A Generalization of Clique Polynomials and Graph Homomorphism

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Abstract. The clique polynomial of a graph $G$ is the ordinary generating function of the number of complete subgraphs (cliques) of $G$. In this paper, we introduce a new vertex-weighted version of these polynomials. We also show that these weighted clique polynomials have always a real root provided that the weights are non-negative real numbers. As an application, we obtain a no-homomorphism criteria based on the largest real root of our vertex-weighted clique polynomial.

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1. Introduction

The dependence polynomial was first introduced by Fisher [1], while working on the problem of counting the number of words of length $n$ from the alphabet of $m$ letters so that some pairs of them can commute. Fisher and Solow [2] introduced the dependence polynomial, as follows:

$$f_G(x) = 1 - c_1 x + c_2 x^2 - c_3 x^3 + \cdots + (-1)^\omega c_\omega x^\omega$$

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where $\omega$ is the size of the largest clique in $G$ and $c_i$ denotes the number of complete subgraphs of size $i$ in $G$. Fisher [1], showed that the generating function of the above word-counting problem is $\frac{1}{f_G(x)}$.

If we change the sign of all negative coefficients in $f_G(x)$ to positive signs, we obtain a polynomial which is called the clique polynomial and denoted by $C(G, x)$. Hajiabolhasan and Mehrabadi [3] showed that for any simple graph $G$, the clique polynomial of $G$ has always a real root using basic counting techniques, induction and the intermediate value theorem. As an immediate consequence, they obtained a new generating function proof of Mantel’s theorem [4, p.41] for triangle-free graphs. In this paper, we will continue the same line of research by introducing a new weighted version of the clique polynomial. Our main goal here is to show that how one can use the largest real root of this new graph polynomial to obtain a no-homomorphism criteria.

2. Weighted Clique Polynomials

Throughout the paper we will assume that $G$ is a simple undirected graph. The graph terminology that we use is standard and generally follows from [4]. For a given graph $G$, we denoted by $V(G)$ its set of vertices and by $E(G)$ its set of edges. When $S \subseteq V(G)$, the induced subgraph $G[S]$ consists of $S$ and all edges whose endpoints are connected in $S$. The neighborhood of a vertex $u$, written $N(u)$, is the set of vertices adjacent to $u$. We write $G - u$ for the subgraph of $G$ obtained by deleting a vertex $u$. We also write $G - uv$ for the subgraph obtained by deleting an edge $uv \in E(G)$. Here by an $i$-clique, $i \geq 1$, we mean a complete subgraph of $G$ with $i$ vertices. The clique number of a graph $G$ denoted by $\omega$ is the size of the largest clique in $G$. For simplicity of our arguments, we will assume that $V(G) = [n] = \{1, 2, \ldots, n\}$.

For a given set $A$, it’s power set denoted by $\mathcal{P}(A)$, is the set of all subsets of $A$. We will associate a real weight function $w : V(G) \mapsto \mathbb{R}$ with the given graph $G$, by associating an indeterminate $w_i$ to each vertex $i$ of $G$ which can be viewed as the weight of the vertex $i$. More precisely, we
have

\[ w : V(G) \mapsto \mathbb{R} \]
\[ w(i) = w_i, \quad \forall i \in V(G). \]

The above mapping can be simply extended to the following multiplicative weight function (we will use the same symbol for the extended version)

\[ w : \mathcal{P}(V(G)) \mapsto \mathbb{R} \]
\[ w(S) = \prod_{i \in S} w_i, \quad \forall S \subseteq V(G). \]

That is, the weight of any subset of vertices will be obtained as the product of the weights of those vertices. By convention, the empty product (corresponding to the set \( S = \emptyset \)) is defined to be 1.

**Remark 2.1.** It is worthy to note that the above weight function on vertices of a graph \( G \) can be considered as the multiplicative version of the following additive weight function:

\[ w : \mathcal{P}(V(G)) \mapsto \mathbb{R} \]
\[ w(S) = \sum_{i \in S} w_i, \quad \forall S \subseteq V(G). \]

Now, we are ready to give the definition of the weighted clique polynomial.

**Definition 2.2.** Let \( G \) be a graph on \( n \) vertices associated with the multiplicative weight function \( w : V(G) \mapsto \mathbb{R} \). We define the weighted clique polynomial of \( G \) denoted by \( C(G, x; w) \), as follows

\[ C(G, x; w) = \sum_{i=0}^{\omega} c_i(G, w)x^i \]

where \( w = (w_1, \ldots, w_n) \) is the weight vector of vertices of \( G \) and \( c_i(G, w), i \geq 1 \), the weighted-sum of all \( i \)-cliques in \( G \), is defined by

\[ c_i(G, w) = \sum_{S : S \subseteq V(G), \ G[S] \text{ is an } i \text{-clique}} w(S). \]
By convention, we assume $c_0(G, \mathbf{w}) = 1$ for any weight vector $\mathbf{w}$ and any graph $G$. In particular, if all weights are equal to one then we obtain the clique polynomial of $G$ [3].

**Example 2.3.** Consider the graph $G_1$ with $\mathbf{w} = (1, 1, 1, 1, 1)$ as depicted in Fig.1, we obtain

$$C(G_1, x; \mathbf{w}) = 1 + 5x + 3x^2 + x^3.$$  

The above polynomial has at least one real root, because of having odd degree. Moreover, since the quadratic polynomial $\frac{d}{dx} C(G_1, x; \mathbf{w}) = 5 + 6x + 3x^2$ has the discriminate $\Delta = 9 - 15 = -6 < 0$, then by the first derivative criteria $C(G_1, x, \mathbf{w})$ is an increasing function on its domain and hence the clique polynomial of $G_1$ has only one real root.

![Figure 1](image_url)

**Figure 1.** The clique polynomial of the graph $G_1$ has only one real root.

It seems that as in the case of the clique polynomials, we have always at least a real root for the weighted clique polynomials for any arbitrary choices of non-negative real weights. Next, we present the necessary tools for proving this interesting result.

The following counting lemma is key for proving the existence of a real root for the weighted clique polynomials with non-negative real weights. Here, for the given edge $e = \{u, v\}$, the notation $N(e)$ stands
for $N(e) = N(u) \cap N(v)$. We also define the multiplicative weight function associated with the subgraph $H$ of $G$ simply by restricting the weight function associated with $G$ to $H$. More precisely, we only need to choose the weight of a vertex which is not in $V(H)$ to be zero.

**Lemma 2.4.** Let $G$ be a graph and $u, v \in V(G)$ with non-negative real weights $w_u$ and $w_v$. Then, we have

\begin{align}
\text{i) } C(G, x; w) &= C(G - u, x; w_1) + w_u x C(G[N(u)], x; w_2), \quad (1) \\
\text{ii) } C(G, x; w) &= C(G - e, x; w_3) + w_u w_v x^2 C(G[N(e)], x; w_4) \quad (2)
\end{align}

where $e = uv \in E(G)$ and $w_1, w_2, w_3$ and $w_4$ are the weight vectors for the subgraphs $G - u$, $G[N(u)]$, $G - e$ and $G[N(e)]$, respectively.

**Proof.** We first note that the following recurrence relations

\begin{align}
c_i(G, w) &= c_i(G - u, w_1) + w_u c_{i-1}(G[N(u)], w_2), \quad (i \geq 1) \\
c_i(G, w) &= c_i(G - e, w_3) + w_u w_v c_{i-2}(G[N(e)], w_4), \quad (i \geq 2),
\end{align}

can be easily proved using a simple counting and case analysis arguments. Now, by multiplying both sides of the above relations to $x^i$ and summing over all $i$’s we obtain the desired results. \qed

The join of two simple graphs $G$ and $H$, written $G \vee H$, is defined as a graph with the vertex set $V(G) \cup V(H)$ and the edge set $E(G) \cup E(H) \cup \{xy | x \in V(G) \land y \in V(H)\}$. Considering the definition of the weighted clique polynomials and using Lemma 2.4, we get the following multiplicative property for the weighted clique polynomials.

**Proposition 2.5.** Let $G$ and $H$ be arbitrary graphs with their weight vectors $w_1 = (w_1^{(g)}, \ldots, w_m^{(g)})$ and $w_2 = (w_1^{(h)}, \ldots, w_n^{(h)})$. Then, we have

\[ C(G \vee H, x; w) = C(G, x; w_1)C(H, x; w_2) \]

where

\[ w = (w_1^{(g)}, \ldots, w_m^{(g)}, w_1^{(h)}, \ldots, w_n^{(h)}). \]
**Definition 2.6.** Let $G$ be a graph and $\mathcal{Z}(G)$ be the set of all negative real roots of $C(G, x; w)$. We define $\zeta_G$ by

$$
\zeta_G = \begin{cases} 
\max \mathcal{Z}(G) & \text{if } \mathcal{Z}(G) \neq \emptyset, \\
-\infty & \text{otherwise.}
\end{cases}
$$

**Theorem 2.7.** Let $G$ be a graph and $H$ be the induced subgraph of $G$. Then $\zeta_H \leq \zeta_G$.

**Proof.** We proceed by strong induction on $|V(G)| = n$. If $n = 1, 2$, then the assertion is easily followed. Moreover, if $H = G$ then we are also done. Therefore, we will assume that $H$ is a proper induced subgraph of $G$. Since any proper induced subgraph $H$ of $G$ can be obtained from it by a sequence of vertex-deletion operations and the binary relation $\leq$ is transitive, it is sufficient to prove the assertion for $H = G - u$. If $\mathcal{Z}(G-u) = \emptyset$, by the definition of $\zeta_{G-u}$, we are done. Otherwise, plugging $\alpha = \zeta_{G-u} (< 0)$ into both sides of (1), we get

$$
C(G, \alpha; w) = w_u \alpha C(G[N(u)], \alpha; w_2).
$$

Now, we distinguish between two cases.

**Case 1.** If $C(G[N(u)], \alpha; w_2) \geq 0$, then by positivity of the weights we conclude that $C(G, \alpha; w) \leq 0$. This implies that, using intermediate value theorem and $C(G, 0, w) = 1$, the polynomial $C(G, x; w)$ has a real root in the interval $[\alpha, 0)$. Hence, we immediately obtain $\zeta_{G-u} = \alpha \leq \zeta_G$.

**Case 2.** Otherwise, $C(G[N(u)], \alpha; w_2) < 0$. Now, we claim that this case is impossible. To do this, note that the last inequality implies that $C(G[N(u)], x; w_2)$ has a real root in the interval $[\alpha, 0)$ (let say $\beta$), once again by applying intermediate value theorem. Hence we conclude that

$$
\zeta_{G-u} = \alpha \leq \beta \leq \zeta_{G[N(u)]}.
$$

But this last inequality is impossible by the induction hypothesis, since $G[N(u)]$ is an induced subgraph of $G - u$ and we obtain the inequality

$$
\zeta_{G[N(u)]} \leq \zeta_{G-u}.
$$
Thus, the only possibility is Case 1 which implies the desired inequality $\zeta_{G-u} \leq \zeta_G$. This completes the proof by mathematical induction. □

**Corollary 2.8.** For any graph $G$, let $w_u$ be the weight of the vertex $u$ which has the maximum weights among all vertices. Then, $-\frac{1}{w_u} \leq \zeta_G < 0$.

**Proof.** First of all, by definition of $\zeta_G$ it is clear that $\zeta_G < 0$. Next, let $u$ be the vertex of $G$ with the maximum weight $w_u$ and let $H$ be the induced subgraph $G[u]$. Then, clearly $C(H, x; w_u) = 1 + w_u x$. Hence, $\zeta_H = -\frac{1}{w_u}$. Now applying Theorem 2.7, we get $-\frac{1}{w_u} \leq \zeta_G$. □

**Remark 2.9.** It is worth to note that the above corollary shows that the weighted clique polynomial has always a real root, provided that the weights are non-negative real numbers and the weight vector is not identically zero.

As we already saw, when $H$ is an induced subgraph of $G$ we obtain $\zeta_H \leq \zeta_G$. Next, we show that for a spanning subgraph $H$ of $G$ we have the reverse inequality; that is, $\zeta_H \geq \zeta_G$. Recall that a spanning subgraph $H$ of a given graph $G$ is the one with the same vertex-set as $G$; that is, $V(H) = V(G)$.

**Theorem 2.10.** Let $G$ be a graph and $H$ be the spanning subgraph of $G$. Then $\zeta_H \geq \zeta_G$.

**Proof.** We proceed by strong induction on the number of edges. It is sufficient to prove the assertion for the case $H = G - e$, where $e = uv$ is an edge of $G$. Now by substituting $\zeta_G$ in both sides of (2), we get

$$C(G - uv, \zeta_G; w_3) = -w_u w_v \zeta_G^2 C(G[N(e)], \zeta_G; w_4).$$

(3)

Since $G[N(e)]$ is an induced subgraph of $G$, then by Theorem 2.7 the right-hand side of (3) is negative which implies that $C(G - uv, \zeta_G; w_3)$ is also negative. Considering the fact that $C(G - uv, 0; w_3) = 1$ and applying the intermediate value theorem, we get the desired result. □

**Definition 2.11.** An independent set in a graph is a set of pairwise nonadjacent vertices. The independence number of a graph $G$, written $\alpha(G)$, is the maximum size of an independent set of vertices.
Proposition 2.12. Let $G$ be a graph with $n$ vertices and $\alpha(G)$ its independence number. Let $w = (w_1, \ldots, w_n)$ be the weight vector of $G$ with $w = \min_{1 \leq i \leq n} w_i$. Then, we have $\alpha(G) \leq -\frac{1}{w_G}$.

Proof. Assume that $S = \{i_1, i_2, \ldots, i_k\}$ is an independent set of size $\alpha(G) = k$ in $G$ and $H$ is the induced subgraph $G[S]$. Since $H$ has no edges, we obtain

$$C(H, x; w) = 1 + (w_{i_1} + w_{i_2} + \cdots + w_{i_k})x.$$ 

Now, set $w = \min_{1 \leq i \leq n} w_i$. Then $\xi_H = -\frac{1}{w_{i_1} + \cdots + w_{i_k}} \geq -\frac{1}{\alpha(G)w}$, and since $\zeta_H \leq \zeta_G$ by Theorem 2.7, we finally get

$$\alpha(G) \leq -\frac{1}{w_G}. \quad \square$$

3. Weighted Clique Polynomials and Homomorphisms

In this section we will discuss about one of the applications of the weighted clique polynomials for obtaining a no-homomorphism criteria. We first review some basics of graph homomorphism. The reader may consult the reference [5].

Definition 3.1. Let $G$ and $H$ be two simple graphs. A homomorphism of $G$ to $H$, written as $f : G \rightarrow H$ is a mapping $f : V(G) \rightarrow V(H)$ such that $f(u)f(v) \in E(H)$ whenever $uv \in E(G)$. A homomorphism of $G$ to $H$ is also called an $H$-coloring of $G$. we shall call a homomorphism $f : G \rightarrow H$ surjective, if the mapping $f : V(G) \rightarrow V(H)$ is surjective.

Definition 3.2. Let $G$ and $H$ be two simple graphs and $f : G \rightarrow H$ a homomorphism. We associate a partition $\theta_f$ with $f$ consisting of the preimages of $f$, i.e., the set $f^{-1}(x), x \in V(H)$. Clearly the set $S_x = f^{-1}(x)$ is an independent set, if there is no loop at vertex $x \in V(H)$. Thus, the mapping $\theta_f$ partitions the vertex set $V(G)$ into independent sets.

Remark 3.3. It is not hard to see that every weighted clique polynomial with non-negative integer weights can be viewed as the clique polynomial
with clusters of vertices. To do this, we need the following definition of blow-up graphs.

**Definition 3.4.** For a given graph $G = (V, E)$ with the non-negative integer weight function $w : V(G) \rightarrow \mathbb{R}$ and the vertex set $V = \{1, 2, \ldots, n\}$, the blow-up graph $G_b$ of $G$ is defined as a graph $G_b = (V_b, E_b)$ such that $V_b = \{A_1, \ldots, A_n\}$ where $A_i$ is the set of $w_i$ vertices with no edges among them, that we will call it a cluster of vertices ($|A_i| = w_i$). There is an edge $A_iA_j \in E_b$ if there is an edge between vertices $i, j \in V$.

**Remark 3.5.** Note that to obtain a blow-up graph $G_b$ form a graph $G = (V, E)$, we replace each vertex $i \in V$ with a cluster of vertices of size $w_i$ (blowing-up process) and then we replace each edge $e = ij \in E$ in $G$ with a complete bipartite graph $K_{w_i,w_j}$ with bipartition $(A_i, A_j)$ between two clusters of vertices of sizes $w_i$ and $w_j$ (see Fig 2). Now, using a simple counting argument based on inclusion - exclusion principle, one can show that

$$C(G, x; w) = C(G_b, x),$$

provided that the weights are non-negative integers.

**Example 3.6.** In the following picture we depicted a graph $G$ with the weight vector $w = (1, 2, 1, 3)$, and its blow-up graph $G_b$. Note that $G_b$ has four clusters of vertices of sizes 1, 2, 1 and 3. It is not hard to see that

$$C(G, x; w) = 1 + 7x + 8x^2 + 2x^3 = C(G_b, x).$$

![Figure 2](image_url) A graph $G$ and its blow-up graph $G_b$. 
Now we are at position to state the main result of this section.

**Theorem 3.7.** Let $G$ and $H$ be two simple graphs and $f : G \rightarrow H$ be a surjective homomorphism. Then, we have

$$\zeta_G \geq \zeta_H.$$  

**Proof.** Let $V(H) = \{1, 2, \ldots, m\}$. Set

$$w = (|f^{-1}(1)|, |f^{-1}(2)|, \ldots, |f^{-1}(m)|).$$

Since $f : G \rightarrow H$ is a homomorphism, the partition function $\theta_f$ partitions the vertex set $V(G)$ into independent sets $A_i, i = 1, \ldots, m$, with $A_i = f^{-1}(i)$. Now the blow-up graph $G_b$ of the graph $G$ with clusters of vertices $A_i$'s has the clique polynomial $C(G_b, x)$. By surjectivity of $f$, its clear that the blow-up graph $H_b$ of the graph $H$ is an inducted subgraph of $G_b$. Therefore, using Theorem 2.7, we get

$$\zeta_{H_b} \leq \zeta_{G_b},$$

which is equivalent to $\zeta_H \leq \zeta_G$, applying the identity (4). □

**Corollary 2.8.** Let $G$ and $H$ be two simple graphs such that $\zeta_G < \zeta_H$. Then, there is no surjective homomorphism from $G$ to $H$.

**References**


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