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On 2-Absorbing Semiprimary Submodules

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Abstract. In this paper, we introduce the concept of 2-absorbing semiprimary submodules in modules over a commutative ring with nonzero identity which is a generalization of 2-absorbing primary submodule. Let N be a proper submodule of an R-module M. Then N is said to be a 2-absorbing semiprimary submodule of M if whenever $a_1a_2 \in R, m \in M$ and $a_1a_2m \in N$, then $a_1a_2 \in \sqrt{(N :_R M)}$ or $a_1m \in N$ or $a_2^n m \in N$, for some positive integer n. We have given an example and proved number of results concerning 2-absorbing semiprimary submodules.

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1. Introduction

Throughout this paper, we assume that all rings are commutative with $1 \neq 0$. Let R be a commutative ring and M be an R-module. We will denote by $(N :_R M)$ the residual of N by M, that is, the set of all $r \in R$ such that $rM \subseteq N$. In 1974, Fuchs [10] introduced the concept of primary ideals of rings. He defined a primary ideal P of R with identity to be a proper ideal of ring R and if $a, b \in R$ and $ab \in P$, then $a \in P$ or

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$b^n \in P$, for some positive integer n. The concept of 2-absorbing ideals, which is a generalization of prime ideal, was introduced by Badawi in [1]. He defined a 2-absorbing ideal P of a commutative ring R with identity to be a proper ideal of R and if $a, b, c \in R$ such that $abc \in P$, then $ab \in P$ or $bc \in P$ or $ac \in P$. In 2007, Badawi et al. [2] introduced the concept of 2-absorbing primary ideals of commutative rings with identity, which is a generalization of primary ideals, and investigated some properties. Recall that a proper ideal P of R is called a 2-absorbing primary ideal of R if whenever $a, b, c \in R$ and $abc \in P$, then $ab \in P$ or $ac \in \sqrt{P}$ or $bc \in \sqrt{P}$.

The concept of prime submodule was introduced and studied by Feller and Swokowski [9]. We recall that a proper submodule N of M is called a prime submodule, if $rm \in N$, where $r \in R, m \in M$, then $m \in N$ or $r \in (N :_R M)$. The idea of decomposition of submodules into classical prime submodules were introduced by Behboodi in [4, 5]. He defined a classical prime submodule N of M to be a proper submodule of M and if $a, b \in R, m \in M$ and $abm \in N$, then $am \in N$ or $bm \in N$. The concept of classical primary submodule, which is a generalization of classical prime submodule, was introduced by Baziar and Behboodi in [3]. Recall from [3] that a proper submodule N of M is said to be a classical primary submodule of M if whenever $a, b \in R$ and $m \in M$ with $abm \in N$, then $am \in N$ or $b^n m \in N$, for some positive integer n. In 2011, Darani and Soheilnia [6] introduced the concept of 2-absorbing submodules of modules over commutative rings with identities. Recall that a proper submodule N of M is called a 2-absorbing submodule of M as in [6] if whenever $abm \in N$ for some $a, b \in R$ and $m \in M$, then $ab \in (N :_R M)$ or $am \in N$ or $bm \in N$. The concept of 2-absorbing primary submodules, a generalization of primary submodules was introduced and investigated in [7]. A proper submodule N of M is called a 2-absorbing primary submodule of M if whenever $abm \in N$ for some $a, b \in R$ and $m \in M$, then $am \in N$ or $bm \in N$ or $ab \in \sqrt{(N :_R M)}$. The paper is organized as follows. In the last section, we introduce the notion of 2-absorbing semiprimary submodules and give some characterizations of the same. The properties of 2-absorbing semiprimary submodule are also studied in some specific domain.

2. Preliminaries

In this section we refer to [1, 10, 8, 11, 12, 13] for some elementary aspects and quote few definitions and examples which are essential to step up this study. For more details we refer to the papers in the references.

Definition 2.1. *[10] A proper ideal P of R is said to be primary if whenever $a_1, a_2 \in R$ and $a_1a_2 \in I$, then $a_1 \in I$ or $a_2^n \in P$, for some positive integer n.*

Definition 2.2. *[2] A proper ideal I of R is said to be 2-absorbing primary if whenever $a_1, a_2, a_3 \in R$ and $a_1a_2a_3 \in I$, then $a_1a_2 \in I$ or $a_1a_3 \in \sqrt{I}$ or $a_2a_3 \in \sqrt{I}$.*

Remark 2.3. *It is easy to see that every primary ideal is 2-absorbing primary.*

Definition 2.4. *[8] Let N be a submodule of an R-module M. The residual of N by M, denoted $(N :_R M)$, is the ideal*

$$(N :_R M) = \{r \in R : rM \subseteq N\}.$$

If $m \in M$, then the ideal $(N :_R m)$ is defined by $(N :_R m) = \{r \in R : rm \in N\}$.

Remark 2.5. *It is clear that if N is a submodule of an R-module M, then $(N :_R M)$ $((N :_R m))$ is an ideal of R.*

Definition 2.6. *[8] Let M be an R-module, I be an ideal of R and N be a submodule of M. The residual of N by I, denoted by $(N :_M I)$, is the submodule $(N :_M I) = \{m \in M : Im \subseteq N\}$. If I consists of one element, say a, then $(N :_M a) = \{m \in M : am \in N\}$.*

Lemma 2.7. *[8] Let K, L and N be submodules of an R-module M and let A and B be ideals of R. Then*

1. *$(L \cap N :_R M) = (L :_R M) \cap (N :_R M)$.*
2. *If $A \subseteq B$, then $(N :_M B) \subseteq (N :_M A)$.*
3. *$((N :_M A) :_M B) = (N :_M AB)$.*
4. *$(L \cap N :_M A) = (L :_M A) \cap (N :_M A)$.*

5. *If $L \subseteq N$, then $(L :_M A) \subseteq (N :_M A)$ and $(N :_M K)$.*

Proof. The proof is available in [8]. $\square$

Definition 2.8. *[8] Let M be an R-module. A proper submodule N of M is said to be irreducible if N is not the intersection of two submodules of M that properly contain it.*

Definition 2.9. *[8] A nonempty subset S of a ring R is said to be multiplicatively closed if $1 \in S$ and $ab \in S$, whenever $a, b \in S$.*

Definition 2.10. *[11] Let R be a ring and M be an R-module. The set of zero divisors of M, denoted by $Zd(M)$ is defined by*

$$Zd(M) = \{r \in R : rm = 0, \exists m \in M - \{0\}\}.$$

Lemma 2.11. *[8] Let R be a ring and M be an R-module. Let S be a multiplicatively closed set in R. Let T be the set of all pairs (x, s), where $x \in M$ and $s \in S$. Define a relation on T by $(x, s) \sim (\acute{x}, \acute{s})$, if and only if there exists $t \in S$ such that $t(s\acute{x} - \acute{s}x) = 0$. Then $\sim$ is an equivalence relation on T.*

Proof. The proof is available in [8]. $\square$

Definition 2.12. *[8] For $(x, s) \in T$ which defined in lemma 2.11, denote the equivalence classes of $\sim$ which contains (x, s) by $\frac{x}{s}$. Let $S^{-1}M$ denote the set of all equivalence classes of T with respect to this relation. We can make $S^{-1}M$ into an R-module by setting $\frac{x}{s} + \frac{y}{t} = \frac{tx+sy}{st}$ and $a(\frac{x}{s}) = \frac{ax}{s}$, where $x, y \in M$ and $t, s, a \in S$. The module $S^{-1}M$ is called the module of fractions of M with respect to S.*

Remark 2.13. *Let R be a ring, T be a multiplicatively closed subset of R, M be an R-module. We know that every submodule of $S^{-1}M$ is of the form $S^{-1}N$ for some submodule N of M [12].*

Note [13] Let N be a submodule of R-module M, we define $N(S) = \{m \in M : sm \in N, \exists s \in S\}$. Then $N(S)$ is a submodule of M containing N and $S^{-1}(N(S)) = S^{-1}N$.

Definition 2.14. *[8] Since R may be considered as an R-module we can form the quotient ring $S^{-1}R$. An element of $S^{-1}R$ has the form*

$\frac{a}{s}$, *where* $a \in R$ *and* $s \in S$. *We can make* $S^{-1}R$ *into a ring by setting* $\left(\frac{a}{s}\right)\left(\frac{b}{t}\right) = \frac{ab}{st}$, *where* $a, b \in R$ *and* $s, t \in S$. *The ring* $S^{-1}R$ *is called the ring of fractions of* R *with respect to* S.

3. Properties of 2-Absorbing Semiprimary Submodules

The results of the following theorems seem to play an important role to study 2-absorbing semiprimary submodules of modules over commutative rings; these facts will be used frequently and normally we shall make no reference to this definition.

Remark 3.1. *It is easy to see that every* 2*-absorbing primary submodule is* 2*-absorbing semiprimary.*

The following example shows that the converse of Remark 3.1 is not true.

Example 3.2. Let $R = \mathbf{Z}$ and $M = \mathbf{Z}$. Consider the submodule $N = 2^2 \cdot 3\mathbf{Z}$ of M. It is easy to see that N is a 2-absorbing semiprimary submodule of M. Notice that $2 \cdot 2 \cdot 3 \in N$, but $2 \cdot 3 \notin N$ and $(2 \cdot 2)^n \notin (N :_R M)$, for all positive integer n. Therefore N is not a 2-absorbing primary submodule of M.

Theorem 3.3. *Let* N *be a proper submodule of an* R*-module* M.

1. *If* N *is a* 2*-absorbing semiprimary submodule of* M, *then* $(N :_R m)$ *is a* 2*-absorbing primary ideal of* R, *for every* $m \in M - N$.

2. *For every* $m \in M - N$, *if* $(N :_R m)$ *is a primary ideal of* R, *then* N *is a* 2*-absorbing semiprimary submodule of* M.

Proof. 1. Suppose that N is a 2-absorbing semiprimary submodule of M. Let $a_1, a_2, a_3 \in R$ such that $a_1a_2a_3 \in (N :_R m)$. Clearly, $a_1a_3(a_2m) \in N$. By hypothesis, $a_1a_3 \in \sqrt{(N :_R M)} \subseteq \sqrt{(N :_R m)}$ or $a_1a_2m \in N$ or $a_3^n a_2 m \in N$ for some positive integer n. Therefore $a_1a_2 \in (N :_R m)$ or $a_2a_3 \in \sqrt{(N :_R m)}$ or $a_1a_3 \in \sqrt{(N :_R m)}$. Hence $(N :_R m)$ is a 2-absorbing primary ideal of R.

2. Assume that $(N :_R m)$ is a primary ideal of R for every $m \in M - N$. Let $a_1, a_2 \in R$ such that $a_1a_2m \in N$. Then $a_1a_2 \in (N :_R m)$. Thus $a_1 \in (N :_R m)$ or $a_2^n \in (N :_R m)$ for some positive integer n. Therefore $a_1m \in N$ or $a_2^n m \in N$ for some positive integer n. Hence N is a 2-absorbing semiprimary submodule of M. $\square$

Theorem 3.4. *Let N be a proper submodule of an R-module M. If N is a 2-absorbing semiprimary submodule of M, then $(N :_M r)$ is a 2-absorbing semiprimary submodule of M containing N for every $r \in R - (N :_R M)$.*

Proof. Suppose that N is a 2-absorbing semiprimary submodule of M. Let $a_1, a_2 \in R$ and $m \in M$ such that $a_1a_2m \in (N :_M r)$. Then $a_1a_2(rm) = ra_1a_2m \in N$. By hypothesis, $a_1a_2 \in \sqrt{(N :_R M)}$ or $a_1rm \in N$ or $a_2^n rm \in N$ for some positive integer n. Therefore $a_1a_2 \in \sqrt{(N :_R M)}$ or $a_1m \in (N :_M r)$ or $a_2^n m \in (N :_M r)$ for some positive integer n. Hence $(N :_M r)$ is a 2-absorbing semiprimary submodule of M. $\square$

Theorem 3.5. *Let $M, \acute{M}$ be two R-modules and $f : M \to \acute{M}$ be a homomorphism of R-modules.*

1. *If $\acute{N}$ is a 2-absorbing semiprimary submodule of $\acute{M}$, then $f^{-1}(\acute{N})$ is a 2-absorbing semiprimary submodule of M.*

2. *Let f be surjectivity. If N is a 2-absorbing semiprimary submodule of M, then $f(N)$ is a 2-absorbing semiprimary submodule of $\acute{M}$.*

Proof. 1. Let $a_1, a_2 \in R$ and $m \in M$ such that $a_1a_2m \in f^{-1}(\acute{N})$. Since f is a homomorphism, we have $a_1a_2f(m) = f(a_1a_2m) \in \acute{N}$. By hypothesis, $a_1a_2 \in \sqrt{(\acute{N} :_R \acute{M})}$ or $a_1f(m) \in \acute{N}$ or $a_2^n f(m) \in \acute{N}$ for some positive integer n. Therefore $a_1a_2 \in \sqrt{(f^{-1}(\acute{N}) :_R M)}$ or $a_1m \in f^{-1}(\acute{N})$ or $a_2^n m \in f^{-1}(\acute{N})$ for some positive integer n. Hence $f^{-1}(\acute{N})$ is a 2-absorbing semiprimary submodule of M.

2. Let $a_1, a_2 \in R$ and $\acute{m} \in \acute{M}$ such that $a_1a_2\acute{m} \in f(N)$. Since f is homomorphism, there exists $m \in M$ such that $\acute{m} = f(m)$. Therefore $f(a_1a_2m) = a_1a_2f(m) \in f(N)$ and so $a_1a_2m \in N$. By hypothesis,

$a_1a_2 \in \sqrt{(N :_R M)}$ or $a_1m \in N$ or $a_2^n m \in N$ for some positive integer n. Thus $a_1a_2 \in \sqrt{(f(N) :_R \acute{M})}$ or $a_1\acute{m} \in f(N)$ or $a_2^n\acute{m} \in f(N)$ for some positive integer n. Hence $f(N)$ is a 2-absorbing semiprimary submodule of $\acute{M}$. $\square$

Theorem 3.6. *Let M be an R-module and $N \subseteq K$ be two submodules of M. Then K is a 2-absorbing semiprimary submodule of M if and only if K/N is a 2-absorbing semiprimary submodule of M/N.*

Proof. Suppose that K is a 2-absorbing semiprimary submodule of M. Let $a_1, a_2 \in R$ and $m \in M$ such that $a_1a_2(m+N) \in (K/N)$. Then $a_1a_2m \in K$. By hypothesis, $a_1a_2 \in \sqrt{(K :_R M)}$ or $a_1m \in K$ or $a_2^n m \in K$ for some positive integer n. Therefore $a_1a_2 \in \sqrt{(K/N :_R M/N)}$ or $a_1(m+N) \in K/N$ or $a_2^n(m+N) \in K/N$ for some positive integer n. Hence K/N is a 2-absorbing semiprimary submodule of M/N.

Conversely, assume that K/N is a 2-absorbing semiprimary submodule of M/N. Let $a_1, a_2 \in R$ and $m \in M$ such that $a_1a_2m \in K$. Then $a_1a_2(m+N) = a_1a_2m+N \in K/N$. By hypothesis, $a_1a_2 \in \sqrt{(K/N :_R M/N)}$ or $a_1(m+N) \in K/N$ or $a_2^n(m+N) \in K/N$ for some positive integer n. Thus $a_1a_2 \in \sqrt{(K :_R M)}$ or $a_1m \in K$ or $a_2^n m \in K$ for some positive integer n. Hence K is a 2-absorbing semiprimary submodule of M. $\square$

Theorem 3.7. *Let N be a submodule of an R-module M and S be a multiplicative subset of R. If N is a 2-absorbing semiprimary submodule of M such that $(N :_R M) \cap S = \emptyset$, then $S^{-1}N$ is a 2-absorbing semiprimary submodule of $S^{-1}M$.*

Proof. Let $a_1, a_2 \in R, s_1, s_2, s_3 \in S$ and $m \in M$ such that $\frac{a_1}{s_1}\frac{a_2}{s_2}\frac{m}{s_3} \in S^{-1}N$. Then there exists $s \in S$ such that $sa_1a_2m \in N$. By hypothesis, $a_1a_2 \in \sqrt{(N :_R M)}$ or $a_1sm \in N$ or $a_2^n sm \in N$ for some positive integer n. Thus $\frac{a_1}{s_1}\frac{a_2}{s_2} \in \sqrt{(S^{-1}N :_R S^{-1}M)}$ or $\frac{a_1}{s_1}m = \frac{a_1sm}{s_1s} \in S^{-1}N$ or $(\frac{a_2}{s_2})^n m = \frac{a_2^n sm}{s_2^n s} \in S^{-1}N$ for some positive integer n. Hence $S^{-1}N$ is a 2-absorbing semiprimary submodule of $S^{-1}M$. $\square$

Theorem 3.8. *Let N be a submodule of an R-module M and S be a multiplicative subset of R. If $S^{-1}N$ is a 2-absorbing semiprimary submodule of $S^{-1}M$ such that $S \cap Zd(N) = \emptyset$ and $S \cap Zd(M/N) = \emptyset$, then*

N is a 2-absorbing semiprimary submodule of M.

Proof. Let $a_1, a_2 \in R$ and $m \in M$ such that $a_1a_2m \in N$. Then $\frac{a_1}{1}\frac{a_2}{1}\frac{m}{1} \in S^{-1}N$. By hypothesis, $\frac{a_1}{1}\frac{a_2}{1} \in \sqrt{(S^{-1}N :_R S^{-1}M)}$ or $\frac{a_1}{1}\frac{m}{1} \in S^{-1}N$ or $(\frac{a_2}{1})^n\frac{m}{1} \in S^{-1}N$ for some positive integer n. If $\frac{a_1}{1}\frac{a_2}{1} \in \sqrt{(S^{-1}N :_R S^{-1}M)}$, then $(\frac{a_1}{1}\frac{a_2}{1})^t \in (S^{-1}N :_R S^{-1}M)$ for some positive integer t. Thus there exists $s \in S$ such that $s(a_1a_2)^tM \subseteq N$ for some positive integer t. Since $S \cap Zd(M/N) = \emptyset$, we have $(a_1a_2)^tM \subseteq N$ so $a_1a_2 \in \sqrt{(N :_R M)}$. If $\frac{a_1}{1}\frac{m}{1} \in S^{-1}N$, then there exists $s \in S$ such that $sa_1m \in N$. Thus $s(a_1m + N) = sa_1m + N = N$. But $S \cap Zd(M/N) = \emptyset$, it follows that $a_1m \in N$. If $(\frac{a_2}{1})^n\frac{a_m}{1} \in N$, then there exists $s \in S$ such that such that $sa_1^nm \in N$. Thus $s(a_2^nm+N) = sa_2^nm+N = N$. Since $S\cap Zd(M/N) = \emptyset$, we have $a_2^nm \in N$. Therefore N is a 2-absorbing semiprimary submodule of M. $\square$

Theorem 3.9. *Let N be a proper submodule of an R-module M. The following conditions are equivalent:*

1. *N is a 2-absorbing semi-primary submodule of M.*
2. *For every $a, b \in R - (N :_R M)$ if $ab \in R - \sqrt{(N :_R M)}$, then $(N :_R ab) \subseteq (N :_R a) \cup (N :_R b^n)$ for some positive integer n.*

Proof. $(1 \Rightarrow 2)$ Suppose that N is a 2-absorbing semi-primary submodule of M. Let $m \in (N :_R ab)$. Then $abm \in N$. By hypothesis, $ab \in \sqrt{(N :_R M)}$ or $am \in N$ or $b^nm \in N$ for some positive integer n. But $ab \in R - \sqrt{(N :_R M)}$, $m \in (N :_R a)$ or $m \in (N :_R b^n)$ for some positive integer n. Therefore $m \in (N :_R a) \cup (N :_R b^n)$ for some positive integer n. Hence $(N :_R ab) \subseteq (N :_R a) \cup (N :_R b^n)$ for some positive integer n.

$(2 \Rightarrow 1)$ It is clear. $\square$

Theorem 3.10. *Let N be a proper submodule of an R-module M. The following conditions are equivalent:*

1. *N is a 2-absorbing semi-primary submodule of an R-module M.*
2. *For every $a, b \in R - (N :_R M)$ if $ab \in R - \sqrt{(N :_R M)}$, then $(N :_R ab) \subseteq (N :_R a) \cup (N :_R b^n)$ for some positive integer n.*

3. *Let R be a u-ring. For every $a, b \in R - (N :_R M)$ if $ab \in R - \sqrt{(N :_R M)}$, then $(N :_R ab) \subseteq (N :_R a)$ or $(N :_R ab) \subseteq (N :_R b^n)$ for some positive integer n.*

Proof. It is clear from Theorem 3.9. $\square$

Theorem 3.11. *Let N be a proper submodule of an R-module M. The following conditions are equivalent:*

1. *N is a 2-absorbing semi-primary submodule of an R-module M.*

2. *For every $a \in R - (N :_R M)$ and every ideal I of R such that $I \nsubseteq (N :_R M)$, if $aI \nsubseteq \sqrt{(N :_R M)}$, then $(N :_R aI) \subseteq (N :_R a) \cup (N :_R I^n)$ for some positive integer n.*

3. *Let R be a u-ring. For every $a \in R - (N :_R M)$ and every ideal I of R such that $I \nsubseteq (N :_R M)$, if $aI \nsubseteq \sqrt{(N :_R M)}$, then $(N :_R aI) \subseteq (N :_R a)$ or $(N :_R aI) \subseteq (N :_R I^n)$ for some positive integer n.*

4. *For every ideals I, J of R such that $I, J \nsubseteq (N :_R M)$ if $IJ \nsubseteq \sqrt{(N :_R M)}$, then $(N :_R IJ) \subseteq (N :_R I) \cup (N :_R J^n)$ for some positive integer n.*

5. *Let R be a u-ring. For every ideals I, J of R such that $I, J \nsubseteq (N :_R M)$ if $IJ \nsubseteq \sqrt{(N :_R M)}$, then $(N :_R IJ) \subseteq (N :_R I)$ or $(N :_R IJ) \subseteq (N :_R J^n)$ for some positive integer n.*

Proof. It is clear from Theorem 3.9. $\square$

Theorem 3.12. *Let N be a proper submodule of an R-module M. The following conditions are equivalent:*

1. *N is a 2-absorbing semi-primary submodule of M.*

2. *For every $a \in R - (N :_R M)$ and $m \in M$ if $am \notin N$, then $(N :_R am) \subseteq (\sqrt{((N :_R M)} :_R a) \cup \sqrt{(N :_R m)}$.*

Proof. $(1 \Rightarrow 2)$ Let $a \in R - (N :_R M)$ and $m \in M$ such that $am \notin N$. Assume that $r \in (N :_R am)$. Then $ram \in N$. By hypothesis, $ar \in$

$\sqrt{(N :_R M)}$ or $am \in N$ or $r^n m \in N$ for some positive integer n. Since $am \notin N$, we have $r \in (\sqrt{(N :_R M)} :_R a)$ or $r \in \sqrt{(N :_R m)}$. Thus $r \in (\sqrt{(N :_R M)} :_R a) \cup \sqrt{(N :_R m)}$.
Therefore $(N :_R am) \subseteq (\sqrt{((N :_R M)} :_R a) \cup \sqrt{(N :_R m)}$.
$(2 \Rightarrow 1)$ It is clear. $\square$

Corollary 3.13. *Let N be a proper submodule of an R-module M. The following conditions are equivalent:*

1. *N is a 2-absorbing semi-primary submodule of M.*
2. *For every ideal I of R such that $I \subseteq R - (N :_R M)$ and $m \in M$ if $Im \nsubseteq N$, then $(N :_R Im) \subseteq (\sqrt{(N :_R M)} : I) \cup \sqrt{(N :_R m)}$.*

Proof. It is clear from Theorem 3.12. $\square$

Lemma 3.14. *Let N be a 2-absorbing semi-primary submodule of an R-module M. For all $m_1 \in M$ and $m_2 \in M - N$ if $rs \in (N :_R m_1) - \sqrt{(N :_R m_2)}$, then $(N :_R rsm_2) \subseteq (N :_R rm_2) \cup \sqrt{(N :_R s^n m_2)}$ for some positive integer n.*

Proof. Suppose that $rs \in (N :_R m_1) - (N :_R m_2)$, where $m_1 \in M$ and $m_2 \in M - N$. Let $a \in (N :_R rsm_2)$. Then $(ars)m_2 = a(rsm_2) \in N$ so $ars \in (N :_R m_2)$. Clearly, $ar \in (N :_R m_2)$ or $as \in \sqrt{(N :_R m_2)}$ or $rs \in \sqrt{(N :_R m_2)}$. By the assumption, $ar \in (N :_R m_2)$ or $as \in \sqrt{(N :_R m_2)}$. Thus $a \in (N :_R rm_2)$ or $a \in \sqrt{(N :_R s^n m_2)}$ for some positive integer n. Therefore $(N :_R rsm_2) \subseteq (N :_R rm_2) \cup \sqrt{(N :_R s^n m_2)}$ for some positive integer n. $\square$

Proposition 3.15. *Let N be an irreducible submodule of an R-module M. For each $r \in R$ if $(N :_R r) = (N :_R r^2)$, then N is a 2-absorbing semi-primary submodule of M.*

Proof. Let $a, b \in R$ and $m \in M$ such that $abm \in N$. Suppose that $ab \notin \sqrt{(N :_R M)}$ and $am \notin N$ and $b^n m \notin N$ for all positive integer n. Clearly, $N \subseteq (N + abM) \cap (N + Ram) \cap (N + Rb^n m)$ for all positive integer n. Let $m_0 \in (N + abM) \cap (N + Ram) \cap (N + Rb^n m)$. This implies that $m_0 \in N + abM$ and $m_0 \in N + Ram$ and $m_0 \in N + Rb^n m$. Then there exist $r_1, r_2 \in R, m_1 \in M$ and $n_1, n_2 \in N$ such that $n_1 + abm_1 =$

$m_0 = n_2 + r_1 am = m_0 = n_3 + b_2^n m$. Consider

$$\begin{aligned} an_1 + a_1^2 bm_1 &= am_0 \\ &= an_2 + r_1 a_1^2 m \\ &= am_0 \\ &= an_3 + ab_2^n m. \end{aligned}$$

We have $a_1^2 r_1 m \in N$. It follows that $r_1 m \in (N :_R a_1^2)$. By the assumption, $r_1 m \in (N :_R a)$ so that $r_1 am \in N$. Thus $N = (N + abM) \cap (N + Ram) \cap (N + Rb^n m)$. Now since N is an irreducible, we have $N + abM \subseteq N$ or $am \in N + Ram \subseteq N$ or $b^n m \in N + Rb^n m \subseteq N$, a contradiction. Hence N is a 2-absorbing semi-primary submodule of M. $\square$

Theorem 3.16. *Let N_1 be a proper submodule of R_1-module M_1 and let M_2 be an R_2-module. Then $N_1 \times M_2$ is a 2-absorbing semi-primary submodule of $M_1 \times M_2$ if and only if N_1 is a 2-absorbing semi-primary submodule of M_1.*

Proof. Suppose that $N_1 \times M_2$ is a 2-absorbing semi-primary submodule of $M_1 \times M_2$. Let $a, b \in R_1$ and $m \in M_1$ such that $abm \in N_1$. Then $(a, 0)(b, 0)(m, 0) = (abm, 0) \in N_1 \times M_2$.
By hypothesis, $(ab, 0) = (a, 0)(b, 0) \in \sqrt{(N_1 \times M_2 :_R M_1 \times M_2)}$ or $(am, 0) = (a, 0)(m, 0) \in N_1 \times M_2$ or $(b^n m, 0) = (b, 0)^n (m, 0) \in N_1 \times M_2$ for some positive integer n. This implies that $ab \in \sqrt{(N_1 :_R M_1)}$ or $am \in N_1$ or $b^n m \in N_1$ for some positive integer n. Hence N_1 is a 2-absorbing semi-primary submodule of M_1.

Conversely, suppose N_1 is a 2-absorbing semi-primary submodule of M_1. Let $a, b \in R$ and $(m_1, m_2) \in M_1 \times M_2$ such that $(abm_1, abm_2) = ab(m_1, m_2) \in N_1 \times M_2$.
Then $abm_1 \in N_1$. By assumption, $ab \in \sqrt{(N_1 :_R M_1)}$ or $am_1 \in N_1$ or $b^n m_1 \in N_1$ for some positive integer n. So $ab \in \sqrt{(N_1 \times M_2 :_R M_1 \times M_2)}$ or $a(m_1, m_2) = (am_1, am_2) \in N_1 \times M_2$ or $b^n (m_1, m_2) = (b^n m_1, b^n m_2) \in N_1 \times M_2$ for some positive integer n and thus we are done. $\square$

Corollary 3.17. *Let M_i be an R-module and N_i be a proper submodule of M_i for all $i = \{1, 2, \ldots, k\}$. Then the following conditions are equivalent:*

1. $M_1 \times M_2 \times \ldots \times M_{i-1} \times N_i \times M_{i+1} \times M_k$ *is a* 2*-absorbing semi-primary submodule of* $M_1 \times M_2 \times \ldots \times M_k$.
2. N_i *is a* 2*-absorbing semi-primary submodule of* M_i.

Proof. This follows from Theorem 3.16. $\square$

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