The Construction of Fraction Gamma Rings and Local Gamma Rings by Using Commutative Gamma Rings

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Abstract. One of the first constructions of algebra is the quotient field of a commutative integral domain, constructed as a set of fractions, which can lead to a very useful technique in commutative ring theory. In this article the researchers considered rings of fractions for gamma rings and some new characterizations were developed in gamma rings of fractions.

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1. Introduction

The notation of a gamma ring was first introduced by Nobusawa [10] as a generalization of a classical ring and afterward Barnes [2] improved the concepts of Nobusawa’s Γ-ring and developed the more general Γ-ring in which all classical rings were contained in this Γ-ring. We know, quotient fields are applied for making valuation rings and Dedekind domains and Dedekind domains are used in numbers theory [9].

In this paper the researchers constructed fraction Γ-ring and discussed their characteristics and relations by using local Γ-rings.

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Let $R$ and $\Gamma$ be two additive abelian groups and there exists a mapping $(x, \gamma, y) \mapsto x\gamma y$ of $R \times \Gamma \times R \rightarrow R$, which satisfies the conditions:

\begin{align*}
(i) \quad (x + y)\gamma z &= x\gamma z + y\gamma z, \quad x(\gamma_1 + \gamma_2)y = x\gamma_1 y + x\gamma_2 y, \\
(ii) \quad x\gamma(y + z) &= x\gamma y + x\gamma z,
\end{align*}

for all $x, y, z \in R$ and $\gamma, \gamma_1, \gamma_2 \in \Gamma$. Then $R$ is called a gamma ring.

If there exists $1_R \in R$ and $\gamma_0 \in \Gamma$ such that for all $r \in R$,

$1_R\gamma_0 r = r\gamma_0 1_R = r$, then $1 = 1_R$ is called identity element [13].

Let $R$ be a $\Gamma$-ring with $1$. An element $a \in R$ is called invertible if there exists $b \in R$ such that $a\gamma_0 b = b\gamma_0 a = 1$, also $b$ is unique and called the multiplicative inverse of $a$ and is denoted by $a^{-1}$.

An element $a \in R$ is said to be zero-divisor if there exists $b \neq 0$ such that $a\gamma_0 b = b\gamma_0 a = 0$.

Let $R$ be $\Gamma$-ring. If for all $a, b \in R$ and for all $\gamma \in \Gamma, a\gamma b = b\gamma a$, then $R$ is called commutative $\Gamma$-ring.

A subset $I$ of $\Gamma$-ring $R$ is said left(or right) gamma ideal if $I$ is an additive subgroup of $R$ and $R\Gamma I \subseteq I$ (or $\Gamma R I \subseteq I$) [11].

Let $R$ be a $\Gamma$-ring. The ideal generated by $a \in R$ is the intersection of all ideals contain $a$ and

$\langle a \rangle = \{ na + x_1 a_1 + \ldots + x_k a_k | n, k \in \mathbb{Z}, a, x_1, \ldots, x_k \in R, a_1, \ldots, a_k \in \Gamma \}$.

A $\Gamma$-ring homomorphism [6] is a mapping $f$ of $\Gamma$-ring $R$ to $\Gamma$-ring $R'$ such that:

\begin{align*}
(i) \quad f(x + y) &= f(x) + f(y), \quad (ii) \quad f(x\gamma y) &= f(x)\gamma f(y), \text{ for all } x, y \in R \text{ and } \gamma \in \Gamma.
\end{align*}

A multiplicatively closed subset of $\Gamma$-ring $R$ is a subset $S$ of $R$ such that $1 \in S \text{ and } s_1 \Gamma s_2 \subseteq S$, for all $s_1, s_2 \in S$.

Let $R$ be a $\Gamma$-ring with $1$ and $*: R \times \Gamma \times R \rightarrow R$ be a map on $R$ such that $(R - \{0\}, *)$ be a group. Then $R$ is called $\Gamma$-field.

We consider the following assumptions

\begin{align*}
(*) \quad x\alpha y\beta z &= x\beta y\alpha z, \text{ for all } x, y, z \in R \text{ and } \alpha, \beta \in \Gamma, \\
(\ast\ast) \quad (s_1 \alpha s_2)\gamma_0 (s_1 \alpha s_2)\gamma_0 (x\beta y) + (s_1 \beta s_2)\gamma_0 (s_1 \beta s_2)\gamma_0 (x\alpha y) &= 0,
\end{align*}

for all $x, y, z \in R, s_1, s_2 \in S, \alpha, \beta \in \Gamma$ [4].
2. Fractions of Gamma Rings

Throughout this section, the word gamma ring $R$ means a commutative gamma ring with 1 and without zero-divisor.

**Proposition 2.1.** If $a$ and $b$ are invertible in $R$, so is $a\gamma_0 b$ and

$$(a\gamma_0 b)^{-1} = b^{-1}\gamma_0 a^{-1}.$$ 

**Proof.**

$$(a\gamma_0 b)\gamma_0 (b^{-1}\gamma_0 a^{-1}) = a\gamma_0 (b\gamma_0 b^{-1})\gamma_0 a^{-1}$$

$$= a\gamma_0 1\gamma_0 a^{-1}$$

$$= (a\gamma_0 1)\gamma_0 a^{-1}$$

$$= a\gamma_0 a^{-1}$$

$$= 1,$$

and similarly $(b^{-1}\gamma_0 a^{-1})\gamma_0 (a\gamma_0 b) = 1$. □

**Proposition 2.2.** Let $R$ be a $\Gamma$-ring and $S = R – \{0\}$. We define relation $\sim$ on $R \times S$ as follows:

$$(a,s) \sim (b,t) \iff a\gamma_0 t - b\gamma_0 s = 0,$$

for $a, b \in R$ and $s, t \in S$. Then $\sim$ is an equivalence relation.

**Proof.** We show that the relation $\sim$ is reflexive, symmetric and transitive.

Since for all $a \in R$ and $s \in S, a\gamma_0 s - a\gamma_0 s = 0$, so $(a, s) \sim (a, s)$.

If $(a, s) \sim (b, t)$, then $a\gamma_0 t - b\gamma_0 s = 0$, and so $b\gamma_0 s - a\gamma_0 t = 0$. Thus $(b, t) \sim (a, s)$.

If $(a, s) \sim (b, t)$ and $(b, t) \sim (c, u)$, then we have

$$a\gamma_0 t - b\gamma_0 s = 0,$$

$$b\gamma_0 u - c\gamma_0 t = 0.$$  

(1)

On the other hand, a multiplication by $\gamma_0 u$ of (1) and $\gamma_0 s$ of (2) gives

$$a\gamma_0 t\gamma_0 u - b\gamma_0 s\gamma_0 u = 0,$$

$$b\gamma_0 u\gamma_0 s - c\gamma_0 t\gamma_0 s = 0.$$  

(3)

(4)

Sum of (3) and (4), we obtain

$$a\gamma_0 t\gamma_0 u - c\gamma_0 t\gamma_0 s = 0.$$  

(5)

By using commutativity, we have

$$(a\gamma_0 u - c\gamma_0 s)\gamma_0 t = 0.$$  

(6)
We have \( t \neq 0 \) and \( R \) is without zero-divisor, which gives \( a_0u - c_0s = 0 \) and thus \((a, s) \sim (c, u)\).

Hence, the proof is complete. \( \square \)

**Theorem 2.3.** Let \([a, s]\) denote the equivalence class of \((a, s)\), and \(S^{-1}R\) denote the set of equivalence classes. If \(R\) satisfies the conditions \((\ast)\) and \((\ast\ast)\), we define addition and multiplication of these fractions as follows:

\[
\begin{aligned}
S^{-1}R \times \Gamma \times S^{-1}R & \to S^{-1}R \\
[r, s] + [r', s'] & = [r_0s' + s_0r', s_0s'] \\
[r, s]γ[r', s'] & = [γr', sγs'],
\end{aligned}
\]

then

(i) these definitions are well-defined.

(ii) \(S^{-1}R\) is a \(\Gamma\)-ring with identity element \([1, 1]\).

**Proof.** (i): If \([r_1, s_1] = [r_1', s_1']\) and \([r_2, s_2] = [r_2', s_2']\), then we have

\[
\begin{aligned}
r_1γs_1 - s_1γr_1' & = 0, \\
r_2γs_2 - s_2γr_2' & = 0.
\end{aligned}
\]

A multiplication by \(s_2γs_2'\) of \((7)\) and \(s_1γs_1'\) of \((8)\) gives

\[
\begin{aligned}
r_1γs_1'γs_2γs_2' - s_1γr_1'γs_2γs_2' & = 0, \\
r_2γs_2'γs_1γs_1' - s_2γr_2'γs_1γs_1' & = 0.
\end{aligned}
\]

Sum of \((9)\) and \((10)\), we obtain

\[
r_1γs_1'γs_2γs_2' - s_1γr_1'γs_2γs_2' + r_2γs_2'γs_1γs_1' - s_2γr_2'γs_1γs_1' = 0.
\]

Since \(R\) is commutative \(\Gamma\)-ring, we have

\[
r_1γs_2γs_1'γs_2' + r_2γs_1γs_1'γs_2' - r_1'γs_2'γs_1γs_1' - s_1'γr_2'γs_1γs_2 = 0,
\]

therefore

\[
(r_1γs_2 + r_2γs_1)γs_1γs_2 - (r_1'γs_2' + s_1'γr_2')γs_1γs_2 = 0,
\]

\[
[r_1γs_2 + r_2γs_1, s_1γs_2] = [r_1'γs_2' + s_1'γr_2', s_1'γs_2'],
\]

\[
[r_1, s_1] + [r_2, s_2] = [r_1', s_1'] + [r_2', s_2'].
\]

\[
[r_1γs_2 + r_2γs_1, s_1γs_2] = [r_1'γs_2' + s_1'γr_2', s_1'γs_2']\]

\[
[r_1, s_1] + [r_2, s_2] = [r_1', s_1'] + [r_2', s_2']\]
Thus addition is well-defined.

Now, let \([r_1, s_1] = [r_2, s_2], [r'_1, s'_1] = [r'_2, s'_2]\) and \(\gamma = \gamma_1 = \gamma_2\), we have

\[
\begin{align*}
 r_1\gamma_0s_2 - s_1\gamma_0r_2 &= 0, \\
 r'_1\gamma_0s'_2 - s'_1\gamma_0r'_2 &= 0.
\end{align*}
\]  

(14)  

(15)

On the other hand, a multiplication by \(r'_1\gamma s'_2\) of (14) and \(r_2\gamma s_1\) of (15) gives

\[
\begin{align*}
 r_1\gamma_0s_2 r'_1\gamma s'_2 - r'_1\gamma s'_2 \gamma s_1\gamma_0 r_2 &= 0, \\
 r'_1\gamma_0 s'_2 r_2\gamma s_1 - s'_1\gamma_0 r'_2 \gamma r_2 s_1 &= 0.
\end{align*}
\]  

(16)  

(17)

By using the sum of (16) and (17) and applying the condition (\(\ast\)), we obtain

\[
 r_1\gamma r'_1\gamma_0 s_2 \gamma s'_2 - s_1\gamma s'_1 \gamma_0 r_2 r'_2 = 0,
\]

(18)

or

\[
[r_1\gamma r'_1, s_1\gamma s'_1] = [r_2 r'_2, s_2 \gamma s'_2],
\]

(19)

therefore \([r_1, s_1] [r'_1, s'_1] = [r_2, s_2] [r'_2, s'_2]\).

Thus the multiplication is well-defined.

**Proof (ii).** Since \(S\) is a multiplicatively closed subset of \(R\) and \(R\) is \(\Gamma -\) ring, therefore \(r\gamma r' \in R\) and \(s\gamma s' \in S\), for all \(r, r' \in R, s, s' \in S\) and \(\gamma \in \Gamma\). Thus \([r, s] [r', s'] = [r\gamma r', s\gamma s'] \in S^{-1}R\).

For \([r_1, s_1], [r_2, s_2], [r_3, s_3] \in S^{-1}R\) and \(\alpha \in \Gamma\), we have

\[
([r_1, s_1] + [r_2, s_2])\alpha [r_3, s_3] = [r_1, s_1]\alpha [r_3, s_3] + [r_2, s_2]\alpha [r_3, s_3],
\]

because

\[
([r_1, s_1] + [r_2, s_2])\alpha [r_3, s_3] = [r_1\gamma_0 s_2 + s_1\gamma_0 r_2, s_1\gamma_0 s_2]\alpha [r_3, s_3]
= [r_1\gamma_0 s_2 \alpha r_3 + s_1\gamma_0 r_2 \alpha r_3, s_1\gamma_0 s_2 \alpha s_3].
\]

Also

\[
[r_1, s_1]\alpha [r_3, s_3] + [r_2, s_2]\alpha [r_3, s_3] = [r_1\alpha r_3, s_1\alpha s_3] + [r_2 \alpha r_3, s_2 \alpha s_3]
= [r_1 \alpha r_3 \gamma_0 s_2 \alpha s_3 + s_1 \alpha s_3 \gamma_0 r_2 \alpha r_3, s_1 \alpha s_3 \gamma_0 s_2 \alpha s_3].
\]

It is easy to see that

\[
[r_1\gamma_0 s_2 \alpha r_3 + s_1\gamma_0 r_2 \alpha r_3, s_1\gamma_0 s_2 \alpha s_3] = [r_1 \alpha r_3 \gamma_0 s_2 \alpha s_3 + s_1 \alpha s_3 \gamma_0 r_2 \alpha r_3, s_1 \alpha s_3 \gamma_0 s_2 \alpha s_3]
\]


Now, we show that \([r_1, s_1](\alpha + \beta)[r_2, s_2] = [r_1, s_1]\alpha[r_2, s_2] + [r_1, s_1]\beta[r_2, s_2]\), we have
\[
[r_1, s_1](\alpha + \beta)[r_2, s_2] = [r_1(\alpha + \beta)r_2, s_1(\alpha + \beta)s_2] \\
= [r_1\alpha r_2 + r_1\beta r_2, s_1\alpha s_2 + s_1\beta s_2],
\]
\[
[r_1, s_1]\alpha[r_2, s_2] + [r_1, s_1]\beta[r_2, s_2] = [r_1\alpha r_2, s_1\alpha s_2] + [r_1\beta r_2, s_1\beta s_2] \\
= [(r_1\alpha r_2)\gamma_0(s_1\beta s_2) + (s_1\alpha s_2)\gamma_0(r_1\beta r_2, s_1\alpha s_2\gamma_0 s_1\beta s_2].
\]

We prove that
\[
[r_1\alpha r_2 + r_1\beta r_2, s_1\alpha s_2 + s_1\beta s_2] = [r_1\alpha r_2\gamma_0 s_1\beta s_2 + s_1\alpha s_2\gamma_0 r_1\beta r_2, s_1\alpha s_2\gamma_0 s_1\beta s_2],
\]
or
\[
r_1\alpha r_2\gamma_0 s_1\alpha s_2\gamma_0 s_1\beta s_2 + r_1\beta r_2\gamma_0 s_1\alpha s_2\gamma_0 s_1\beta s_2 - r_1\alpha r_2\gamma_0 s_1\beta s_2 s_1\alpha s_2 \\
- s_1\alpha s_2\gamma_0 r_1\beta r_2\gamma_0 s_1\alpha s_2 - r_1\alpha r_2\gamma_0 s_1\beta s_2 s_1\alpha s_2\gamma_0 r_1\beta r_2\gamma_0 s_1\beta s_2 = 0.
\]

But by using commutativity and the condition (**), the above relation is satisfied.

Also we have
\[
([r_1, s_1]\alpha[r_2, s_2])\beta[r_3, s_3] = [r_1\alpha r_2, s_1\alpha s_2]\beta[r_3, s_3] \\
= [(r_1\alpha r_2)\beta r_3, (s_1\alpha s_2)\beta s_3] \\
= [r_1\alpha(r_2\beta r_3), s_1\alpha(s_2\beta s_2)] \\
= [r_1, s_1]\alpha[r_2\beta r_3, s_2\beta s_3] \\
= [r_1, s_1]\alpha([r_2, s_2]\beta[r_3, s_3]).
\]

For all \([r, s] \in S^{-1}R\), we have
\[
[r, s]\gamma_0[1, 1] = [r\gamma_0 1, s\gamma_0 1] = [r, s],
\]
and similarly since \([1, 1]\gamma_0[r, s] = [r, s],\) thus \([1, 1] \in S^{-1}R\) is an identity element, the proof is complete. □

The \(\Gamma\) -ring \(S^{-1}R\) is called the \(\Gamma\)-ring of fraction of \(R\) with respect to \(S\).

**Proposition 2.4.** Let \(R, S\) be in Proposition 2.2. Then
(i) \([0, s] = [0, 1], \) for all \(s \in S\).
(ii) \([r, s] = [r\gamma r', s\gamma r'], \) for all \(r, r' \in R, s \in S\) and \(\gamma \in \Gamma\).
(iii) \((-x\alpha y) = x(-\alpha)y, \) for all \(x, y \in R, \) \(\alpha \in \Gamma\).
(iv) \([r, r] = [1, 1], \) for all \(r \in R\).
Proof. (i) Since \( 0\gamma_0 s - 1\gamma_0 0 = 0 \), so \( [0,1] = [0, s] \).

(ii) Since \( R \) is commutative \( \Gamma \)-ring, then \( r\gamma_0 s\gamma r' - s\gamma_0 r\gamma r' = 0 \) and therefore
\[
[r, s] = [r\gamma r', s\gamma r'].
\]

(iii) We have \( x(-\alpha)y + x(\alpha)y = x(-\alpha + \alpha)y = 0 \), thus \(-x\alpha y = x(-\alpha)y \).

(iv) Since \( r\gamma_0 1 - 1\gamma_0 r = r - r = 0 \), so \([r, r] = [1, 1]\), for all \( r \in R \). \( \square \)

**Theorem 2.5.** If \( R \) is a \( \Gamma \)-ring and \( S = R - \{0\} \), then \( S^{-1}R \) is a \( \Gamma \)-field.

**Proof.** By using Theorem 2.1, \( (S^{-1}R, +, \cdot) \) is a \( \Gamma \)-ring with identity element \([1, 1]\), thus for every \( r, s \in S \), we prove that \([r, s]^{-1} = [s, r]\). By using commutativity \( \Gamma \)-ring \( R \) and Proposition 2.3 \((iv)\), we have
\[
[r, s]\gamma_0 [s, r] = [r\gamma_0 s, s\gamma_0 r] = [r\gamma_0 s, r\gamma_0 s] = [1, 1].
\]

Similarly \([s, r]\gamma_0 [r, s] = [1, 1]\).

Now, we prove that \((S^{-1}R, \cdot)\) is associative. Since \( R \) is a \( \Gamma \)-ring, we have
\[
([r_1, s_1]\gamma_1 [r_2, s_2])\gamma_2 [r_3, s_3] = [r_1\gamma_1 r_2, s_1\gamma_1 s_2]\gamma_2 [r_3, s_3]
= [(r_1\gamma_1 r_2)\gamma_2 r_3, (s_1\gamma_1 s_2)\gamma_2 s_3]
= [r_1\gamma_1 (r_2\gamma_2 s_3), s_1\gamma_1 (s_2\gamma_2 s_3)]
= [r_1, s_1]\gamma_1 ([r_2, s_2]\gamma_2 [r_3, s_3]).
\]

At the end, we prove that \((S^{-1}R, +, \cdot)\) is commutative. Since \( R \) is a commutative \( \Gamma \)-ring, for every \( \gamma \in \Gamma, r_1, r_2 \in R, s_1, s_2 \in S \) we have
\[
[r_1, s_1]\gamma_1 [r_2, s_2] = [r_1\gamma r_2, s_1\gamma s_2]
= [r_2\gamma r_1, s_2\gamma s_1]
= [r_2, s_2]\gamma [r_1, s_1].
\]

Hence \((S^{-1}R, +, \cdot)\) is a \( \Gamma \)-field. \( \square \)

At the end of this section, we give an example of matrices that are not rings under addition and matrix multiplication, but we will make a gamma ring of them.

**Example 2.6.** Let \( \mathbb{Z} \) be integers rings and \( M_{m \times n}(\mathbb{Z}) \) be the set of all \( m \times n \) matrices with entries in \( \mathbb{Z} \). We consider
\[
R = \{[ x \ x ] \mid x \in \mathbb{Z} \} \subseteq M_{1 \times 2} \text{ and } \Gamma = \{\begin{bmatrix} n \\ o \end{bmatrix} \mid n \in \mathbb{Z} \} \subseteq M_{2 \times 1}.
\]

and we define
\[
\begin{cases}
. : R \times \Gamma \times R & \longrightarrow R \\
[x \ x \ \begin{bmatrix} n \\ o \end{bmatrix} \begin{bmatrix} y \ y \end{bmatrix} = \begin{bmatrix} nxy \ nxy \end{bmatrix}]
\end{cases}
\]
for all \[
\begin{bmatrix}
  x \\
  x
\end{bmatrix}, \begin{bmatrix}
  y \\
  y
\end{bmatrix}
\] in \(R\) and for all \[
\begin{bmatrix}
  n \\
  o
\end{bmatrix}
\] in \(\Gamma\).

It is easy to see that \(R\) is a \(\Gamma\)-ring. We show that \(R\) is integral domain with \(1_R = \begin{bmatrix}
  1 \\
  1
\end{bmatrix}\) and \(\gamma_0 = \begin{bmatrix}
  1 \\
  0
\end{bmatrix}\).

Hence, if we consider \(S = R - \{0\}\), then by using Theorem 2.2, \(S^{-1}R\) is a \(\Gamma\)-field.

**Proof.** For \[
\begin{bmatrix}
  x \\
  x
\end{bmatrix} \neq \begin{bmatrix}
  0 \\
  0
\end{bmatrix},
\] then \(x \neq 0\) and if \[
\begin{bmatrix}
  y \\
  y
\end{bmatrix} \in R,
\] we have \[
\begin{bmatrix}
  x \\
  x
\end{bmatrix}. \begin{bmatrix}
  1 \\
  o
\end{bmatrix}. \begin{bmatrix}
  y \\
  y
\end{bmatrix} = \begin{bmatrix}
  0 \\
  0
\end{bmatrix} \Rightarrow \begin{bmatrix}
  xy \\
  xy
\end{bmatrix} = \begin{bmatrix}
  0 \\
  0
\end{bmatrix}
\]
\[
\Rightarrow xy = 0
\]
\[
\Rightarrow y = 0,
\]
then \[
\begin{bmatrix}
  y \\
  y
\end{bmatrix} = \begin{bmatrix}
  0 \\
  0
\end{bmatrix}.
\]

Also for all \[
\begin{bmatrix}
  x \\
  x
\end{bmatrix} \in R
\] we have
\[
\begin{bmatrix}
  1 \\
  1
\end{bmatrix}. \begin{bmatrix}
  1 \\
  o
\end{bmatrix}. \begin{bmatrix}
  x \\
  x
\end{bmatrix} = 1. \begin{bmatrix}
  x \\
  x
\end{bmatrix} = \begin{bmatrix}
  x \\
  x
\end{bmatrix}
\] and
\[
\begin{bmatrix}
  x \\
  x
\end{bmatrix}. \begin{bmatrix}
  1 \\
  o
\end{bmatrix}. \begin{bmatrix}
  1 \\
  1
\end{bmatrix} = x. \begin{bmatrix}
  1 \\
  1
\end{bmatrix} = \begin{bmatrix}
  x \\
  x
\end{bmatrix},
\] hence \(R\) has identity element.

With simple calculations, we get the equivalence class of \([\begin{bmatrix}
  x \\
  x
\end{bmatrix}, \begin{bmatrix}
  y \\
  y
\end{bmatrix}]\) is \(\{[\begin{bmatrix}
  z \\
  z
\end{bmatrix}, \begin{bmatrix}
  t \\
  t
\end{bmatrix}] | xt = yz, x, z \in \mathbb{Z}, y, t \in \mathbb{Z} - \{0\}\} \).

\[\square\]

### 3. Homomorphisms of Gamma Rings

In this section, the notion homomorphism of gamma rings is defined and some theorems will be proved.

**Theorem 3.1.** Let \(R\) be a \(\Gamma\)-ring and \(S^{-1}R\) be a \(\Gamma\)-ring of fraction in Theorem 2.1. Then the mapping \(f : R \rightarrow S^{-1}R\) such that \(f(r) = [r, 1]\) is a \(\Gamma\)-ring homomorphism.

**Proof.** At first we show that \(f\) is well-defined. If \(r_1 = r_2\), then \(r_1 - r_2 = 0\). Since \(r_1 = r_1\gamma_01\) and \(r_2 = r_2\gamma_01\), then \(r_1\gamma_01 - r_2\gamma_01 = 0\) and so \([r_1, 1] = [r_2, 1]\).

Now we prove that \(f(r_1 + r_2) = f(r_1) + f(r_2)\) and \(f(r_1\gamma_2) = f(r_1)\gamma f(r_2)\), for all \(r_1, r_2 \in R\) and \(\gamma \in \Gamma\). We have
\[
f(r_1) + f(r_2) = [r_1, 1] + [r_2, 1] = [r_1\gamma_01 + r_2\gamma_01, 1\gamma_01] = [r_1 + r_2, 1] = f(r_1 + r_2)
\]
We have \( f(r_1)\gamma f(r_2) = [r_1, 1]\gamma [r_2, 1] = [r_1\gamma r_2, 1\gamma 1] \), but \( f(r_1\gamma r_2) = [r_1\gamma r_2, 1] \), we get that

\[
[r_1\gamma r_2, 1] = [r_1\gamma r_2, 1\gamma 1] \iff r_1\gamma r_2\gamma 01\gamma 1 - r_1\gamma r_2\gamma 01 = 0 \\
\iff r_1\gamma r_2\gamma 1 - r_1\gamma r_2 = 0.
\]

If we put \( \alpha = -\gamma, \beta = \gamma_0, x = r_1, y = r_2 \) and \( s_1 = s_2 = 1 \), in condition \((**)\), we have

\[
1(-\gamma)1\gamma_01(-\gamma)1\gamma_01\gamma r_1\gamma_01\gamma_01\gamma_01\gamma_01 = 0.
\]

By Proposition 2.3 \((iii)\), we obtain

\[
1\gamma_1r_1\gamma_0r_2 - r_1\gamma r_2 = 0.
\]

Also by the condition \((*)\), we have

\[
1\gamma_1\gamma_0r_1\gamma r_2 - r_1\gamma r_2 = 0.
\]

Since \( 1\gamma_0r_1 = r_1 \), we get that

\[
1\gamma r_1\gamma r_2 - r_1\gamma r_2 = 0.
\]

Hence the theorem is proved.

**Proposition 3.2.** Let \( R \) and \( R' \) be \( \Gamma \)-rings with identity elements and \( f : R \rightarrow R' \) be a \( \Gamma \)-ring epimorphism. Then \( f(1_R) = 1_{R'} \).

**Proof.** We prove that \( f(1_R)\gamma_0r' = r'\gamma_0f(1_R) = r' \), for all \( r' \in R' \). Since \( f \) is surjective, there exists \( r \in R \) such that \( f(r) = r' \). We have

\[
f(1_R)\gamma_0r' = f(1_R)\gamma_0f(r) = f(1_R\gamma_0r) = f(r) = r',
\]

\[
r'\gamma_0f(1_R) = f(r)\gamma_0f(1_R) = f(r\gamma_01_R) = f(r) = r'.
\]

Hence \( f(1_R) = 1_{R'} \).

**Proposition 3.3.** If \( R \) and \( R' \) are \( \Gamma \)-rings with identity elements without zero-divisor and \( f : R \rightarrow R' \) is a non-zero \( \Gamma \)-ring homomorphism. Then \( f(1_R) = 1_{R'} \).

**Proof.** We have

\[
f(1_R) = f(1_R\gamma_01_R) = f(1_R)\gamma_0f(1_R)
\]

\[
\Rightarrow f(1_R) - f(1_R)\gamma_0f(1_R) = 0
\]

\[
\Rightarrow f(1_R)\gamma_0(1_{R'} - f(1_R)) = 0
\]

\[
\Rightarrow 1_{R'} - f(1_R) = 0.
\]
Hence $f(1_R) = 1_{R'}$. □

**Proposition 3.4.** Let $R$ and $R'$ be $\Gamma$-rings with identity elements, without zero-divisor and $f : R \rightarrow R'$ is a non-zero $\Gamma$-ring homomorphism. Then $f(a^{-1}) = (f(a))^{-1}$.

**Proof.** Suppose $a \in R$ is invertible and $a^{-1}$ is inverse of $a$. We have

\[
f(a\gamma a^{-1}) = f(1_R) = f(a^{-1}\gamma a) \\
\Rightarrow f(a)\gamma f(a^{-1}) = 1_{R'} = f(a^{-1})\gamma f(a) \\
\Rightarrow f(a^{-1}) = (f(a))^{-1}.
\]

4. Local Gamma Rings

In this section, local gamma rings is defined and will be given several conditions equivalent for local gamma rings.

**Definition 4.1.** A $\Gamma$-ideal $P$ in $\Gamma$-ring $R$ is prime [15], if $P \neq R$ and if $A\Gamma B \subseteq P$, then $A \subseteq P$ or $B \subseteq P$, for every $\Gamma$-ideals $A$ and $B$ in $R$.

**Theorem 4.2.** If $R$ is a commutative $\Gamma$- ring and $P$ is a $\Gamma$-ideal such that $P \neq R$ and $a\gamma b \in P$, for $a, b \in R$ and all $\gamma \in \Gamma$ it implies that $a \in P$ or $b \in P$, then $P$ is prime and conversely.

**Proof.** $\Rightarrow$ If $A$ and $B$ are gamma ideals in $R$ such that $A\Gamma B \subseteq P$, but $A \nsubseteq P$ and $B \nsubseteq P$, then there are $a_0 \in A$ and $b_0 \in B$ such that $a_0$ and $b_0$ are not in $P$.

Since $A\Gamma B \subseteq P$, then for every $\gamma \in \Gamma$, $a_0\gamma b_0 \in A\Gamma B \subseteq P$ and by assumption $a_0 \in P$ or $b_0 \in P$, this is a contradiction. Thus $A \subseteq P$ or $B \subseteq P$.

$\Leftarrow$ Let $P$ be a prime gamma ideal and $a\gamma b \in P$ for every $a, b \in R$ and for all $\gamma \in \Gamma$, then $\langle a \rangle \Gamma \langle b \rangle \subseteq P$ and therefore $\langle a \rangle \subseteq P$ or $\langle b \rangle \subseteq P$, but $a \in \langle a \rangle$ and $b \in \langle b \rangle$ thus $a \in P$ or $b \in P$. □

**Theorem 4.3.** In a commutative $\Gamma$-ring $R$ with identity, an ideal $P$ is prime if and only if $S = R - P$ is multiplicatively closed subset.

**Proof.** $\Rightarrow$ Let gamma ideal $P$ be prime in $R$ and $s_1, s_2 \in S$. Then $s_1$ and $s_2$ aren’t in $P$ ($S = R - P$). Since $P$ is prime, for every $\gamma \in \Gamma$, $s_1\gamma s_2$ isn’t in $P$, hence for all $\gamma \in \Gamma$, $s_1\gamma s_2 \in S$, therefore $s_1\Gamma s_2 \subseteq S$.

$\Leftarrow$ Suppose $S = R - P$ is a multiplicatively closed subset in $R$, then $1 \in S$ and so $S \neq \emptyset$, i.e $P \neq R$. □
If $a\gamma b \in P$ for every $a, b \in R$ and for every $\gamma \in \Gamma$, then $a\gamma b$ isn’t in $S$. Since $S$ is multiplicatively closed subset, then $a$ or $b$ aren’t in $S$, i.e $a \in P$ or $b \in P$. □

**Notation**

Let gamma ideal $P$ be prime in $R$ and $S = R - P$. Then we write $A_{\Gamma P} = S^{-1}R$.

**Theorem 4.4.** In a commutative $\Gamma$-ring $R$ with identity, if gamma ideal $P$ is prime and $S = R - P$, then the set $M = \{[a, s] \mid a \in P, s \in S\}$ is an ideal of $A_{\Gamma P}$.

**Proof.** Since $0 \in P$, then $[0, s] \in M$, for $s \in S$ and so $M \neq \emptyset$. To show that $(M, +)$ is subgroup, for every $a, b \in P$ and $s, s' \in S$, we have $a\gamma_0 s'$ and $b\gamma_0 s \in P$ ($P$ is an $\Gamma$-ideal) and $s\gamma_0 s' \in S$ ($S$ is a multiplicatively closed subset).
Thus $[a, s] - [b, s'] = [a, s] + [-b, s'] = [a\gamma_0 s - b\gamma_0 s, s\gamma_0 s'] \in M$.
To show that $MTA_{\Gamma P} \subseteq M$, we consider $[a, s]\gamma [b, s'] \in MTA_{\Gamma P}$.
Since $P$ is an ideal in $R$ and $S$ is a multiplicatively closed subset, then $a\gamma b \in P$ and $s\gamma s' \in S$. Thus $[a, s]\gamma [b, s'] = [a\gamma b, s\gamma s'] \in M$, i.e $MTA_{\Gamma P} \subseteq M$. □

**Theorem 4.5.** Let $M$ be the set of all non-invertible elements of $\Gamma$-ring $R$, then the following properties are equivalent:
(1) $M$ is additively closed ($\forall a_1, a_2 \in M, a_1 + a_2 \in M$),
(2) $M$ is a two-sided gamma ideal of $R$,
(3r) $M$ is the largest proper right gamma ideal,
(3l) $M$ is the largest proper left gamma ideal,
(4r) In gamma ring $R$ there exists a largest proper right ideal,
(4l) In gamma ring $R$ there exists a largest proper left ideal,
(5r) For every $r \in R$ either $r$ or $1 - r$ is right invertible,
(5l) For every $r \in R$ either $r$ or $1 - r$ is left invertible,
(6) For every $r \in R$ either $r$ or $1 - r$ is invertible.

**Proof.** (1) ⇒ (2): Let $M$ be additively closed. At first, we show that every right (left) invertible element is invertible. If $b \in R$ is right invertible, then there exists $b' \in R$ such that $b\gamma_0 b' = 1$, to show that $b'\gamma_0 b = 1$, we have two cases.

**Case 1.** If $b'\gamma_0 b$ isn’t in $M$, then there is $s \in R$ with $s\gamma_0 (b'\gamma_0 b) = 1$. A right multiplication by $\gamma_0 b'$ gives
$s\gamma_0 b' \gamma_0 b = 1\gamma_0 b' \Rightarrow s\gamma_0 b' \gamma_0 1 = b' \Rightarrow s\gamma_0 b' = b' \Rightarrow b'\gamma_0 b = 1$. 

(1) ⇒ (3r): Let $M$ be additively closed. At first, we show that every right invertible element is invertible. If $b \in R$ is right invertible, then there exists $b' \in R$ such that $b\gamma_0 b' = 1$, to show that $b'\gamma_0 b = 1$, we have two cases.

**Case 1.** If $b'\gamma_0 b$ isn’t in $M$, then there is $s \in R$ with $s\gamma_0 (b'\gamma_0 b) = 1$. A right multiplication by $\gamma_0 b'$ gives
$s\gamma_0 b' \gamma_0 b = 1\gamma_0 b' \Rightarrow s\gamma_0 b' \gamma_0 1 = b' \Rightarrow s\gamma_0 b' = b' \Rightarrow b'\gamma_0 b = 1$. 

(1) ⇒ (4r): Let $M$ be additively closed. At first, we show that every right invertible element is invertible. If $b \in R$ is right invertible, then there exists $b' \in R$ such that $b\gamma_0 b' = 1$, to show that $b'\gamma_0 b = 1$, we have two cases.

**Case 1.** If $b'\gamma_0 b$ isn’t in $M$, then there is $s \in R$ with $s\gamma_0 (b'\gamma_0 b) = 1$. A right multiplication by $\gamma_0 b'$ gives
$s\gamma_0 b' \gamma_0 b = 1\gamma_0 b' \Rightarrow s\gamma_0 b' \gamma_0 1 = b' \Rightarrow s\gamma_0 b' = b' \Rightarrow b'\gamma_0 b = 1$. 

(1) ⇒ (5r): Let $M$ be additively closed. At first, we show that every right invertible element is invertible. If $b \in R$ is right invertible, then there exists $b' \in R$ such that $b\gamma_0 b' = 1$, to show that $b'\gamma_0 b = 1$, we have two cases.

**Case 1.** If $b'\gamma_0 b$ isn’t in $M$, then there is $s \in R$ with $s\gamma_0 (b'\gamma_0 b) = 1$. A right multiplication by $\gamma_0 b'$ gives
$s\gamma_0 b' \gamma_0 b = 1\gamma_0 b' \Rightarrow s\gamma_0 b' \gamma_0 1 = b' \Rightarrow s\gamma_0 b' = b' \Rightarrow b'\gamma_0 b = 1$. 

(1) ⇒ (6): Let $M$ be additively closed. At first, we show that every right invertible element is invertible. If $b \in R$ is right invertible, then there exists $b' \in R$ such that $b\gamma_0 b' = 1$, to show that $b'\gamma_0 b = 1$, we have two cases.
Case 2. If $b'\gamma_0b \in M$, then $1 - b'\gamma_0b$ isn’t in $M$, otherwise if $1 - b'\gamma_0b \in M$, we have $b'\gamma_0b \in M$ and $M$ is an additively closed set, then

$$1 = (1 - b'\gamma_0b) + b'\gamma_0b \in M.$$ 

It is a contradiction.

Thus there exists $s \in R$ such that $s\gamma_0(1 - b'\gamma_0b) = 1$. The right multiplication by $\gamma_0b'$ gives

$$s\gamma_0(1 - b'\gamma_0b)\gamma_0b' = 1\gamma_0b' \implies s\gamma_0(1\gamma_0b' - b'\gamma_0b\gamma_0b') = b'$$

$$\implies s\gamma_0(b' - b'\gamma_01) = b'$$

$$\implies s\gamma_0(b' - b') = b'$$

$$\implies 0 = b',$$

it is contradiction to $b\gamma_0b' = 1$. Hence by using case 1, $b$ is invertible.

Now, we prove that for every $m \in M$, $r \in R$ and $\gamma \in \Gamma$, $r\gamma m \in M$ and $m\gamma r \in M$.

Suppose $r\gamma m$ is not in $M$, then there exists $s \in R$ such that $r\gamma m\gamma_0 s = 1$. By using case 1, $s\gamma_0 r\gamma m = 1$ and by the contradiction (*), $s\gamma r\gamma_0 m = 1$. Thus $s\gamma r$ is inverse of $m$, in contradiction with $m \in M$. Hence $r\gamma m \in M$ and similarly $m\gamma r \in M$.

Let $\sum_{i=1}^{n} r_i\gamma_i m_i \in R\Gamma M$. Since $r_i\gamma_i m_i \in M$, for every $1 \leq i \leq n$ and $M$ is an additively closed set, then $\sum_{i=1}^{n} r_i\gamma_i m_i \in M$, i.e $R\Gamma M \subseteq M$ and similarly $M\Gamma R \subseteq M$. Hence $M$ is two-sided gamma ideal of $R$.

(2) $\implies$ (3r): Let $M$ be two-sided gamma ideal in $R$. Then $M$ is right gamma ideal. Since 1 isn’t in $M$, then $M \not\subseteq R$.

Let $B$ be proper right gamma ideal in $R$. We show that $B \subseteq M$. If $b \in B$, then $b\Gamma R$ is right gamma ideal of $B$ and therefore $b\Gamma R$ is a proper right gamma ideal in $R$. Thus $b$ isn’t invertible and hence $b \in M$, i.e $B \subseteq M$.

(3r) $\implies$ (4r): It is clearly that $M$ is a largest proper right gamma ideal.

(4r) $\implies$ (5r). Let $N$ be the largest proper right ideal. Let $r \in R$ and $r$ and $1 - r$ aren’t invertible. Then $r\Gamma R$ and $(1 - r)\Gamma R$ are proper gamma ideals of $R$, hence $r\Gamma R \subseteq N$ and $(1 - r)\Gamma R \subseteq N$.

We have $1 = (1 - r)\gamma_01 + r\gamma_01 \in (1 - r)\Gamma R + r\Gamma R \subseteq N$, i.e $1 \in N$, in contradiction with $N \not\subseteq R$.

(5r) $\implies$ (6): It suffices to show that every right invertible element is invertible. Let $b$ has right inverse like $b'$. Then $b\gamma_0 b' = 1$.

Let $b'\gamma_0b \in R$. We have two cases:

Case 1. $b'\gamma_0b$ is right invertible, hence there is $s \in R$ such that $b'\gamma_0b\gamma_0s = 1$. 

The left multiplication by \((b\gamma_0)\) gives

\[
b\gamma_0 b' \gamma_0 b \gamma_0 s = b\gamma_0 1 \implies 1\gamma_0 b\gamma_0 s = b
\]
\[
\implies b\gamma_0 s = b
\]
\[
\implies b' \gamma_0 b = 1.
\]

Case 2. \((1 - b' \gamma_0 b)\) is right invertible, hence there is \(s \in R\) with \((1 - b' \gamma_0 b)\gamma_0 s = 1\). The left multiplication by \((b\gamma_0)\) gives

\[
b\gamma_0 (1 - b' \gamma_0 b) \gamma_0 s = b\gamma_0 1 \implies (b\gamma_0 1 - b\gamma_0 b' \gamma_0 b) \gamma_0 s = b
\]
\[
\implies (b - 1\gamma_0 b) \gamma_0 s = b
\]
\[
\implies (b - b) \gamma_0 s = b
\]
\[
\implies 0 = b.
\]

It is in contradiction to \(b\gamma_0 b' = 1\). Hence by using case 1, \(b' \gamma_0 b = 1\).

(6) \implies (1). Suppose \(m_1, m_2 \in M\), we show that \(m_1 + m_2 \in M\).
If \(m_1 + m_2\) isn’t in \(M\), then \(m_1 + m_2\) is invertible, so there is \(s \in R\) with \((m_1 + m_2)\gamma_0 s = 1\) thus \(m_1\gamma_0 s = 1 - m_2\gamma_0 s\).
But \(m_1\gamma_0 s \in M\) must be held, otherwise \(m_1\gamma_0 s\) is invertible, i.e there is \(b \in R\) such that \(m_1\gamma_0 s\gamma_0 b = 1\) and then \(s\gamma_0 b\) is right inverse of \(m_1\). Since (6) \implies (5r) holds, we can use the fact that every right invertible element is invertible. Hence \(m_1\) isn’t in \(M\), there is contradiction.
Similarly it is proved that \(m_2\gamma_0 s \in M\) and by using (6), \((1 - m_2\gamma_0 s)\) is invertible and therefore \(m_1\gamma_0 s\) is invertible, in contradiction with \(m_1\gamma_0 s \in M\).

**Definition 4.6.** A gamma ring \(R\) which satisfies the equivalent properties of Theorem 4.4 is called local gamma ring.

**Corollary 4.7.** \(A_{\Gamma P}\) is a local \(\Gamma\)-ring.

**Proof.** It follows from Theorem 4.3 and Theorem 4.4. \(\square\)

**References**


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