Absolutely Extendable Property and Stable Elements in $\Gamma$-Semihyperrings

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Abstract. The concept of $\Gamma$-semihyperrings is a generalization of a semiring, a generalization of a $\Gamma$-semiring, and a generalization of a semihyperering. In this paper, we define the notions of complex product, extension property and flat $\Gamma$-semihyperrings and some of their properties are obtained. In addition, we prove that every flat $\Gamma$-semihyperring is absolutely extendable. Finally, we give some characterization of stable elements.

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1. Introduction

The theory of hyperstructures was introduced by Marty [17] in 1934 during the 8th Congress of the Scandinavian Mathematicians. Algebraic hyperstructures are a generalization of classical algebraic structures. In
a classical algebraic structure the composition of two elements is an element, while in an algebraic hyperstructure the composition of two elements is a non-empty set.

Let $H$ be a non-empty set. Then, the map $\circ : H \times H \rightarrow \mathcal{P}^*(H)$ is called a hyperoperation, where $\mathcal{P}^*(H)$ is the family of non-empty subsets of $H$. $(H, \circ)$ is called a semihypergroup if for every $x, y \in H$, we have $x \circ (y \circ z) = (x \circ y) \circ z$. If for every $x \in H$, $x \circ H = H = H \circ x$, then $(H, \circ)$ is called a hypergroup. In the above definition, if $A$ and $B$ are two non-empty subsets of $H$ and $x \in H$, then we define

$$A \circ B = \bigcup_{a \in A, b \in B} a \circ b, \quad x \circ A = \{x\} \circ A \quad \text{and} \quad A \circ x = A \circ \{x\}.$$ 

Since then, hundreds of papers and several books have been written on this topic; see [2, 3, 6, 20]. A recent book on hyperstructures points out on their applications in cryptography, codes, automata, probability, geometry, lattices, binary relations, graphs and hypergraphs. Another book [6] is devoted especially to the study of hyperring theory; several kinds of hyperrings are introduced and analyzed, and the volume ends with an outline of applications in chemistry and physics, analyzing several special kinds of hyperstructures: e-hyperstructures and transposition hypergroups. A well known type of a hyperring is called the Krasner hyperring [16] and then some researchers such as Davvaz et al. [1, 5, 4, 7, 8, 14, 15, 18, 22], Gontineac [13], Sen and Dasgupta [19], Vougiouklis [20, 21] and others followed him.

**Definition 1.1.** A Krasner hyperring is an algebraic structure $(R, +, \cdot)$ which satisfies the following axioms:

1. $(R, +)$ is a canonical hypergroup, i.e.,
   
   (i) for every $x, y, z \in R$, $x + (y + z) = (x + y) + z$,
   
   (ii) for every $x, y \in R$, $x + y = y + x$,
   
   (iii) there exists $0 \in R$ such that $0 + x = x$.
   
   (iv) for every $x \in R$ there exists a unique element $-x \in R$ such that $0 \in x + (-x)$.
   
   (v) $z \in x + y$ implies that $y \in -x + z$ and $x \in -y + z$. 
(2) Relating to the multiplication, \((R, \cdot)\) is a semigroup having zero as a bilaterally absorbing element,

(3) The multiplication is distributive with respect to the hyperoperation +.

Recently, the concept of \(\Gamma\)-hyperstructures such as \(\Gamma\)-semihypergroups, \(\Gamma\)-hypergroups, \(\Gamma\)-semihyperrings and \(\Gamma\)-hypermodules study by many reseachers. The concept of \(\Gamma\)-semihyperrings is a generalization of semi-hyperrings, generalization of a \(\Gamma\)-semirings and a generalization of semirings. This concept consider by Dehkordi and Davvaz [9, 10, 11, 12]. They introduced rough ideals, fundamental relations and complex product on \(\Gamma\)-semihyperrings. By the concept fundamental relation on \(\Gamma\)-semihyperrings they introduced covariant functor between the category \(\Gamma\)-semihyperrings and the category semirings.

We know that homological algebra is a efficient toll in the study of rings and modules. This research work deals with certain algebraic systems that is non-additive modification of classical homological structure. Motivated by the definition of flat rings in the category of rings, we define flat \(\Gamma\)-semihyperrings in the category of \(\Gamma\)-semihyperrings. We introduce the notions of complex systems on \(\Gamma\)-semihypergroups, then we prove some results in respect. Also, we introduce the notions of right(left) flat \(\Gamma\)-semihyperring, extension property and absolutely extendable. We prove that every flat \(\Gamma\)-semihyperring is absolutely extendable. Finally, we obtain a characterization of stable elements in \(\Gamma\)-semihyperrings.

2. \(\Gamma\)-Semihyperrings and Complex Product

In [10, 11], Dehkordi and Davvaz introduced the concept of \(\Gamma\)-semihyperrings. Now, in this section, we shall explain more about \(\Gamma\)-semihyperrings. We investigate the concept of left (right) \(\Gamma\)-funs and complex product.

**Definition 2.1.** Let \(R\) and \(\Gamma\) be additive hypergroup and semihyper-group, respectively. Then, \(R\) is called a \(\Gamma\)-semihyperring if there exists a hyperoperation \(R \times \Gamma \times R \rightarrow \mathcal{P}^*(R)\) (the image of \((x, \alpha, y)\) is denoted by \(x\alpha y\), for \(x, y \in R\) and \(\alpha, \beta \in \Gamma\)) satisfies the following conditions:
(1) $x_1\alpha(x_2 + x_3) = x_1\alpha x_2 + x_1\alpha x_3$,

(2) $(x_1 + x_2)\alpha x_3 = x_1\alpha x_3 + x_2\alpha x_3$,

(3) $x_1(\alpha + \beta)x_2 = x_1\alpha x_2 + x_1\beta x_2$,

(4) $(x_1\alpha x_2)\beta x_3 = x_1\alpha(x_2\beta x_3)$,

for all $x_1, x_2, x_3 \in R$ and $\alpha \in \Gamma$.

A $\Gamma$-semihyperring $R$ is called $\Gamma$-hyperring if $R$ is a canonical hyper-group. It is obvious that every Krasner hyperring is a $\Gamma$-hyperring where $x\alpha y$ denotes the product of the elements $x, y \in R$.

**Example 2.2.** Let $R = \{a, b\}$ and $\Gamma = \{\alpha, \beta\}$ be two sets with the following operations and hyperoperation. Then, $R$ is a $\Gamma$-hyperring.

|   | $a$ | $b$ | $\alpha$ | $a$ | $b$ | $\beta$ | $a$ | $b$ | $\alpha$ | $\alpha$ | $\alpha$ | $\alpha$ | $\beta$ | $\alpha$ | $\alpha$ | $\alpha$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $+$ | $a$ | $b$ | $\alpha$ | $a$ | $b$ | $\beta$ | $a$ | $b$ | $\alpha$ | $\alpha$ | $\alpha$ | $\alpha$ | $\beta$ | $\alpha$ | $\alpha$ | $\alpha$ |
| $a$ | $a$ | $R$ | $a$ | $a$ | $a$ | $a$ | $a$ | $a$ | $a$ | $a$ | $a$ | $a$ | $a$ | $a$ | $a$ |
| $b$ | $R$ | $b$ | $b$ | $a$ | $a$ | $b$ | $a$ | $R$ | $\beta$ | $\alpha$ | $\alpha$ | $\alpha$ | $\alpha$ | $\alpha$ | $\alpha$ | $\alpha$ |

**Example 2.3.** Let $S = \{a_1, a_2, a_3, a_4\}$, $\Gamma = \{\alpha, \beta\}$. Then, $S$ is a $\Gamma$-semihyperring with respect to the following operations and hyperoperations:

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for every $x, y \in S$, $x\alpha y = a_1$. 
Example 2.4. Let $R$ be the Krasner hyperring, $R_{m \times n}$ be of all matrices over $R$ and $\Gamma$ be additive semihypergroup of all $n \times m$ matrices over $R$. Then, $R_{m \times n}$ is a $\Gamma$-hyperring where $a \alpha b$ denoted the usual matrix product of $a, \alpha, b$ where $a, b \in R_{m \times n}$ and $\alpha \in \Gamma$.

Example 2.5. Let $\mathbb{R}$ be the set of real numbers. Then, $\mathbb{R}$ is a $\widehat{\mathbb{Z}}$-semihyperring with respect to the following hyperaddition and hyperoperation:

$$x_1 \oplus x_2 = \{z : [x_1] + [x_2] \leq [x_1] + [x_2] + 1\},$$

$$x_1 \widehat{\alpha} x_2 = \{z : \alpha[x_1][x_2] \leq \alpha[x_1][x_2] + 1\},$$

for every $x_1, x_2 \in \mathbb{R}$ and $\alpha \in \widehat{\mathbb{Z}}$, where $\widehat{\mathbb{Z}} = \{\widehat{\alpha} : \alpha \in \mathbb{Z}\}$.

Let $A$ and $B$ be non-empty subsets of $\Gamma$-semihyperring $R$. We define

$$A \Gamma^\Sigma B = \{x \in R : x \in \sum_{i=1}^{n} a_i \alpha_i b_i : a_i \in A, b_i \in B, n \in \mathbb{N}\}.$$ 

Let $\Gamma$ be a semihypergroup and $n$ be a nonzero natural number. Then, we say that

$$x \beta_n y \iff \exists x_1, x_2, \ldots, x_n \in \Gamma : \{x, y\} \subseteq \prod_{i=1}^{n} x_i.$$ 

Let $\beta = \bigcup_{n \geq 1} \beta_n$. Clearly, the relation $\beta$ is reflexive and symmetric. Denote by $\beta^*$ the transitive closure of $\beta$.

Let $R$ be a $\Gamma$-semihyperring and $\mathcal{U}$ be a finite sum of elements of $R$. We define a relation $\gamma$ on $R$ as follows:

$$(a, b) \in \gamma \iff a, b \in u,$$

where $u \in \mathcal{U} = U_R \cup R \Gamma^\Sigma R \cup (U_R + R \Gamma^\Sigma R)$. We denote the transitive closure $\gamma$ by $\gamma^*$ and this equivalence relation is called fundamental equivalence relation on $R$. We denote the equivalence class of the element $a$ by $\gamma^*(a)$.

Hence, $\gamma^*(a_1) = \gamma^*(a_2)$ if and only if there exist $x_1, x_2, \ldots x_{n+1}$ with $x_1 = a_1, x_{n+1} = a_2$ and $u_1, u_2, \ldots u_n \in \mathcal{U}$ such that $\{x_i, x_{i+1}\} \subseteq u_i$, for some $i \in \{1, 2, \ldots n\}$.

Let $R$ be a $\Gamma$-semihyperring. We define a relation $\theta$ on

$$\left\{ \prod_{i=1}^{n} (\gamma^*(x_i), \beta^*(\alpha_i)) : n \in \mathbb{N}, x_i \in R, \alpha_i \in \Gamma \right\},$$
as follows:

\[
\left( \prod_{i=1}^{n} (\gamma^*(x_i), \beta^*(\alpha_i)), \prod_{j=1}^{m} (\gamma^*(x'_j), \beta^*(\alpha'_j)) \right) \in \theta
\]

\[\iff \sum_{i=1}^{n} \gamma^*(x_i)\beta^*(\alpha_i)\gamma^*(x) = \sum_{j=1}^{m} \gamma^*(x'_j)\beta^*(\alpha'_j)\gamma^*(x),\]

for every \(\gamma^*(x) \in [R : \gamma^*]\), where \(\gamma^*\) is a fundamental relation on \(R\). Let \(R\) be a \(\Gamma\)-semihyperring and there exists an element

\[
\theta \left( \prod_{i=1}^{n} (\gamma^*(e_i), \beta^*(\delta_i)) \right),
\]

such that \(\sum_{i=1}^{n} \gamma^*(e_i)\beta^*(\delta_i)\gamma^*(x) = \gamma^*(x)\), for all \(\gamma^*(x) \in [R : \gamma^*]\). We say that this element is an identity element (or just an identity) of \(F(R)\) and \(F(R)\) is a \(\Gamma\)-semihyperring with identity.

Let \(F(R) = \left\{ \theta \left( \prod_{i=1}^{n} (\gamma^*(x_i), \beta^*(\alpha_i)) \right) : x_i \in R, \alpha_i \in \Gamma, n \in \mathbb{N} \right\}\) and \(S\) be a non-empty set. We say that \(S\) is a left \(\Gamma\)-fun if there exists an action

\[
F(R) \times S \rightarrow S
\]

\[
(\theta \left( \prod_{i=1}^{n} (\gamma^*(x_i), \beta^*(\alpha_i)) \right), y) \mapsto \theta \left( \prod_{i=1}^{n} (\gamma^*(x_i), \beta^*(\alpha_i)) \right) y,
\]

with the following property:

\[
\theta \left( \prod_{i,j} (\gamma^*(x_i)\beta^*(\alpha_i)\gamma^*(y_j), \beta^*(\gamma_j)) \right) y
\]

\[
= \theta \left( \prod_{i=1}^{n} (\gamma^*(x_i), \beta^*(\alpha_i)) \right) \left( \theta \left( \prod_{j=1}^{m} (\gamma^*(y_j), \beta^*(\gamma_j)) \right) y, \right),
\]

\[
\left( \theta \left( \prod_{i=1}^{n} (\gamma^*(e_i), \beta^*(\delta_i)) \right), s \right) = s,
\]

where \(\theta \left( \prod_{i=1}^{n} (\gamma^*(x_i), \beta^*(\alpha_i)) \right), \theta \left( \prod_{j=1}^{m} (\gamma^*(y_j), \beta^*(\gamma_j)) \right)\) are elements of \(F(R)\) and \(s \in S\). In the same way, we can define right \(\Gamma\)-fun. Also, if \(R_1\) and \(R_2\) are \(\Gamma_1\)- and \(\Gamma_2\)-semihyperrings respectively, we say that \(S\) is a \((\Gamma_1, \Gamma_2)\)-
fun if it is a left $\Gamma_1$-fun and a right $\Gamma_2$-fun, and
\[
\left(\theta\left(\prod_{i=1}^{n} (\gamma^* (x_i), \beta^* (\alpha_i))\right) y\right) \left(\theta\left(\prod_{j=1}^{m} (\gamma^* (y_j), \beta^* (\gamma_j))\right)\right)
= \theta\left(\prod_{i=1}^{n} (\gamma^* (x_i), \beta^* (\alpha_i))\right) \left(y \theta\left(\prod_{j=1}^{m} (\gamma^* (y_j), \beta^* (\gamma_j))\right)\right),
\]
where $\theta\left(\prod_{i=1}^{n} (\gamma^* (x_i), \beta^* (\alpha_i))\right) \in F(R_1)$, $\theta\left(\prod_{j=1}^{m} (\gamma^* (y_j), \beta^* (\gamma_j))\right) \in F(R_2)$.

It is clear that the cartesian product $X_1 \times X_2$ of a left $\Gamma_1$-fun $X_1$ and a right $\Gamma_2$-fun $X_2$ becomes $(\Gamma_1, \Gamma_2)$-fun if we make the obvious definition:
\[
\theta\left(\prod_{i=1}^{n} (\gamma^* (x_i), \beta^* (\alpha_i))\right) (x_1, x_2) = \left(\theta\left(\prod_{i=1}^{n} (\gamma^* (x_i), \beta^* (\alpha_i))\right) x_1, x_2\right),
\]
\[
(x_1, x_2) \theta\left(\prod_{i=1}^{n} (\gamma^* (x_i), \beta^* (\alpha_i))\right) = \left(x_1, x_2 \theta\left(\prod_{i=1}^{n} (\gamma^* (x_i), \beta^* (\alpha_i))\right)\right).
\]

Suppose that $A$ is a $(\Gamma_1, \Gamma_2)$-fun and $B$ is a $(\Gamma_2, \Gamma_3)$-fun. Hence, $A \times B$ is a $(\Gamma_1, \Gamma_3)$-fun. A map $\varphi : A \times B \rightarrow C$ is called a $(\Gamma_1, \Gamma_3)$-map if for all $a \in A$, $b \in B$ and $\theta\left(\prod_{i=1}^{n} (\gamma^* (x_i), \beta^* (\alpha_i))\right) \in F(R_2)$,
\[
\varphi\left(a \theta\left(\prod_{i=1}^{n} (\gamma^* (x_i), \beta^* (\alpha_i))\right), b\right) = \varphi\left(a, \theta\left(\prod_{i=1}^{n} (\gamma^* (x_i), \beta^* (\alpha_i))\right) b\right).
\]

**Example 2.6.** Let $R$ be a $\Gamma$-semihyperring, $S$ be the set of all one-one and onto functions on $F(R)$. Then, $S$ is a left $\Gamma$-fun.

**Example 2.7.** Let $I$ be an ideal of $\Gamma$-semihyperring $R$. Then,
\[
T(I) = \left\{ \theta\left(\prod_{i=1}^{n} (\gamma^* (x_i), \beta^* (\alpha_i))\right) \in F(R) : \omega\left(\theta\left(\prod_{i=1}^{n} (\gamma^* (x_i), \beta^* (\alpha_i))\right)\right) \subseteq \gamma^* (I) \right\},
\]
is a left $\Gamma$-fun, where
\[
\omega\left(\theta\left(\prod_{i=1}^{n} (\gamma^* (x_i), \beta^* (\alpha_i))\right)\right)
= \left\{ \bigoplus, \gamma^* (x_i) \beta^* (\alpha_i) \gamma^* (x) : 1 \leq i \leq n, \ x \in R \right\}.
\]
We say that \((\Gamma_1, \Gamma_3)\)-func \(C\) is a complex product of \(A\) and \(B\) over \(F(R_2)\) if there is a \((\Gamma_1, \Gamma_3)\)-map \(\varphi : A \times B \rightarrow C\) such that for every \((\Gamma_1, \Gamma_3)\)-func \(D\) and every \((\Gamma_1, \Gamma_3)\)-map \(\beta : A \times B \rightarrow D\) there exists a unique \((\Gamma_1, \Gamma_3)\)-map \(\overline{\beta} : C \rightarrow D\) such that \(\overline{\beta} \circ \varphi = \beta\).

Suppose that \(\rho^*\) is an equivalence relation on \(A \times B\) generated by the following relation:

\[
\rho = \{ (\left( a \theta \left( \prod_{i=1}^{n} (\gamma^*(x_i), \beta^*(\alpha_i)) \right), \left( a, \theta \left( \prod_{i=1}^{n} (\gamma^*(x_i), \beta^*(\alpha_i)) \right) \right) \) ) : a \in A, b \in B, \theta \left( \prod_{i=1}^{n} (\gamma^*(x_i), \beta^*(\alpha_i)) \right) \in F(R_2) \}.
\]

We define \(C(A, B) = [A \times B : \rho^*]\) and denote a typical element \(\rho^*(a, b)\) of \(C(A, B)\) by \(C(a, b)\). By definition of \(\rho\) we have that

\[
C\left( a \theta \left( \prod_{i=1}^{n} (\gamma^*(x_i), \beta^*(\alpha_i)) \right), b \right) = C\left( a, \theta \left( \prod_{i=1}^{n} (\gamma^*(x_i), \beta^*(\alpha_i)) \right) \right) b,
\]

for all \(a \in A\) and \(b \in B\).

**Proposition 2.7.** Let \(A\) be a \((\Gamma_1, \Gamma_2)\)-func and \(B\) be a \((\Gamma_2, \Gamma_3)\)-func. Then, \(C(A, B)\) is a complex product of \(A\) and \(B\) over \(F(R_2)\).

**Theorem 2.8.** The complex product of \(A\) and \(B\) over \(F(R_2)\) is unique up to isomorphism.

### 3. Flat \(\Gamma\)-Semihypergerrings and Stable Elements

Motivated by the definition flat rings in the category of ring, we define flat \(\Gamma\)-semihypergerrings in the category \(\Gamma\)-semihypergerrings. This concept is a efficient tolls in the study of \(\Gamma\)-semihypergerrings. In this section, we introduce the concept of flat \(\Gamma\)-semihypergerrings, absolutely extendable, stable elements. Moreover, we prove that every flat \(\Gamma\)-semihyppererring is absolutely extendable and we obtain a characterization for stable elements.
Definition 3.1. Let $R$ be a $\Gamma$-semihyperring and $X_1$, $X_2$ be left $\Gamma$-funs. Then by a morphism or $\Gamma$-morphism from a left $\Gamma$-fun $X_1$ into a left $\Gamma$-fun $X_2$ we mean a map $\psi : X_1 \rightarrow X_2$ with the following property:

$$
\psi\left(\prod_{i=1}^{n}(\gamma^*(x_i), \beta^*(\alpha_i))\right)x_1 = \theta\left(\prod_{i=1}^{n}(\gamma^*(x_i), \beta^*(\alpha_i))\right)\psi(x_1),
$$

for every $\theta\left(\prod_{i=1}^{n}(\gamma^*(x_i), \beta^*(\alpha_i))\right) \in F(R)$ and $x_1 \in X_1$.

A congruence relation on a left $\Gamma$-fun $X$ is an equivalence relation on $X$ with the following property:

$$
x_1 \rho x_2 \implies \theta\left(\prod_{i=1}^{n}(\gamma^*(x_i), \beta^*(\alpha_i))\right)x_1 \rho \theta\left(\prod_{i=1}^{n}(\gamma^*(x_i), \beta^*(\alpha_i))\right)x_2,
$$

for every $x_1, x_2 \in X$ and $\theta\left(\prod_{i=1}^{n}(\gamma^*(x_i), \beta^*(\alpha_i))\right) \in F(R)$.

The quotient $[X : \rho]$ is a left $\Gamma$-fun structure by the following definition:

$$
\theta\left(\prod_{i=1}^{n}(\gamma^*(x_i), \beta^*(\alpha_i))\right) \cdot \rho(x) = \rho\left(\theta\left(\prod_{i=1}^{n}(\gamma^*(x_i), \beta^*(\alpha_i))\right)x\right).
$$

We can generalize the notion of complex product for three $\Gamma$-funs. Suppose that $X_1$, $X_2$ and $X_3$ are $(\Gamma_1, \Gamma_2)$-, $(\Gamma_2, \Gamma_3)$- and $(\Gamma_3, \Gamma_4)$-funs, respectively. A map $\varphi : X_1 \times X_2 \times X_3 \rightarrow X$ is called a triple map or $(\Gamma_1, \Gamma_4)$-map, if for $x_1 \in X_1$, $x_2 \in X_2$ and $x_3 \in X_3$

$$
\varphi\left(x_1 \theta\left(\prod_{i=1}^{n}(\gamma^*(y_i), \beta^*(\alpha_i))\right)x_2, x_3\right)
= \varphi\left(x_1, \theta\left(\prod_{i=1}^{n}(\gamma^*(y_i), \beta^*(\alpha_i))\right)x_2, x_3\right),
$$

where $\theta\left(\prod_{i=1}^{n}(\gamma^*(y_i), \beta^*(\alpha_i))\right) \in F(R_2)$, and

$$
\varphi\left(x_1, x_2 \theta\left(\prod_{j=1}^{m}(\gamma^*(x_j), \beta^*(\gamma_j))\right)x_3\right)
= \varphi\left(x_1, x_2, \theta\left(\prod_{j=1}^{m}(\gamma^*(x_j), \beta^*(\gamma_j))\right)x_3\right),
$$
where \( \theta \left( \prod_{j=1}^{m} (\gamma^*(x_j), \beta^*(\gamma_j)) \right) \in F(R_3). \)

We say that \( P \) is a complex product of \( X_1, X_2 \) and \( X_3 \) if there exists a unique \((\Gamma_1, \Gamma_4)\)-map \( \psi : X_1 \times X_2 \times X_3 \rightarrow P \) such that for every \((\Gamma_1, \Gamma_4)\)-fun \( X \) and \((\Gamma_1, \Gamma_4)\)-map \( \varphi : P \rightarrow D, \varphi \circ \psi = \varphi \). One can see that \( C(C(X_1, X_2), X_3) \) is a complex product of \( X_1 \times X_2 \times X_3 \) and

\[
C(C(X_1, X_2), X_3) \cong C(X_1, C(X_2, X_3)).
\]

Let \( R \) be a \( \Gamma \)-semihyperring. We say that \( R \) is left flat if for every left \( \Gamma \)-fun \( X \) and monomorphism \( \psi : X_1 \rightarrow X_2 \) of right \( \Gamma \)-funs, the induced map \( \psi_C : C(X_1, X) \rightarrow C(X_2, X) \) is injective. In the same way, we can define a right flat \( \Gamma \)-semihyperring.

Suppose that \( R_1 \) is a \( \Gamma \)-subsemihyperring of \( R \). We say that \( R_1 \) has the extension property in \( R \) if for every right \( \Gamma \)-fun \( X_1 \) and left \( \Gamma \)-fun \( X_2 \) in \( R_1 \), the following map is injective:

\[
\psi : C_{F(R_1)}(X_1, X_2) \rightarrow C_{F(R_1)}(C_{F(R_1)}(X_1, F(R)), X_2)
\]

\[
C(x_1, x_2) \mapsto C \left( C \left( x_1, \theta \left( \prod_{i=1}^{n} (\gamma^*(e_i), \beta^*(\delta_i)) \right) \right), x_2 \right).
\]

A \( \Gamma \)-semihyperring \( R \) is called absolutely extendable if it has extension property in every \( \Gamma \)-semihyperring \( R' \) containing it as \( \Gamma \)-subsemihyperring.

**Example 3.2.** Let \((R, +, \ast)\) be a Krasner hyperring, \((\Gamma, +)\) be a subsemihypergroup of \((R, +)\) and \( \{ A_g \mid g \in R \} \) be a family of disjoint non-empty sets. Then, \( S = \bigcup_{g \in R} A_g \) is a \( \Gamma \)-semihyperring with respect to the following hyperoperations:

\[
x \oplus y = \bigcup_{t \in g_1 + g_2} A_t, \quad x \alpha y = \bigcup_{i = g_1 + \alpha + g_2} A_t,
\]

where \( x \in A_{g_1} \) and \( y \in A_{g_2} \). Also, \( R \) is a left \( \Gamma \)-fun by

\[
F(S) \times R \rightarrow R
\]

\[
\theta \left( \left( \prod_{i=1}^{n} (\gamma^*(s_i), \beta^*(\alpha_i)) \right), x \right) \rightarrow x,
\]
where \( x \in R, \gamma^*(s_i) \in [S : \gamma^*] \) and \( \beta^*(\alpha_i) \in [\Gamma : \beta^*] \). Let \( X_1 \) and \( X_2 \) be left \( \Gamma \)-funs and \( \psi : X_1 \longrightarrow X_2 \) be a monomorphism. Then, \( \psi_C : C(X_1, R) \longrightarrow C(X_2, R) \) is injective. Indeed,

\[
\psi_C(\rho^*(x_1, r_1)) = \psi_C(\rho^*(x_2, r)),
\]

where \( x_1 \in X \) and \( r \in R \). By definition, \( \rho^* \), we have \( \psi(x_1) = \psi(x_2) \) and \( r_1 = r_2 \). Since \( \psi \) is one to one, we have \( x_1 = x_2 \). Therefore, \( \rho^*(x_1, r_1) = \rho^*(x_2, r_2) \). Therefore, \( S \) is a flat and absolutely extendable \( \Gamma \)-semihyperring.

**Proposition 3.3.** Every flat \( \Gamma \)-semihyperring is absolutely extendable.

**Proof.** Suppose that \( R \) is a flat \( \Gamma \)-semihyperring and \( R_1 \) is a \( \Gamma \)-semihyperring containing \( R \) as a \( \Gamma \)-subsemihyperring. We show that the map

\[
\psi : C_{F(R)}(X_1, X_2) \longrightarrow C_{F(R)}(C_{F(R)}(X_1, F(R_1)), X_2),
\]

is injective. We note that the map

\[
X_1 \cong C_{F(R)}(X_1, F(R)) \longrightarrow C_{F(R)}(X_1, F(R_1)),
\]

is injective. Since \( R \) is flat, the following map is one-one. Hence,

\[
C_{F(R)}(X_1, X_2) \cong C_{F(R)}(C_{F(R)}(X_1, F(R)), X_2) \longrightarrow C_{F(R)}(C_{F(R)}(X_1, F(R_1)), X_2).
\]

Therefore, \( R \) has the extension property in \( R_1 \). This completes the proof.

Let \( R_1 \) be a \( \Gamma \)-subsemihyperring of \( R \) such that \( \theta\left(\prod_{i=1}^{n} (\gamma^*(x_i), \beta^*(\alpha_i))\right) \in F(R) \). We say that \( \theta\left(\prod_{i=1}^{n} (\gamma^*(x_i), \beta^*(\alpha_i))\right) \) is stable element by \( R_1 \) if for every \( \Gamma' \)-semihyperring \( R' \) and homomorphism \( \psi_1, \psi_2 : F(R) \longrightarrow F(R') \)

\[
\psi_1\left(\theta\left(\prod_{j=1}^{n} (\gamma^*(y_j), \beta^*(\gamma_j))\right)\right) = \psi_2\left(\theta\left(\prod_{j=1}^{n} (\gamma^*(y_j), \beta^*(\gamma_j))\right)\right),
\]

for every \( \theta\left(\prod_{j=1}^{n} (\gamma^*(y_j), \beta^*(\gamma_j))\right) \in F(R_1) \) which implies that

\[
\psi_1\left(\theta\left(\prod_{i=1}^{n} (\gamma^*(x_i), \beta^*(\alpha_i))\right)\right) = \psi_1\left(\theta\left(\prod_{i=1}^{n} (\gamma^*(y_i), \beta^*(\gamma_i))\right)\right).
\]
The set of elements of $F(R)$ stable by $R_1$ denoted by $St_R(R_1)$. It is easy to see that $F(R_1) \subseteq St_R(R_1) \quad \Box$.

**Theorem 3.4.** Let $R_1$ be a $\Gamma$-subsemihyperring of $R$ and

$$\theta\left(\prod_{i=1}^{n}(\gamma^*(x_i), \beta^*(\alpha_i))\right) \in F(R).$$

Then,

$$C_{F(R_1)}\left(\theta\left(\prod_{i=1}^{n}(\gamma^*(x_i), \beta^*(\alpha_i))\right), \theta\left(\prod_{i=1}^{n}(\gamma^*(e_i), \beta^*(\delta_i))\right)\right),$$

$$= C_{F(R_1)}\left(\theta\left(\prod_{i=1}^{n}(\gamma^*(e_i), \beta^*(\delta_i))\right), \theta\left(\prod_{i=1}^{n}(\gamma^*(x_i), \beta^*(\alpha_i))\right)\right),$$

implies that $\theta\left(\prod_{i=1}^{n}(\gamma^*(x_i), \beta^*(\alpha_i))\right)$ is stable by $R_1$.

**Proof.** Suppose that

$$C_{F(R_1)}\left(\theta\left(\prod_{i=1}^{n}(\gamma^*(x_i), \beta^*(\alpha_i))\right), \theta\left(\prod_{i=1}^{n}(\gamma^*(e_i), \beta^*(\delta_i))\right)\right),$$

$$= C_{F(R_1)}\left(\theta\left(\prod_{i=1}^{n}(\gamma^*(e_i), \beta^*(\delta_i))\right), \theta\left(\prod_{i=1}^{n}(\gamma^*(x_i), \beta^*(\alpha_i))\right)\right).$$

Let we have $\Gamma'$-semihyperring $R'$ and homomorphism $\psi_1, \psi_2 : F(R) \rightarrow F(R')$ such that for every $\theta\left(\prod_{j=1}^{m}(\gamma^*(s_j), \beta^*(\varepsilon_j))\right) \in F(R_1),$

$$\psi_1\left(\theta\left(\prod_{j=1}^{m}(\gamma^*(s_j), \beta^*(\varepsilon_j))\right)\right) = \psi_2\left(\theta\left(\prod_{j=1}^{m}(\gamma^*(s_j), \beta^*(\varepsilon_j))\right)\right).$$

We define

$$\left(\theta\left(\prod_{j=1}^{m}(\gamma^*(s_j), \beta^*(\varepsilon_j))\right)\right) \cdot \theta\left(\prod_{j=1}^{m}(\gamma^*(z_j), \beta^*(\gamma_j))\right)$$

$$= \psi_1\left(\theta\left(\prod_{i=1}^{n}(\gamma^*(x_i), \beta^*(\alpha_i))\right)\right) \theta\left(\prod_{j=1}^{m}(\gamma^*(z_j), \beta^*(\gamma_j))\right).$$
and

\[
\theta\left(\prod_{j=1}^{m}(\gamma^*(z_j), \beta^*(\gamma_j))\right) \cdot \theta\left(\prod_{j=1}^{m}(\gamma^*(s_j), \beta^*(\epsilon_j))\right) = \theta\left(\prod_{j=1}^{m}(\gamma^*(z_j), \beta^*(\gamma_j))\right) \psi_2\left(\theta\left(\prod_{j=1}^{m}(\gamma^*(s_j), \beta^*(\epsilon_j))\right)\right),
\]

where \(\theta\left(\prod_{j=1}^{m}(\gamma^*(s_j), \beta^*(\epsilon_j))\right) \in F(R_1)\) and \(\theta\left(\prod_{j=1}^{m}(\gamma^*(z_j), \beta^*(\gamma_j))\right) \in F(R')\). Hence, \(F(R')\) is a \((\Gamma_1, \Gamma_1)\)-fun in \(R_1\). We define \(\psi : F(R) \times F(R) \to F(R')\) by the rule that

\[
\psi\left(\theta\left(\prod_{i=1}^{n}(\gamma^*(t_i), \beta^*(\delta_i))\right), \theta\left(\prod_{j=1}^{m}(\gamma^*(y_j), \beta^*(\gamma_j))\right)\right) = \psi_1\left(\theta\left(\prod_{i=1}^{n}(\gamma^*(t_i), \beta^*(\delta_i))\right)\right) \psi_2\left(\theta\left(\prod_{j=1}^{m}(\gamma^*(y_j), \beta^*(\gamma_j))\right)\right).
\]

Then, \(\psi\) is a \((\Gamma_1, \Gamma_1)\)-map in \(R_1\). Indeed,

\[
\psi\left(\theta\left(\prod_{i,j}(\gamma^*(t_i)\beta^*(\delta_i)\gamma^*(y_j), \beta^*(\gamma_j))\right), \theta\left(\prod_{r=1}^{m}(\gamma^*(z_r), \beta^*(\omega_r))\right)\right)
\]

\[
= \psi_1\left(\prod_{i,j}(\gamma^*(t_i)\beta^*(\delta_i)\gamma^*(y_j), \beta^*(\gamma_j))\right) \psi_2\left(\theta\left(\prod_{r=1}^{m}(\gamma^*(z_r), \beta^*(\omega_r))\right)\right)
\]

\[
= \psi_1\left(\theta\left(\prod_{i=1}^{n}(\gamma^*(t_i), \beta^*(\delta_i))\right)\right) \psi_1\left(\theta\left(\prod_{j=1}^{m}(\gamma^*(y_j), \beta^*(\gamma_j))\right)\right); \psi_2\left(\theta\left(\prod_{r=1}^{m}(\gamma^*(z_r), \beta^*(\omega_r))\right)\right)
\]

\[
= \psi_1\left(\theta\left(\prod_{i=1}^{n}(\gamma^*(t_i), \beta^*(\delta_i))\right)\right) \psi_2\left(\prod_{r,j}(\gamma^*(y_j)\beta^*(\gamma_j)\gamma^*(z_r), \beta^*(\omega_r))\right).
\]

Hence, there exists a map \(\bar{\psi} : C_{F(R_1)}(F(R), F(R)) \to F(R')\) such that

\[
\bar{\psi}\left(C_{F(R_1)}\left(\theta\left(\prod_{i=1}^{n}(\gamma^*(t_i), \beta^*(\delta_i))\right), \theta\left(\prod_{j=1}^{m}(\gamma^*(y_j), \beta^*(\gamma_j))\right)\right)\right)
\]

\[
= \psi\left(\theta\left(\prod_{i=1}^{n}(\gamma^*(t_i), \beta^*(\delta_i))\right), \theta\left(\prod_{j=1}^{m}(\gamma^*(y_j), \beta^*(\gamma_j))\right)\right)
\]

\[
= \psi_1\left(\theta\left(\prod_{i=1}^{n}(\gamma^*(t_i), \beta^*(\delta_i))\right)\right) \psi_2\left(\theta\left(\prod_{j=1}^{m}(\gamma^*(y_j), \beta^*(\gamma_j))\right)\right).
\]
Now, by assumption
\[
\psi_1\left(\theta\left(\prod_{i=1}^{n} (\gamma^*(x_i), \beta^*(\alpha_i))\right)\right) \\
= \psi_1\left(\theta\left(\prod_{i=1}^{m} (\gamma^*(x_i), \beta^*(\alpha_i))\right)\right) \psi_2\left(\theta\left(\prod_{i=1}^{n} (\gamma^*(e_i), \beta^*(\delta_i))\right)\right) \\
= \psi\left(C_{F(R_1)}\left(\theta\left(\prod_{i=1}^{n} (\gamma^*(x_i), \beta^*(\alpha_i))\right), \theta\left(\prod_{i=1}^{n} (\gamma^*(e_i), \beta^*(\delta_i))\right)\right)\right) \\
= \psi\left(C_{F(R_1)}\left(\theta\left(\prod_{i=1}^{n} (\gamma^*(x_i), \beta^*(\alpha_i))\right), \theta\left(\prod_{i=1}^{n} (\gamma^*(e_i), \beta^*(\alpha_i))\right)\right)\right) \\
= \psi_2\left(\theta\left(\prod_{i=1}^{n} (\gamma^*(x_i), \beta^*(\alpha_i))\right)\right) .
\]

This completes the proof. \(\square\)

**Theorem 3.5.** Let \(R_1\) be a \((\Gamma, \Gamma)\)-subsemihyperring of \(R\). Then,
\[
\theta\left(\prod_{j=1}^{m} (\gamma^*(x_j), \beta^*(\alpha_j))\right) \in St_R(R_1)
\]

implies that
\[
C_{F(R_1)}\left(\theta\left(\prod_{j=1}^{m} (\gamma^*(x_j), \beta^*(\alpha_j))\right), \theta\left(\prod_{i=1}^{n} (\gamma^*(e_i), \beta^*(\delta_i))\right)\right) \\
= C_{F(R_1)}\left(\theta\left(\prod_{i=1}^{n} (\gamma^*(e_i), \beta^*(\delta_i))\right)\theta\left(\prod_{j=1}^{m} (\gamma^*(x_j), \beta^*(\alpha_j))\right)\right). 
\]

**Proof.** We know that \(C_{F(R_1)}(F(R), F(R))\) is a \((\Gamma, \Gamma)\)-fun in \(R\) as follows:
\[
\theta\left(\prod_{i=1}^{n} (\gamma^*(y_i), \beta^*(\gamma_i))\right) \left(C_{F(R_1)}\left(\theta\left(\prod_{j=1}^{m} (\gamma^*(z_j), \beta^*(\alpha_j))\right)\right) \\
\cdot \theta\left(\prod_{j=1}^{m'} (\gamma^*(z'_j), \beta^*(\alpha'_j))\right)\right) \\
= C_{F(R_1)}\left(\theta\left(\prod_{i,j} \left(\gamma^*(y_i)\beta^*(\gamma_i)\gamma^*(z_j), \beta^*(\alpha_j)\right)\right), \theta\left(\prod_{j=1}^{m'} (\gamma^*(z'_j), \beta^*(\alpha'_j))\right)\right); 
\]
It is easy to see that this binary relation is associative. In fact
One can see that \( \Omega \) is a \((\Gamma, \Gamma)\)-fun in \( R \). We define a binary relation on \( F(R) \times \Omega \) as follows:

\[
C_{F(R_1)} \left( \theta \left( \prod_{j=1}^{m} (\gamma^*(z_j), \beta^*(\alpha_j)) \right), \theta \left( \prod_{j=1}^{m'} (\gamma^*(z'_j), \beta^*(\alpha'_j)) \right) \right)
\]

\[
\cdot \theta \left( \prod_{i=1}^{n} (\gamma^*(y_i), \beta^*(\gamma_i)) \right)
\]

\[
= C_{F(R_1)} \left( \theta \left( \prod_{j=1}^{m} (\gamma^*(z_j), \beta^*(\alpha_j)) \right), \theta \left( \prod_{i,j} (\gamma^*(z_j), \beta^*(\alpha_j) \gamma^*(y_i), \beta^*(\gamma_i)) \right) \right).
\]

Let \( \Omega \) be the set of all finite combinations

\[
\sum_{i=1}^{n} n_i \left( C_{F(R_1)} \left( \theta \left( \prod_{j=1}^{m} \gamma^*(x_{ij}), \beta^*(\alpha_{ij}) \right) \right), \theta \left( \prod_{j=1}^{m'} \gamma^*(x'_{ij}), \beta^*(\alpha'_{ij}) \right) \right).
\]

One can see that \( \Omega \) is a \((\Gamma, \Gamma)\)-fun in \( R \). We define a binary relation on \( F(R) \times \Omega \) as follows:

\[
\left( \theta \left( \prod_{t=1}^{n} (\gamma^*(y_t), \beta^*(\alpha_t)) \right), \right.
\]

\[
\sum_{i=1}^{n} n_i \left( C_{F(R_1)} \left( \theta \left( \prod_{j=1}^{m} \gamma^*(x_{ij}), \beta^*(\alpha_{ij}) \right) \right), \theta \left( \prod_{j=1}^{m'} \gamma^*(x'_{ij}), \beta^*(\alpha'_{ij}) \right) \right).
\]

\[
\left( \theta \left( \prod_{t=1}^{n} (\gamma^*(y'_t), \beta^*(\alpha'_t)) \right), \right.
\]

\[
\sum_{i=1}^{m} s_i \left( C_{F(R_1)} \left( \theta \left( \prod_{j=1}^{m} \gamma^*(z_{ij}), \beta^*(\gamma_{ij}) \right) \right), \theta \left( \prod_{j=1}^{m'} \gamma^*(z'_{ij}), \beta^*(\gamma'_{ij}) \right) \right)
\]

\[
\left. \right) \left( \theta \left( \prod_{t,r} (\gamma^*(y_t) \beta^*(\alpha_t) \gamma^*(y'_t), \beta^*(\alpha'_t)) \right) \right).
\]

\[
\sum_{i=1}^{n} n_i C_{F(R_1)} \left( \theta \left( \prod_{j,t} (\gamma^*(y_t) \beta^*(\alpha_t) \gamma^*(z_{ij}), \beta^*(\gamma_{ij})) \right), \theta \left( \prod_{j=1}^{m'} \gamma^*(z'_{ij}), \beta^*(\gamma'_{ij}) \right) \right)
\]

\[
+ \sum_{i=1}^{n} s_i C_{F(R_1)} \left( \theta \left( \prod_{r,j} (\gamma^*(x_{ij}) \beta^*(\alpha_{ij}) \gamma^*(y'_j), \beta^*(\alpha'_{ij})) \right), \right.
\]

\[
\theta \left( \prod_{j=1}^{m'} \gamma^*(x'_{ij}), \beta^*(\alpha'_{ij}) \right) \right).
\]

It is easy to see that this binary relation is associative. In fact \( F(R) \times \Omega \) is a groupoid with identity \( \left( \theta \left( \prod_{i=1}^{n} (\gamma^*(e_i), \beta^*(\delta_i)) \right), 0 \right) \). Suppose that
By a routine process, we see that

\[ \theta \left( \prod_{j=1}^{m} (\gamma^*(x_j), \beta^*(\alpha_j)) \right) \in St_R(R_1). \]

We consider two homomorphisms \( \psi_1 \) and \( \psi_2 \) from \( F(R) \) into \( F(R) \times \Omega \) and show that they coincide on \( R_1 \). We define

\[
\psi_1 \left( \theta \left( \prod_{i=1}^{n} (\gamma^*(y_i), \beta^*(\alpha_i)) \right) \right) = \left( \theta \left( \prod_{i=1}^{n} (\gamma^*(y_i), \beta^*(\alpha_i)) \right), 0 \right)
\]

\[
\psi_2 \left( \theta \left( \prod_{i=1}^{n} (\gamma^*(y_i), \beta^*(\alpha_i)) \right) \right) = \left( \theta \left( \prod_{i=1}^{n} (\gamma^*(y_i), \beta^*(\alpha_i)) \right), \right)
\]

\[
\theta \left( \prod_{i,j} \gamma^*(y_i)\beta^*(\alpha_i)\gamma^*(e_j), \beta^*(\delta_j) \right) = \theta \left( \prod_{i,j} \gamma^*(e_j)\gamma^*(\delta_j)\gamma^*(y_i), \beta^*(\alpha_i) \right).
\]

By a routine process, we see that \( \psi_1 \) and \( \psi_2 \) are homomorphisms. Let

\[ \theta \left( \prod_{i=1}^{n} (\gamma^*(z_i), \beta^*(\alpha_i)) \right) \in F(R_1). \]

This implies that

\[
C_{F(R_1)} \left( \theta \left( \prod_{i=1}^{n} (\gamma^*(z_i), \beta^*(\alpha_i)) \right), \theta \left( \prod_{i=1}^{n} (\gamma^*(e_i), \beta^*(\delta_i)) \right) \right)
\]

\[
= C_{F(R_1)} \left( \theta \left( \prod_{i=1}^{n} (\gamma^*(e_i), \beta^*(\delta_i)) \right), \theta \left( \prod_{i=1}^{n} (\gamma^*(z_i), \beta^*(\alpha_i)) \right) \right)
\]

in \( C_{F(R_1)}(F(R), F(R)) \) and so

\[
\psi_1 \left( \theta \left( \prod_{i=1}^{n} (\gamma^*(z_i), \beta^*(\alpha_i)) \right) \right) = \psi_2 \left( \theta \left( \prod_{i=1}^{n} (\gamma^*(z_i), \beta^*(\alpha_i)) \right) \right).
\]

Moreover, \( \theta \left( \prod_{j=1}^{m} (\gamma^*(x_j), \beta^*(\alpha_j)) \right) \in St_R(R_1) \) implies that

\[
\psi_1 \left( \theta \left( \prod_{j=1}^{m} (\gamma^*(x_j), \beta^*(\alpha_j)) \right) \right) = \psi_2 \left( \theta \left( \prod_{j=1}^{m} (\gamma^*(x_j), \beta^*(\alpha_j)) \right) \right),
\]

and so

\[
C_{F(R_1)} \left( \theta \left( \prod_{j=1}^{m} (\gamma^*(x_j), \beta^*(\alpha_j)) \right), \theta \left( \prod_{i=1}^{n} (\gamma^*(e_i), \beta^*(\delta_i)) \right) \right),
\]

\[
= C_{F(R_1)} \left( \theta \left( \prod_{i=1}^{n} (\gamma^*(e_i), \beta^*(\delta_i)) \right), \theta \left( \prod_{j=1}^{m} (\gamma^*(x_j), \beta^*(\alpha_j)) \right) \right).
\]
This completes the proof. □

**Proposition 3.6.** Let \( R \) be a \( \Gamma \)-semihyperring, such that \( X_1, X_2 \) and \( X_3 \) be \( \Gamma \)-funs and \( \varphi_1 : X_1 \to X_2, \varphi_2 : X_1 \to X_3 \) be \( \Gamma \)-morphisms. Then, there exists a \( \Gamma \)-fun \( X \) and \( \psi_1 : X_2 \to X, \psi_2 : X_3 \to X \) such that \( \psi_1 \) and \( \psi_2 \) are \( \Gamma \)-homomorphisms. Moreover, if \( \psi_1(x_2) = \psi_2(x_3) \), then \( x_2 \in \varphi(X_1) \).

**Proof.** Suppose that \( \rho \) is the equivalence relation generated by all pairs \( (x_1, \varphi_1(x_1)), (x_1, \varphi_2(x_1)) \), on \( X = X_2 \cup X_3 \), where \( x_1 \in X_1 \). The maps \( \psi_1 : X_2 \to X, \psi_2 : X_3 \to X \) are given by \( \psi_1(x_2) = \rho(x_2), \psi_1(x_2) = \rho(x_2) \). This complete the proof. □

**Lemma 3.7.** Let \( R_1 \) be a \( \Gamma \)-subsemihyperring of \( R \) and \( R_1 \) has extension property in \( R \), \( \varphi : X_1 \to X_2 \) be a \( \Gamma \)-morphism of right \( \Gamma \)-fun in \( R_1 \) and

\[
C\left(x_2, \theta\left(\prod_{i=1}^n (\gamma^*(e_i), \beta^*(\delta_i))\right)\right) = C\left(\varphi(x_1), \theta\left(\prod_{j=1}^m (\gamma^*(y_j), \beta^*(\alpha_j))\right)\right),
\]
in \( C(X_2, F(R)) \). Then, \( x_2 \in \varphi(X_1) \).

**Proof.** Suppose that \( X \) is a \( \Gamma \)-fun in Proposition 3.6. Consider the following commutative diagram:

\[
\begin{array}{ccc}
X_1 & \xrightarrow{\varphi} & X_2 \\
\downarrow & & \downarrow \\
X_2 & \xrightarrow{\psi_2} & X
\end{array}
\]

where \( \psi_1 : X_2 \to X \) and \( \psi_2 : X_2 \to X \). Hence, the following diagram is commutative:

\[
\begin{array}{ccc}
C(X_1, F(R)) & \xrightarrow{C(\varphi, 1)} & C(X_2, F(R)) \\
\downarrow & & \downarrow \\
C(X_2, F(R)) & \xrightarrow{C(\psi_2, 1)} & C(X, F(R))
\end{array}
\]

By the extension property the map \( x_2 \mapsto C\left(x_2, \theta\left(\prod_{i=1}^n (\gamma^*(e_i), \beta^*(\delta_i))\right)\right) \)
from $X_2 \cong C(X_2, F(R_1))$ to $C(X_2, F(R))$ is one to one. We have

$$C\left(\psi_1(x_2), \theta\left(\prod_{i=1}^{n} (\gamma^*(e_i), \beta^*(\delta_i))\right)\right)$$

$$= C(\psi_1, 1)\left(C\left(x_2, \theta\left(\prod_{i=1}^{n} (\gamma^*(e_i), \beta^*(\delta_i))\right)\right)\right)$$

$$= C(\psi_1, 1)\left(C\left(\varphi(x_1), \theta\left(\prod_{j=1}^{m} (\gamma^*(y_j), \beta^*(\alpha_j))\right)\right)\right)$$

$$= C(\psi_1 \circ \varphi)(x_1), \theta\left(\prod_{j=1}^{m} (\gamma^*(y_j), \beta^*(\alpha_j))\right)$$

$$= C(\psi_1, 1)\left(C\left(\varphi(x_1), \theta\left(\prod_{j=1}^{m} (\gamma^*(y_j), \beta^*(\alpha_j))\right)\right)\right)$$

$$= C(\psi_2, 1)\left(C\left(\varphi(x_1), \theta\left(\prod_{j=1}^{m} (\gamma^*(y_j), \beta^*(\alpha_j))\right)\right)\right)$$

$$= C(\psi_2, 1)\left(x_2, \theta\left(\prod_{i=1}^{n} (\gamma^*(e_i), \beta^*(\delta_i))\right)\right)$$

$$= C\left(\psi_2(x_2), \theta\left(\prod_{j=1}^{m} (\gamma^*(y_j), \beta^*(\alpha_j))\right)\right).$$

Hence, $\psi_1(x_2) = \psi_2(x_2)$ and it follows by Proposition 3.6, $x_2 \in \varphi(X_1)$. □

**Theorem 3.8.** Let $R_1$ be a $\Gamma$-subsemihyperring of $R$ and suppose that $R_1$ has the extension property in $R$. Let $X_1$, $X_2$ be right $\Gamma$-funs in $R_1$ and $\varphi : X_1 \rightarrow X_2$ be $\Gamma$-monomorphism in $R_1$ and $Z$ be a left $\Gamma$-fun in $R_1$ such that $C(\varphi, 1) : C(X_1, Z) \rightarrow C(X_2, Z)$ is also monomorphism. If

$$C\left(\varphi(x_1), \theta\left(\prod_{j=1}^{m} (\gamma^*(y_j), \beta^*(\alpha_j))\right)\right), z' \right)$$

in $C(C(X_2, F(R)), Z)$. Then, there exists $x_1' \in X_1$ and $z_1 \in Z$ such that

$$C\left(\varphi(x_1'), \theta\left(\prod_{i=1}^{n} (\gamma^*(e_i), \beta^*(\delta_i))\right)\right), z_1 \right).$$
**Proof.** Suppose that
\[
C(C(x_2, \theta\left(\prod_{i=1}^{n} (\gamma^*(e_i), \beta^*(\delta_i))\right)), z) = C(C(\varphi(x_1), \theta\left(\prod_{j=1}^{m} (\gamma^*(y_j), \beta^*(\gamma_j))\right)), z'),
\]
in $C(C(X_2, F(R)), Z)$. By Proposition 3.6, we have the following commutative diagram:
\[
\begin{array}{ccc}
X_1 & \xrightarrow{\psi} & X_2 \\
\downarrow & & \downarrow \\
X_2 & \xrightarrow{\psi_2} & X
\end{array}
\]
such that $\psi_1 : X_2 \rightarrow X$ and $\psi_2 : X_2 \rightarrow X$. Hence, the following diagram is commutative:
\[
\begin{array}{ccc}
C(X_1, Z) & \xrightarrow{C(\varphi, 1)} & C(X_2, Z) \\
\downarrow & & \downarrow \\
C(X_2, Z) & \xrightarrow{C(\psi_2, 1)} & C(X, Z)
\end{array}
\]
is commutative. We note that
\[
C(C(\psi_1(x_2), \theta\left(\prod_{i=1}^{n} (\gamma^*(e_i), \beta^*(\delta_i))\right)), z) = C(C(\psi_1, 1, 1)\left(C(C(x_2, \theta\left(\prod_{i=1}^{n} (\gamma^*(e_i), \beta^*(\delta_i))\right)), z)\right)
\]
\[
= (C(\psi_1, 1, 1)\left(C(C(\varphi(x_1), \theta\left(\prod_{j=1}^{m} (\gamma^*(y_j), \beta^*(\gamma_j))\right)), z')\right)
\]
\[
= C(C((\psi_1 \circ \varphi)(x_1), \theta\left(\prod_{j=1}^{m} (\gamma^*(y_j), \beta^*(\gamma_j))\right)), z')
\]
\[
= C(C((\psi_2 \circ \varphi)(x_1), \theta\left(\prod_{j=1}^{m} (\gamma^*(y_j), \beta^*(\gamma_j))\right)), z')
\]
\[
\vdots
\]
\[
= C(C(\psi_2(x_2), \theta\left(\prod_{i=1}^{n} (\gamma^*(e_i), \beta^*(\delta_i))\right)), z).
\]

By the extension property we deduced that $C(\psi_1(x_2), z) = C(\psi_2(x_2), z)$. Hence, by Proposition 3.6, there exists $C(x'_1, z_1) \in C(X_1, Z)$ such that
\[
C(x_2, z) = C(\varphi, 1)(C(x'_1, z_1)) = C(\varphi(x'_1), z_1).
\]
Therefore,
\[
C\left(C\left(x_2, \theta\left(\prod_{i=1}^{n} (\gamma^* (e_i), \beta^* (\delta_i))\right), z\right)\right) = C\left(C\left(\varphi(x'_1), \theta\left(\prod_{i=1}^{n} (\gamma^* (e_i), \beta^* (\delta_i))\right), z_1\right)\right).
\]

This completes the proof. □

References


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