G}).$$

So T_1 maps $L_0^\infty(\mathcal{G})^*$ into $L^1(\mathcal{G})$. Hence the proof will be complete if we show that

$$T_2(m) := (T - T_1)(m) = 0,$$

for all $m \in L_0^\infty(\mathcal{G})^*$. It is suffice to prove that for every $p \in P(L_0^\infty(\mathcal{G})^*)$

$$T_2(p) = 0.$$

First, note that for every $\phi \in L^1(\mathcal{G})$ and $p \in P(L_0^\infty(\mathcal{G})^*)$

$$\begin{aligned} \phi \diamond T_2(p) &= \phi \diamond (T_1(p) - u \diamond T_1(p)) \\ &= \phi \diamond T_1(p) - \phi \diamond u \diamond T_1(p) \\ &= 0. \end{aligned}$$

Hence

$$L^1(\mathcal{G}) \diamond T_2(p) = \{0\},$$

and so

$$T_2(p)|_{C_0(\mathcal{G})} = 0.$$

Let $\varepsilon > 0$ and $p \in P(L_0^\infty(\mathcal{G})^*)$. There exists a complex-valued continuous function g on $\mathcal{G}$ with compact support $\mathcal{C}$ such that $\|g\|_\infty \leqslant 1$ and

$$\|T_1(p)\|_1 \leqslant |\langle T_1(p), g\rangle| + \varepsilon. \tag{1}$$

Let W be a neighborhood in $\mathcal{G}$ with compact closure such that $\mathcal{C} \subseteq W$. Choose a complex-valued continuous function h on $\mathcal{G}$ such that

$$\chi_{\mathcal{C}} \leqslant h \leqslant \chi_V.$$

If $(\mathcal{C}_\gamma)_{\gamma\in\Gamma}$ be the family of compact subsets of $\mathcal{G}$ directed by upward inclusion, then for any $\gamma \in \Gamma$, there is a complex-valued continuous function $\jmath_\gamma$ on $\mathcal{G}$ such that

$$\chi_{\mathcal{C}_\gamma} \leqslant \jmath_\gamma \leqslant 1.$$

Hence $\langle T_2(p), \jmath_\gamma\rangle = 0$ and $\langle T_1(p), \jmath_\gamma\rangle \leqslant \|T_1(p)\|_1$. This shows that

$$\langle T(p), \chi_{\mathcal{C}_\gamma}\rangle \leqslant \langle T(p), \jmath_\gamma\rangle \leqslant \|T_1(p)\|_1.$$

Since $(\chi_{\mathcal{C}_\gamma})_{\gamma\in\Gamma}$ is a bounded approximate identity for $L_0^\infty(\mathcal{G})$, it follows that

$$\lim_\gamma \langle T(p), \chi_{\mathcal{C}_\gamma}\rangle = \|T(p)\|;$$

indeed, $T(p) \in P(L_0^\infty(\mathcal{G})^*)$. Thus

$$\|T(p)\| \leqslant \|T_1(p)\|_1.$$

Let f be an element in the unit ball of $L_0^\infty(\mathcal{G})$ such that

$$\|T_2(p)\| \leqslant |\langle T_2(p), f\rangle| + \varepsilon.$$

Note that since $g - hg \in C_0(\mathcal{G})$, we have

$$\langle T_2(p), f - hg + g\rangle = \langle T_2(p), g - hg\rangle.$$

This implies that

$$\|T_2(p)\| \leqslant |\langle T_2(p), f - hg + g\rangle| + \varepsilon. \tag{2}$$

By a proper choice of $\lambda \in \mathbb{C}$, one gets

$$\begin{aligned} |\langle T_1(p), f - hg\rangle| + |\langle T_1(p), g\rangle| &= |\langle T_1(p), \lambda(f - hg) + g\rangle| \\ &\leqslant \|T_1(p)\|_1. \end{aligned}$$

Combining this with (1) we get

$$|\langle T_1(p), f - hg\rangle| \leqslant \varepsilon.$$

From this and (1) we infer that

$$\|T_1(p)\| \leqslant |\langle T_1(p), g\rangle| - |\langle T_1(p), f - hg\rangle| + 2\,\varepsilon. \tag{3}$$

Inequalities (2) and (3) together with the fact that

$$\|f - hg + g\| \leqslant 1,$$

show that

$$\begin{aligned} \|T_2(p)\| + \|T_1(p)\|_1 &\leqslant |\langle T(p), f - hg + g\rangle| + 3\,\varepsilon \\ &\leqslant \|T(p)\| + 3\,\varepsilon \\ &\leqslant \|T(p)\|_1 + 3\,\varepsilon. \end{aligned}$$

This shows that $\|T_2(p)\| = 0$ and so $T_2(p) = 0$. Consequently,

$$T(m) = T_1(m) = {}_\phi T(m),$$

for all $m \in L_0^\infty(\mathcal{G})^*$. The uniqueness of ϕ follows from the fact that

$${}_\phi T(u) = \phi,$$

for all $\phi \in L^1(\mathcal{G})$ and $u \in \Lambda(L_0^\infty(\mathcal{G})^*)$. $\square$

Before we give some corollaries of Theorem 2.1, let us remark some results from [12].

Remark 2.2. (i) *The following assertions are equivalent.*

(a) *$\mathcal{G}$ is compact;*

(b) *there is a non zero weakly compact left multiplier on $L_0^\infty(\mathcal{G})^*$;*

(c) *$L^1(\mathcal{G})$ is contained in $\mathcal{L}_{wcc}(L_0^\infty(\mathcal{G})^*)$.*

(ii) *every weakly compact left multiplier ${}_\phi T$ on $L_0^\infty(\mathcal{G})^*$ is compact for all $\phi \in L^1(\mathcal{G})$.*

Corollary 2.3. *The set of positive weakly compact left multipliers on $L_0^\infty(\mathcal{G})^*$ is trivial or isometrically isomorphic to $P(L^1(\mathcal{G}))$.*

Proof. Let $\mathcal{M}_{wcl}(L_0^\infty(\mathcal{G})^*)$ denote the set of all positive weakly compact left multipliers on $L_0^\infty(\mathcal{G})^*$. If $T \in \mathcal{M}_{wcl}(L_0^\infty(\mathcal{G})^*)$, then $T = {}_\phi T$ for some $\phi \in P(L^1(\mathcal{G}))$. Define the function

$$\Psi : \mathcal{M}^p_{wcl}(L_0^\infty(\mathcal{G})^*) \to P(L^1(\mathcal{G})),$$

by $\Psi(T) = \phi$. If $u \in \Lambda(L_0^\infty(\mathcal{G})^*)$, then $\|u\| = 1$ and

$$\|\phi\| = \|{}_\phi T(u)\| \leqslant \|{}_\phi T\| \leqslant \|\phi\|,$$

for all $\phi \in L^1(\mathcal{G})$. Thus $\|\Psi({}_\phi T)\| = \|{}_\phi T\|$. That is, Ψ is an isometry. Now if

$$\mathcal{M}_{wcl}(L_0^\infty(\mathcal{G})^*) \neq \{0\},$$

then $\mathcal{G}$ is compact and so Ψ is surjective by Remark 2.2(i). $\square$

Corollary 2.4. *Let T be a weakly compact left multiplier on $L_0^\infty(\mathcal{G})^*$. Then $T = T^+ - T^-$ for some positive weakly compact left multipliers T^+, T^- on $L_0^\infty(\mathcal{G})^*$ if and only if $T = {}_\phi T$ for some self-adjoint element ϕ of $L^1(\mathcal{G})$. In this case, ϕ is unique.*

Proof. If there exist positive weakly compact left multipliers T^+, T^- on $L_0^\infty(\mathcal{G})^*$ such that

$$T = T^+ - T^-,$$

then by Theorem 2.1, there are $\phi, \psi \in P(L^1(\mathcal{G}))$ such that

$$T = {}_\phi T - {}_\psi T = {}_{\phi-\psi} T.$$

Since ϕ and ψ are positive, they are self-adjoint. Conversely, let ϕ be a self-adjoint element of $L^1(\mathcal{G})$. Then

$$\phi = \phi^+ - \phi^-,$$

for some $\phi^+, \phi^- \in P(L^1(\mathcal{G}))$. If $\phi \neq 0$ and $T = {}_\phi T$, then $\mathcal{G}$ is compact and so ${}_{\phi^+}T$ and ${}_{\phi^-}T$ are positive weakly compact left multiplier on $L_0^\infty(\mathcal{G})^*$ by Remark 2.2(i). The proof will be complete if we note that ${}_\phi T = {}_{\phi^+}T - {}_{\phi^-}T$. $\square$
As an immediate corollary of Theorem 2.1 and Corollary 2.4, we present the next result.

Corollary 2.5. *Let m be a self-adjoint element of $\mathcal{L}_{wcc}(L_0^\infty(\mathcal{G})^*)$. Then $m = m^+ - m^-$ for some positive functionals $m^+, m^- \in \mathcal{L}_{wcc}(L_0^\infty(\mathcal{G})^*)$ if and only if $m \in L^1(\mathcal{G})$.*
As an another consequence of Theorem 2.1 we have the following result.

Corollary 2.6. *The zero map is the only positive weakly compact left multiplier from $L_0^\infty(\mathcal{G})^*$ into* $\mathrm{ran}(L_0^\infty(\mathcal{G})^*)$.

Proof. Let $T : L_0^\infty(\mathcal{G})^* \to \mathrm{ran}(L_0^\infty(\mathcal{G})^*)$ be a positive weakly compact left multiplier. Then $T = {}_\phi T$ for some $\phi \in L^1(\mathcal{G})$. If $u \in \Lambda(L_0^\infty(\mathcal{G})^*)$, then

$$\phi = \phi \diamond u = {}_\phi T(u) = T(u) \in \mathrm{ran}(L_0^\infty(\mathcal{G})^*).$$

So $\phi = u \diamond \phi = 0$. $\square$

From Theorem 2.1 and Remark 2.2(ii), one can obtain the following result.

Corollary 2.7. *Every positive weakly compact left multiplier on $L_0^\infty(\mathcal{G})^*$ is compact.*

Let us recall that $L_0^\infty(\mathcal{G})^*$ is a complete Banach lattice; that is, every nonempty and bounded above subset of $L_0^\infty(\mathcal{G})^*$ possesses a least upper bounded; see [2]. For bounded operator $T : L_0^\infty(\mathcal{G})^* \to L_0^\infty(\mathcal{G})^*$, the *modulus* $|T|$ defined by

$$|T|(m) = \sup\{|T(n)| : |n| \leqslant m\},$$

for all $m \in P(L_0^\infty(\mathcal{G})^*)$. Then $|T|$ is a bounded operator. Note that if T is a weakly compact left multiplier, then $|T|$ is a positive weakly compact operator; see Theorem 16.7, Theorem 17.14 of [2]. It is natural to ask whether $|T|$ is also a left multiplier?

Proposition 2.8. *Let T be a weakly compact left multiplier on $L_0^\infty(\mathcal{G})^*$. Then $|T|$ is a left multiplier on $L_0^\infty(\mathcal{G})^*$ if and only if $|T| = {}_\phi T$ for some $\phi \in L^1(\mathcal{G})$.*

Proof. Let $|T|$ be a left multiplier on $L_0^\infty(\mathcal{G})^*$. Then $|T|$ is a positive weakly compact left multiplier on $L_0^\infty(\mathcal{G})^*$. It follows from Theorem 2.1 that

$$|T| = {}_\phi T,$$

for some $\phi \in P(L^1(\mathcal{G}))$. The converse is clear. $\square$

Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be locally compact groups. A linear isomorphisms

$$\rho : \mathcal{L}_{wcc}(L_0^\infty(\mathcal{G}_1)^*) \to \mathcal{L}_{wcc}(L_0^\infty(\mathcal{G}_2)^*)$$

is called *bipositive* if both ρ and ρ^{-1} are positive.

Proposition 2.9. *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be compact groups. If there is a bipositive algebra isomorphism from $\mathcal{L}_{wcc}(L_0^\infty(\mathcal{G}_1)^*)$ onto $\mathcal{L}_{wcc}(L_0^\infty(\mathcal{G}_2)^*)$, then $\mathcal{G}_1$ and $\mathcal{G}_2$ are topological isomorphic.*

Proof. Let

$$\rho : \mathcal{L}_{wcc}(L_0^\infty(\mathcal{G}_1)^*) \to \mathcal{L}_{wcc}(L_0^\infty(\mathcal{G}_2)^*),$$

be a bipositive algebra isomorphism. If $\phi, \psi \in P(L^1(\mathcal{G}_1))$, then ϕ, ψ are positive elements of $\mathcal{L}_{wcc}(L_0^\infty(\mathcal{G}_1)^*)$. Hence $\rho(\phi \diamond \psi)$ is a positive element in

$$\mathcal{L}_{wcc}(L_0^\infty(\mathcal{G}_2)^*).$$

It follows from Corollary 2.5 that

$$\rho(\phi \diamond \psi) \in L^1(\mathcal{G}).$$

In view of Cohen's factorization theorem, we have

$$\rho(L^1(\mathcal{G}_1)) \subseteq L^1(\mathcal{G}_2).$$

Analogously,

$$\rho^{-1}(L^1(\mathcal{G}_2)) \subseteq L^1(\mathcal{G}_1).$$

Consequently,

$$\rho(L^1(\mathcal{G}_1)) = L^1(\mathcal{G}_2).$$

To complete the proof we only need to note that $\mathcal{G}_1$ and $\mathcal{G}_2$ are topological isomorphic if there exists a bipositive algebra isomorphism from $L^1(\mathcal{G}_1)$ onto $L^1(\mathcal{G}_2)$; see [8]. □

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