ON NILPOTENCY OF OUTER POINTWISE INNER ACTOR OF THE LIE ALGEBRA CROSSED MODULES

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Abstract. Let $\mathcal{L}$ be a Lie algebra crossed module and $\mathrm{Act}_{pi}(\mathcal{L})$ be a point wise inner Actor of $\mathcal{L}$. In this paper, we introduce lower and upper central series of $\mathcal{L}$ and show that if $\mathrm{Act}_{pi}(\frac{\mathcal{L}}{Z_j(\mathcal{L})})/\mathrm{InnAct}(\frac{\mathcal{L}}{Z_j(\mathcal{L})})$ is the nilpotent of class k, then $\mathrm{Act}_{pi}(\mathcal{L})/\mathrm{InnAct}(\mathcal{L})$ is the nilpotent of the maximum class $j+k$. Moreover, if $\dim(\mathcal{L}^i/(\mathcal{L}^i \cap Z_j(\mathcal{L}))) \leqslant 1$, then $\mathrm{Act}_{pi}(\mathcal{L})/\mathrm{InnAct}(\mathcal{L})$ is the nilpotent of the maximum class $i+j-1$.

1. Introduction

Crossed modules in groups were introduced by Whitehead [13] in order to study homotopy relations of groups. Lie algebra crossed modules were used by Roisin and Lavendhomme as sufficient coefficients of a non-abelian cohomology of a T-algebra in [10].

A crossed module of Lie algebras is a homomorphism $d : L_1 \longrightarrow L_0$ along with an action of L_0 on L_1 satisfying special conditions. For an introduction and notation, we refer to Casas [2], Casas and Ladra [3, 4] .

Ilgaz et. al. [6] introduced the concept of solvability and nilpotence for Lie algebra crossed modules. In this paper, we introduce the upper and lower central series for Lie algebra crossed modules and show if $\mathrm{Act}_{pi}(\frac{\mathcal{L}}{Z_j(\mathcal{L})})/\mathrm{InnAct}(\frac{\mathcal{L}}{Z_j(\mathcal{L})})$ is the nilpotent of class k, then $\mathrm{Act}_{pi}(\mathcal{L})/\mathrm{InnAct}(\mathcal{L})$ is the nilpotent of the maximum class $k+j$. In addition, if $\dim(\mathcal{L}^i/(\mathcal{L}^i \cap Z_j(\mathcal{L}))) \leqslant 1$, then $\mathrm{Act}_{pi}(\mathcal{L})/\mathrm{InnAct}(\mathcal{L})$ is the nilpotent of the maximum class $i+j-1$.

Note that if $j = 0$, the results would be the same as Jamshidi Rad and Saeedi [7]. The idea of this paper is obtained from papers of Rai [11] and Sah's [12] in groups theory.

The paper is organized as follows. In section 2, we introduce the definitions and elementary symbols of Lie algebra crossed modules. In section 3, we define the upper and lower central series for crossed modules and prove some preliminary lemmas. In section 4, after proving the required lemmas, we express and prove the main theorem.

2. Preliminaries on Crossed Modules

A crossed module of Lie algebras is a homomorphism $d : L_1 \longrightarrow L_0$ along with an action of L_0 on L_1, denoted by $(l_0, l_1) \longrightarrow {}^{l_0} l_1$ for all $l_0 \in L_0$ and $l_1 \in L_1$ such that satisfies the following conditions:

(1) $d({}^{l_0}l_1) = [l_0, d(l_1)]$,

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(2) ${}^{d(l_1)}l_1' = [l_1, l_1']$,

for all $l_0 \in L_0$ and $l_1, l_1' \in L_1$. The crossed module $\mathcal{L}$ is denoted as $\mathcal{L} : (L_1, L_0, d)$. The crossed module $\mathcal{M} : (M_1, M_0, d')$ is called a subcrossed module $\mathcal{L} : (L_1, L_0, d)$ and shown as $\mathcal{M} \leqslant \mathcal{L}$ if M_0 and M_1 are subalgebras L_0 and L_1, respectively and d' is the restriction of d on M_1 and M_0 acts on M_1 as L_0 acts on L_1.
A subcrossed module $\mathcal{M} : (M_1, M_0, d')$ of a crossed module $\mathcal{L} : (L_1, L_0, d)$ is an ideal of $\mathcal{L}$ and shown as $\mathcal{M} \trianglelefteq \mathcal{L}$ if M_0 and M_1 are ideals of L_0 and L_1, respectively and for all $l_0 \in L_0$, $m_0 \in M_0$, $l_1 \in L_1$ and $m_1 \in M_1$

$$ {}^{l_0}m_1 \in M_1 \quad \text{and} \quad {}^{m_0}l_1 \in M_1. $$

Let $\mathcal{M} : (M_1, M_0, d_|)$ and $\mathcal{N} : (N_1, N_0, d_|)$ are two ideals of crossed module $\mathcal{L} : (L_1, L_0, d)$. Then, $\mathcal{M} \cap \mathcal{N}$ is an ideal of $\mathcal{L}$ and defined as

$$ \mathcal{M} \cap \mathcal{N} : (M_1 \cap N_1, M_0 \cap N_0, d_|). $$

Let $\mathcal{L} : (L_1, L_0, d)$ be a Lie algebra crossed module. Then, the center of this crossed module is an ideal of it and shown as $Z(\mathcal{L})$ and defined as

$$ Z(\mathcal{L}) : \left({}^{L_0}L_1, \mathrm{St}_{L_0}(L_1) \cap Z(L_0), d_|\right) $$

in which

$$ {}^{L_0}L_1 = \{l_1 \in L_1 | {}^{l_0}l_1 = 0, \ \forall \ l_0 \in L_0\}, $$
$$ \mathrm{St}_{L_0}(L_1) = \{l_0 \in L_0 | {}^{l_0}l_1 = 0, \ \forall \ l_1 \in L_1\}. $$

The crossed module $\mathcal{L}$ is abelian, if it coincides with its center.
Let $\mathcal{L} : (L_1, L_0, d)$ be a Lie algebra crossed module. The derived crossed module of $\mathcal{L}$ is defined as

$$ \mathcal{L}^2 : \left(D_{L_0}(L_1), L_0^2, d_|\right) $$

in which $D_{L_0}(L_1) = \langle {}^{l_0}l_1 : \ l_0 \in L_0, l_1 \in L_1\rangle$ (see [5]).
A homomorphism between two Lie algebra crossed modules $\mathcal{L} : (L_1, L_0, d)$ and $\mathcal{L}' : (L_1', L_0', d')$ is a pair (f, g) of Lie algebra homomorphisms $f : L_1 \longrightarrow L_1'$ and $g : L_0 \longrightarrow L_0'$ satisfying the following conditions:

(1) $d'f = gd$,
(2) $f({}^{l_0}l_1) = {}^{g(l_0)} f(l_1)$

for all $l_0 \in L_0$ and $l_1 \in L_1$.

Definition. Assume $\mathcal{L} : (L_1, L_0, d)$ is a crossed module. A derivation of $\mathcal{L}$ is a pair $(\alpha, \beta) : \mathcal{L} \longrightarrow \mathcal{L}$ satisfying the following conditions:

(1) $\alpha \in \mathrm{Der}(L_1)$,
(2) $\beta \in \mathrm{Der}(L_0)$,
(3) $d\alpha = \beta d$,
(4) $\alpha({}^{l_0}l_1) = {}^{l_0} \alpha(l_1) + {}^{\beta(l_0)} (l_1)$,

for all $l_0 \in L_0$ and $l_1 \in L_1$.

The set of all derivations of $\mathcal{L}$ is denoted by $\mathrm{Der}(\mathcal{L})$, which is a Lie algebra with bracket as in the following:

$$ [(\alpha, \beta), (\alpha', \beta')] = ([\alpha, \alpha'], [\beta, \beta']) = (\alpha\alpha' - \alpha'\alpha, \beta\beta' - \beta'\beta). $$

Definition. Assume $\mathcal{L} : (L_1, L_0, d)$ is a Lie algebra crossed module. Then a map $\delta : L_0 \longrightarrow L_1$ is called crossed derivation if

$$ \delta([l_0, l_0']) = {}^{l_0} \delta(l_0') - {}^{l_0'} \delta(l_0) $$

for all $l_0, l_0' \in L_0$. The set of all crossed derivations from L_0 to L_1 is denoted by $\mathrm{Der}(L_0, L_1)$, which turns into a Lie algebra via the following bracket:

$$[\delta_1, \delta_2] = \delta_1 d\delta_2 - \delta_2 d\delta_1$$

for all $\delta_1, \delta_2 \in \mathrm{Der}(L_0, L_1)$.

Proposition 2.1. *Every $\delta \in \mathrm{Der}(L_0, L_1)$ induces two derivations $\delta^0 \in \mathrm{Der}(L_0)$ and $\delta^1 \in \mathrm{Der}(L_1)$ defined as*

$$\delta^0 = d\delta \quad \textit{and} \quad \delta^1 = \delta d$$

and satisfy the following identities:

(1) $\delta\delta^0 = \delta^1\delta$,
(2) $\delta^0 d = d\delta^1$,
(3) $(\delta^1, \delta^0) \in \mathrm{Der}(\mathcal{L})$.

Definition. Let $\mathcal{L} : (L_1, L_0, d)$ be a Lie algebra crossed module. Then $\mathrm{Der}(\mathcal{L})$ acts on $\mathrm{Der}(L_0, L_1)$ as follows:

$$^{(\alpha,\beta)}\delta := \alpha\delta - \delta\beta$$

for all $\alpha, \beta \in \mathrm{Der}(\mathcal{L})$ and $\delta \in \mathrm{Der}(L_0, L_1)$. Now the homomorphism $\Delta : \mathrm{Der}(L_0, L_1) \longrightarrow \mathrm{Der}(\mathcal{L})$ defined by $\delta \mapsto (\delta d, d\delta)$ is a crossed module and it is denoted by $\mathrm{Act}(\mathcal{L})$. We have

$$\mathrm{Act}(\mathcal{L}) : (\mathrm{Der}(L_0, L_1), \mathrm{Der}(\mathcal{L}), \Delta).$$

Proposition 2.2. *There always exists a canonical homomorphism of crossed modules as follows:*

$$(\varepsilon, \eta) : \mathcal{L} \longrightarrow \mathrm{Act}(\mathcal{L})$$

in which

$$\begin{array}{rcl} \varepsilon : L_1 & \longrightarrow & \mathrm{Der}(L_0, L_1) \\ l_1 & \longmapsto & \delta_{l_1} \end{array} \qquad \textit{and} \qquad \begin{array}{rcl} \eta : L_0 & \longrightarrow & \mathrm{Der}(\mathcal{L}) \\ l_0 & \longmapsto & (\alpha_{l_0}, \beta_{l_0}) \end{array}$$

with

$$\delta_{l_1}(l_0) = {}^{l_0} l_1, \quad \alpha_{l_0}(l_1) = {}^{l_0} l_1, \quad \beta_{l_0}(l_0') = [l_0, l_0'],$$

for all $l_0, l_0' \in L_0$ and $l_1 \in L_1$. The image of this homomorphism is an ideal of $\mathrm{Act}(\mathcal{L})$*, denoted by* $\mathrm{InnAct}(\mathcal{L})$*, and it is given by*

$$\mathrm{InnAct}(\mathcal{L}) : (\varepsilon(L_1), \eta(L_0), \Delta_|).$$

It can be easily shown that $\ker(\varepsilon, \eta) = Z(\mathcal{L})$.

Definition. Let $\mathcal{L} : (L_1, L_0, d)$ be a Lie algebra crossed module. Then pointwise inner Actor of $\mathcal{L}$ is defined as

$$\mathrm{Act}_{pi}(\mathcal{L}) : (\mathrm{Der}_{pi}(L_0, L_1), \mathrm{Der}_{pi}(\mathcal{L}), \Delta_|)$$

in which

$$\mathrm{Der}_{pi}(L_0, L_1) = \{\delta \in \mathrm{Der}(L_0, L_1) \text{ s.t } \forall l_0 \in L_0 \ \exists l_1 \in L_1 | \ \delta(l_0) = {}^{l_0} l_1\},$$

$$\mathrm{Der}_{pi}(\mathcal{L}) = \left\{ (\alpha, \beta) \in \mathrm{Der}(\mathcal{L}) \ \middle| \ \begin{array}{l} \forall l_1 \in L_1 \ \exists l_0 \in L_0 \text{ s.t } \alpha(l_1) = {}^{l_0} l_1 \\ \forall l_0 \in L_0 \ \exists l_0' \in L_0' \text{ s.t } \beta(l_0) = [l_0', l_0] \end{array} \right\}.$$

It can easily be proved that $\mathrm{Act}_{pi}(\mathcal{L})$ is a subcrossed module of $\mathrm{Act}(\mathcal{L})$ including $\mathrm{InnAct}(\mathcal{L})$. (see [1]).

Definition. Let $\mathcal{L} : (L_1, L_0, d)$ be a Lie algebra crossed module. Then, $\mathrm{ID}^*\mathrm{Act}(\mathcal{L})$ is defined as

$$\mathrm{ID}^*\mathrm{Act}(\mathcal{L}) : (\mathrm{ID}^*(L_0, L_1), \mathrm{ID}^*(\mathcal{L}), \Delta_|)$$

in which

$$\mathrm{ID}^*(L_0, L_1) = \left\{ \delta \in \mathrm{Der}(L_0, L_1) \;\middle|\; \begin{array}{l} \delta(l_0) \in D_{L_0}(L_1),\ \forall l_0 \in L_0, \\ \delta(l_0) = 0,\ \forall l_0 \in \mathrm{St}_{L_0}(L_1) \cap Z(L_0) \end{array} \right\}$$

and

$$\mathrm{ID}^*(\mathcal{L}) = \left\{ (\alpha, \beta) \in \mathrm{Der}(\mathcal{L}) \;\middle|\; \begin{array}{l} \alpha(l_1) \in D_{L_0}(L_1),\ \forall l_1 \in L_1, \\ \alpha(l_1) = 0,\ \forall l_1 \in^{L_0} L_1, \\ \beta(l_0) \in L_0^2,\ \forall l_0 \in L_0, \\ \beta(l_0) = 0,\ \forall l_0 \in \mathrm{St}_{L_0}(L_1) \cap Z(L_0) \end{array} \right\}.$$

It can easily be shown that $\mathrm{ID}^*\mathrm{Act}(\mathcal{L})$ is a subcrossed module of $\mathrm{Act}(\mathcal{L})$ including $\mathrm{Act}_{pi}(\mathcal{L})$ (see [1]).

Definition. Let $\mathcal{L} : (L_1, L_0, d)$ be a Lie algebra crossed module and $\mathcal{N} : (N_1, N_0, d_|)$ be an ideal of $\mathcal{L}$. Then, $\mathrm{Act}^{\mathcal{N}}(\mathcal{L})$ is defined as

$$\mathrm{Act}^{\mathcal{N}}(\mathcal{L}) : (\mathrm{Der}^{\mathcal{N}}(L_0, L_1), \mathrm{Der}^{\mathcal{N}}(\mathcal{L}), \Delta_|)$$

in which

$$\mathrm{Der}^{\mathcal{N}}(L_0, L_1) = \{\delta \in \mathrm{Der}(L_0, L_1) \,|\, \delta(x_0) \in N_1 \ \forall x_0 \in L_0\}\ ,$$

$$\mathrm{Der}^{\mathcal{N}}(\mathcal{L}) = \{(\alpha, \beta) \in \mathrm{Der}(\mathcal{L}) \,|\, \alpha(x_1) \in N_1 \ \forall x_1 \in L_1,\ \beta(x_0) \in N_0 \ \forall x_0 \in L_0\}\ .$$

3. Upper and lower central series of Lie algebra crossed modules

Definition. Let $\mathcal{L} : (L_1, L_0, d)$ be a Lie algebra crossed module. Then the lower central series $\mathcal{L}$ is defined as

$$\mathcal{L}^1 \supseteq \mathcal{L}^2 \supseteq \cdots \supseteq \mathcal{L}^n \supseteq \mathcal{L}^{n+1} \supseteq \cdots$$

in which,

$$\begin{aligned} &\mathcal{L}^1 = \mathcal{L} : (L_1, L_0, d) \\ &\mathcal{L}^2 : (D_{L_0}(L_1), L_0^2, d_|) \\ &\mathcal{L}^3 : (D_{L_0}(D_{L_0}(L_1)), L_0^3, d_|) \\ &\ \vdots \\ &\mathcal{L}^n : (\underbrace{D_{L_0}(D_{L_0}(\cdots(D_{L_0}}_{n-1 \text{ times}}(L_1)))), L_0^n, d_|). \end{aligned}$$

For simplicity we use the $\mathcal{L}^n : (D^n_{L_0}(L_1), L_0^n, d_|)$.

Definition. Let $\mathcal{L} : (L_1, L_0, d)$ be a Lie algebra crossed module. Then the upper central series $\mathcal{L}$ is defined as

$$Z_0(\mathcal{L}) \subseteq Z_1(\mathcal{L}) \subseteq \cdots \subseteq Z_n(\mathcal{L}) \subseteq Z_{n+1}(\mathcal{L}) \subseteq \cdots$$

in which,

$$\begin{aligned}
&Z_0(\mathcal{L}) = 0\\
&Z_1(\mathcal{L}) = Z(\mathcal{L}) : (A_1(\mathcal{L}), B_1(\mathcal{L}) \cap Z_1(L_0), d_|)\\
&Z_2(\mathcal{L}) : (A_2(\mathcal{L}), B_2(\mathcal{L}) \cap Z_2(L_0), d_|)\\
&\quad\vdots\\
&Z_n(\mathcal{L}) : (A_n(\mathcal{L}), B_n(\mathcal{L}) \cap Z_n(L_0), d_|)
\end{aligned}$$

where

$$A_i(\mathcal{L}) = \left\{ x_1 \in L_1 \;\middle|\; {}^{x_{0i}}\!\!\cdot^{\cdot^{\cdot^{x_{01}}}} x_1 = 0 \ \forall x_{0j} \in L_0,\ 1 \leqslant j \leqslant i \right\},$$

$$B_i(\mathcal{L}) = \left\{ x_0 \in L_0 \;\middle|\; \begin{array}{l} {}^{x_0}{}^{x_{01}}\cdot^{\cdot^{\cdot^{x_{0i-1}}}} x_1 = 0, \quad {}^{[x_0,x_{01}]}{}^{x_{02}}\cdot^{\cdot^{\cdot^{x_{0i-1}}}} x_1 = 0,\\ {}^{[x_0,x_{01},x_{02}]}{}^{x_{03}}\cdot^{\cdot^{\cdot^{x_{0i-1}}}} x_1 = 0,\ \ldots,\\ {}^{[x_0,x_{01},\cdots,x_{0i-2}]}{}^{x_{0i-1}} x_1 = 0,\ {}^{[x_0,x_{01},\cdots,x_{0i-1}]} x_1 = 0 \end{array} \quad \forall x_1 \in L_1,\ x_{0j} \in L_0,\ 1 \leqslant j \leqslant i \right\}$$

for $\forall i \in \mathbb{N}$.

Definition. Let $\mathcal{L} : (L_1, L_0, d)$ be a Lie algebra crossed module. If there is $n \in Z^+$ such that $\mathcal{L}^{n+1} = 0$ or $Z_n(\mathcal{L}) = \mathcal{L}$, then $\mathcal{L}$ is the nilpotent of class n.

Lemma 3.1. *Let $\mathcal{L} : (L_1, L_0, d)$ be a Lie algebra crossed module and $x_0 \in L_0^i$. Then*

1. $x_1 \in A_j(\mathcal{L})$ *if and only if* ${}^{x_0}x_1 \in A_{j-i}(\mathcal{L})$;
2. $[x_0, y_0] \in B_{j-i}(\mathcal{L}) \cap Z_{j-i}(L_0) \Longleftrightarrow y_0 \in B_j(\mathcal{L}) \cap Z_j(L_0)$.

Proof. The proof is straightforward. □

Lemma 3.2. [7] *Let $\mathcal{L} : (L_1, L_0, d)$ be a Lie algebra crossed module and for all $k \geqslant 0$, $(\delta, (\alpha, \beta)) \in \mathrm{Act}^k_{pi}(\mathcal{L})$. Then*

1. *For all $x_0 \in L_0$, there are $b_{x_0} \in D^k_{L_0}(L_1)$ and $c_{x_0} \in L_0^k$ so that $\delta(x_0) ={}^{x_0} b_{x_0}$ and $\beta(x_0) = [c_{x_0}, x_0]$;*
2. *For all $x_1 \in L_1$, there is $b_{x_1} \in L_0^k$ so that $\alpha(x_1) ={}^{b_{x_1}} x_1$.*

Lemma 3.3. *Let $\mathcal{L} : (L_1, L_0, d)$ be a Lie algebra crossed module, $(\delta_{x_1}, (\alpha_{x_0}, \beta_{x_0})) \in \mathrm{InnAct}(\mathcal{L})$ and $(\delta', (\alpha', \beta')) \in \mathrm{Act}(\mathcal{L})$ be arbitrary. Then*

1. $[\delta', \delta_{x_1}] = \delta_{\delta'(d(x_1))}$;
2. $[\alpha', \alpha_{l_0}] = \alpha_{\beta'(l_0)}$;
3. $[\beta', \beta_{l_0}] = \beta_{\beta'(l_0)}$.

Proof. (1) Let $l_0 \in L_0$

$$\begin{aligned}
[\delta', \delta_{x_1}](l_0) &= (\delta' d\delta_{x_1} - \delta_{x_1} d\delta')(l_0) = \delta' d\delta_{x_1}(l_0) - \delta_{x_1} d\delta'(l_0)\\
&= \delta' d({}^{l_0}x_1) - \delta_{x_1}(d\delta'(l_0)) = \delta' d({}^{l_0}x_1) -{}^{d\delta'(l_0)} x_1\\
&= \delta'([l_0, d(x_1)]) - [\delta'(l_0), x_1] ={}^{l_0} \delta'(d(x_1)) -{}^{d(x_1)} \delta'(l_0) - [\delta'(l_0), x_1]\\
&={}^{l_0} \delta'(d(x_1)) - [x_1, \delta'(l_0)] - [\delta'(l_0), x_1] ={}^{l_0} \delta'(d(x_1))\\
&= \delta_{\delta'(d(x_1))}(l_0).
\end{aligned}$$

(2) Let $x_1 \in L_1$

$$\begin{aligned}[\alpha', \alpha_{l_0}](x_1) &= (\alpha'\alpha_{l_0} - \alpha_{l_0}\alpha')(x_1) = \alpha'\alpha_{l_0}(x_1) - \alpha_{l_0}\alpha'(x_1) \\ &= \alpha'({}^{l_0}x_1) -{}^{l_0} \alpha'(x_1) ={}^{l_0} \alpha'(x_1) +{}^{\beta'(l_0)} x_1 -{}^{l_0} \alpha'(x_1) \\ &={}^{\beta'(l_0)} x_1 = \alpha_{\beta'(l_0)}(x_1).\end{aligned}$$

(3) Let $x_0 \in L_0$

$$\begin{aligned}[\beta', \beta_{l_0}](x_0) &= (\beta'\beta_{l_0} - \beta_{l_0}\beta')(x_0) = \beta'\beta_{l_0}(x_0) - \beta_{l_0}\beta'(x_0) \\ &= \beta'([l_0, x_0]) - [l_0, \beta'(x_0)] = [\beta'(l_0), x_0] + [l_0, \beta'(x_0)] - [l_0, \beta'(x_0)] \\ &= [\beta'(l_0), x_0] = \beta_{\beta'(l_0)}(x_0).\end{aligned}$$

□

Lemma 3.4. *Let $\mathcal{L} : (L_1, L_0, d)$ be a Lie algebra crossed module. Let $(\delta_{x_1}, (\alpha_{x_0}, \beta_{x_0}))$ and $(\delta_{y_1}, (\alpha_{y_0}, \beta_{y_0}))$ are two arbitrary elements of* $\mathrm{InnAct}(\mathcal{L})$*. Then*

(1) $[\delta_{x_1}, \delta_{y_1}] = \delta_{[y_1, x_1]}$;
(2) $[\alpha_{x_0}, \alpha_{y_0}] = \alpha_{[x_0, y_0]}$;
(3) $[\beta_{x_0}, \beta_{y_0}] = \beta_{[x_0, y_0]}$.

Proof. It can be easily proved similar to Lemma 3.3. □

Lemma 3.5. *Let $\mathcal{L} : (L_1, L_0, d)$ be a Lie algebra crossed module and $\mathcal{H}$ be a sub-crossed module of* $\mathrm{ID}^*\mathrm{Act}(\mathcal{L})$ *contains* $\mathrm{InnAct}(\mathcal{L})$*. Then*

$$\mathcal{H} \cap \mathrm{Act}^{Z(\mathcal{L})}(\mathcal{L}) = Z(\mathcal{H}).$$

Proof. See [8], Corollary 4.3. □

4. Main theorem

In this section, first we state and prove some essential lemma, and then present the main theorem of this paper.

Lemma 4.1. *Let $\mathcal{L} : (L_1, L_0, d)$ be a Lie algebra crossed module and $\mathcal{N} : (N_1, N_0, d)$ an ideal of it. If*

$$(\mathrm{Act}_{pi}(\frac{\mathcal{L}}{\mathcal{N}}))^j \leqslant (\mathrm{Inn}(\mathrm{Act}(\frac{\mathcal{L}}{\mathcal{N}}))^k \qquad j, k \in \mathbb{N}$$

then

$$(\mathrm{Act}_{pi}(\mathcal{L}))^j \leqslant (\mathrm{Act}_{pi}(\mathcal{L}))^k \cap \mathrm{Act}^{\mathcal{N}}(\mathcal{L}) + (\mathrm{InnAct}(\mathcal{L}))^k.$$

Proof. Assume $(\delta, (\alpha, \beta)) \in (\mathrm{Act}_{pi}(\mathcal{L}))^j$. We know that $\delta \in D^j_{\mathrm{Der}_{pi}(\mathcal{L})}(\mathrm{Der}_{pi}(L_0, L_1))$. Now take $\bar{\delta}$ be crossed induced derivation by δ on $\frac{L_0}{N_0}$. Hence

$$\bar{\delta} \in D^j_{\mathrm{Der}_{pi}(\frac{\mathcal{L}}{\mathcal{N}})}(\mathrm{Der}_{pi}(\frac{L_0}{N_0}, \frac{L_1}{N_1})).$$

By the assumption, we have

$$D^j_{\mathrm{Der}_{pi}(\frac{\mathcal{L}}{\mathcal{N}})}(\mathrm{Der}_{pi}(\frac{L_0}{N_0}, \frac{L_1}{N_1})) \subseteq D^k_{\eta(\frac{L_0}{N_0})}(\xi(\frac{L_1}{N_1})) \subseteq D^k_{\mathrm{Der}_{pi}(\frac{\mathcal{L}}{\mathcal{N}})}(\mathrm{Der}_{pi}(\frac{L_0}{N_0}, \frac{L_1}{N_1})).$$

Using the first part of Lemma 3.2, for all $x_0 + N_0 \in \frac{L_0}{N_0}$, there exists $b_{x_0} + N_1 \in D^k_{\frac{L_0}{N_0}}(\frac{L_1}{N_1})$ such that

$$\bar{\delta}(x_0 + N_0) ={}^{x_0+N_0} b_{x_0} + N_1 ={}^{x_0} b_{x_0} + N_1.$$

Therefore,

$$\bar{\delta}(x_0+N_0)=\delta(x_0)+N_1={}^{x_0}b_{x_0}+N_1 \Rightarrow \delta(x_0)={}^{x_0}b_{x_0}+n_1 \quad \text{for } n_1\in N_1.$$

Then

$$\delta(x_0)=\delta_{b_{x_0}}(x_0)+n_1.$$

We take

$$\lambda=\delta+\delta_{-b_{x_0}}.$$

Hence, $\lambda\in \mathrm{Der}^{\mathcal{N}}(L_0,L_1)$. Now without loss of generality, assume $k\leqslant j$, we have

$$\delta\in D^{j}_{\mathrm{Der}_{pi}(\mathcal{L})}(\mathrm{Der}_{pi}(L_0,L_1))\subseteq D^{k}_{\mathrm{Der}_{pi}(\mathcal{L})}(\mathrm{Der}_{pi}(L_0,L_1)).$$

Therefore,

$$\lambda=\delta+\delta_{-b_{x_0}}\in D^{k}_{\mathrm{Der}_{pi}(\mathcal{L})}(\mathrm{Der}_{pi}(L_0,L_1)).$$

Consequently,

$$\delta=\lambda+\delta_{b_{x_0}}\in D^{k}_{\mathrm{Der}_{pi}(\mathcal{L})}(\mathrm{Der}_{pi}(L_0,L_1))\cap \mathrm{Der}^{\mathcal{N}}(L_0,L_1)+D^{k}_{\eta(L_0)}(\xi(L_1)). \tag{1}$$

Let $(\alpha,\beta)\in \mathrm{Der}^{j}_{pi}(\mathcal{L})$. Consider $\bar{\alpha}$ be induced derivation by α on $\dfrac{L_1}{N_1}$. By the assumption, we have

$$\bar{\alpha}\in \mathrm{Der}^{j}_{pi}(\frac{L_1}{N_1})\subseteq \eta^{k}(\frac{L_0}{N_0})\subseteq \mathrm{Der}^{k}_{pi}(\frac{L_1}{N_1}).$$

By using the second part of Lemma 3.2, for all $x_1+N_1\in \frac{L_1}{N_1}$, there exists $b_{x_1}\in L_0^k$ such that

$$\bar{\alpha}(x_1+N_1)={}^{b_{x_1}}x_1+N_1 \Rightarrow \alpha(x_1)+N_1={}^{b_{x_1}}x_1+N_1.$$

Therefore,

$$\alpha(x_1)={}^{b_{x_1}}x_1+n_1 \quad \text{for } n_1\in N_1.$$

We take

$$\gamma=\alpha+\alpha_{-b_{x_1}}.$$

Thus, $\gamma\in \mathrm{Der}^{N_1}(L_1)$. Now without loss of generality, assume $k\leqslant j$, we have

$$\alpha\in \mathrm{Der}^{j}_{pi}(L_1)\subseteq \mathrm{Der}^{k}_{pi}(L_1).$$

Therefore,

$$\gamma=\alpha+\alpha_{-b_{x_1}}\in \mathrm{Der}^{k}_{pi}(L_1).$$

Consequently,

$$\alpha=\gamma+\alpha_{b_{x_1}}\in \mathrm{Der}^{k}_{pi}(L_1)\cap \mathrm{Der}^{N_1}(L_1)+\eta^{k}(L_0). \tag{2}$$

Consider $\bar{\beta}$ be induced derivation by β on $\dfrac{L_0}{N_0}$. By the assumption, we have

$$\bar{\beta}\in \mathrm{Der}^{j}_{pi}(\frac{L_0}{N_0})\subseteq \eta^{k}(\frac{L_0}{N_0})\subseteq \mathrm{Der}^{k}_{pi}(\frac{L_0}{N_0}).$$

Using the first part of Lemma 3.2, for all $x_0+N_0\in \frac{L_0}{N_0}$, there exists $c_{x_0}\in L_0^k$ such that

$$\bar{\beta}(x_0+N_0)=[c_{x_0},x_0]+N_0.$$

Therefore,

$$\bar{\beta}(x_0+N_0)=\beta(x_0)+N_0=[c_{x_0},x_0]+N_0 \Rightarrow \beta(x_0)=[c_{x_0},x_0]+n_0 \quad \text{for } n_0\in N_0.$$

We take

$$Z = \beta + \beta_{-c_{x_0}}.$$

Then, $Z \in \mathrm{Der}^{N_0}(L_0)$. Now without loss of generality, assume $k \leqslant j$, we have

$$\beta \in \mathrm{Der}^{j}_{pi}(L_0) \subseteq \mathrm{Der}^{k}_{pi}(L_0).$$

Therefore,

$$Z = \beta + \beta_{-c_{x_0}} \in \mathrm{Der}^{k}_{pi}(L_0).$$

Consequently,

$$\beta = Z + \beta_{c_{x_0}} \in \mathrm{Der}^{k}_{pi}(L_0) \cap \mathrm{Der}^{N_0}(L_0) + \eta^k(L_0). \tag{3}$$

Now, by using (1), (2) and (3), we get

$$(\mathrm{Act}_{pi}(\mathcal{L}))^j \leqslant (\mathrm{Act}_{pi}(\mathcal{L}))^k \cap \mathrm{Act}^{\mathcal{N}}(\mathcal{L}) + (\mathrm{InnAct}(\mathcal{L}))^k.$$

□

Definition. Let $\mathcal{L} : (L_1, L_0, d)$ be a Lie algebra crossed module. Then

$$[\mathcal{L}, \mathrm{Act}(\mathcal{L})] : ([L_0, \mathrm{Der}(L_0, L_1)] + [L_1, \mathrm{Der}(L_1)], [L_0, \mathrm{Der}(L_0)])$$

in which

$$\begin{aligned} [L_0, \mathrm{Der}(L_0, L_1)] &= \{\delta(x_0) \,|x_0 \in L_0, \delta \in \mathrm{Der}(L_0, L_1)\}\,; \\ [L_1, \mathrm{Der}(L_1)] &= \{\alpha(x_1) \,|x_1 \in L_1, \alpha \in \mathrm{Der}(L_1)\}\,; \\ [L_0, \mathrm{Der}(L_0)] &= \{\beta(x_0) \,|x_0 \in L_0, \beta \in \mathrm{Der}(L_0)\}\,. \end{aligned}$$

Lemma 4.2. *Let $\mathcal{L} : (L_1, L_0, d)$ be a Lie algebra crossed module. Then*

$$[\mathcal{L}^i, \mathrm{Act}^{Z_j(\mathcal{L})}(\mathcal{L})] \subseteq Z_{j-i+1}(\mathcal{L}). \tag{4}$$

Proof. It can be proved by induction on i.

Let $i = 1$, it is clear from definition of $\mathrm{Act}^{Z_j(\mathcal{L})}(\mathcal{L})$.

Assume for i, (4) holds. That is,

$$[L_0^i, \mathrm{Der}^{Z_j(\mathcal{L})}(L_0, L_1)] + [D^i_{L_0}(L_1), \mathrm{Der}^{Z_j(\mathcal{L})}(L_1)] \subseteq A_{j-i+1}(\mathcal{L}),$$

$$[L_0^i, \mathrm{Der}^{Z_j(\mathcal{L})}(L_0)] \subseteq B_{j-i+1}(\mathcal{L}) \cap Z_{j-i+1}(L_0).$$

Now, take $\delta \in \mathrm{Der}^{Z_j(\mathcal{L})}(L_0, L_1)$ and $l_0 \in L_0^{i+1}$. Then, there exist $x_0 \in L_0$ and $y_0 \in L_0^i$ such that $l_0 = [x_0, y_0]$. Thus,

$$\delta(l_0) = \delta([x_0, y_0]) =^{x_0} \delta(y_0) -^{y_0} \delta(x_0).$$

By inductive assumption $\delta(y_0) \in A_{j-i+1}(\mathcal{L})$ and using the Lemma 3.1 ${}^{x_0}\delta(y_0) \in A_{j-i}(\mathcal{L})$. Moreover, since $\delta(x_0) \in A_j(\mathcal{L})$ and $y_0 \in L_0^i$, by using the Lemma 3.1 ${}^{y_0}\delta(x_0) \in A_{j-i}(\mathcal{L})$. Therefore, $\delta(l_0) \in A_{j-i}(\mathcal{L})$. Consequently,

$$[L_0^{i+1}, \mathrm{Der}^{Z_j(\mathcal{L})}(L_0, L_1)] \in A_{j-i}(\mathcal{L}). \tag{5}$$

Let $(\alpha, \beta) \in \mathrm{Der}^{Z_j(\mathcal{L})}(\mathcal{L})$ and $x_1 \in D^{i+1}_{L_0}(L_1)$. Hence, there exist $y_1 \in D^i_{L_0}(L_1)$ and $y_0 \in L_0$ such that $x_1 =^{y_0} y_1$. Thus,

$$\alpha(x_1) = \alpha({}^{y_0}y_1) =^{y_0} \alpha(y_1) +^{\beta(y_0)} y_1.$$

Now, by given inductive assumption and $\beta(y_0) \in \beta_j(\mathcal{L}) \cap Z_j(L_0)$, we conclude that $\alpha(x_1) \in A_{j-i}(\mathcal{L})$. Hence,

$$[D^{i+1}_{L_0}(L_1), \mathrm{Der}^{Z_j(\mathcal{L})}(L_1)] \in A_{j-i}(\mathcal{L}). \tag{6}$$

Take $x_0 \in L_0^{i+1}$, then there exist $y_0 \in L_0^i$ and $z_0 \in L_0$ such that $x_0 = [y_0, z_0]$. Thus,

$$\beta(x_0) = \beta[y_0, z_0] = [\beta(y_0), z_0] + [y_0, \beta(z_0)].$$

By inductive assumption and $\beta(z_0) \in B_j(\mathcal{L}) \cap Z_j(L_0)$, we conclude that $\beta(x_0) \in B_{j-i}(\mathcal{L}) \cap Z_{j-i}(L_0)$. Therefore,

$$[L_0^{i+1}, \mathrm{Der}^{Z_j(\mathcal{L})}(L_0)] \subseteq B_{j-i}(\mathcal{L}) \cap Z_{j-i}(L_0). \tag{7}$$

By using (5), (6) and (7), we obtain

$$[\mathcal{L}^{i+1}, \mathrm{Act}^{Z_j(\mathcal{L})}(\mathcal{L})] \subseteq Z_{j-i}(\mathcal{L}).$$

□

Lemma 4.3. *Let $\mathcal{L} : (L_1, L_0, d)$ be a Lie algebra crossed module. Then*

$$[Z_j(\mathcal{L}), (ID^*\mathrm{Act}(\mathcal{L}))^i] \subseteq Z_{j-i}(\mathcal{L}). \tag{8}$$

Proof. First, take $i = 1$ and prove (8) by induction on j. By definition of $ID^*\mathrm{Act}(\mathcal{L})$, it is clear that $[Z(\mathcal{L}), ID^*\mathrm{Act}(\mathcal{L})] = 0 = Z_0(\mathcal{L})$. Thus, (8) holds for $j = 1$. Now, assume that for j, (8) holds. That is,

$$[B_j(\mathcal{L}) \cap Z_j(L_0), ID^*(L_0, L_1)] + [A_j(\mathcal{L}), ID^*(L_1)] \subseteq A_{j-1}(\mathcal{L}),$$
$$[B_j(\mathcal{L}) \cap Z_j(L_0), ID^*(L_0)] \subseteq B_{j-1}(\mathcal{L}) \cap Z_{j-1}(L_0).$$

Let $\delta \in ID^*(L_0, L_1)$ and $x_0 \in B_{j+1}(\mathcal{L}) \cap Z_{j+1}(L_0)$ we show that $\delta(x_0) \in A_j(\mathcal{L})$. To this end, for all $y_0 \in L_0$, using the Lemma 3.1 we have

$$[x_0, y_0] \in B_j(\mathcal{L}) \cap Z_j(L_0).$$

Also,

$$\delta([x_0, y_0]) =^{x_0} \delta(y_0) -^{y_0} \delta(x_0) \;\Rightarrow\; {}^{y_0}\delta(x_0) =^{x_0} \delta(y_0) - \delta([x_0, y_0]).$$

By given inductive assumption $\delta([x_0, y_0]) \in A_{j-1}(\mathcal{L})$. On the other hand, since $\delta(y_0) \in D_{L_0}(L_1)$ we conclude that ${}^{x_0}\delta(y_0) \in A_{j-1}(\mathcal{L})$. Then, ${}^{y_0}\delta(x_0) \in A_{j-1}(\mathcal{L})$. By using the Lemma 3.1, $\delta(x_0) \in A_j(\mathcal{L})$. Consequently,

$$[B_{j+1}(\mathcal{L}) \cap Z_{j+1}(L_0), ID^*(L_0, L_1)] \subseteq A_j(\mathcal{L}). \tag{9}$$

Let $(\alpha, \beta) \in ID^*(\mathcal{L})$ and $x_1 \in A_{j+1}(\mathcal{L})$ we show that $\alpha(x_1) \in A_j(\mathcal{L})$. To this end, for all $x_0 \in L_0$, using the Lemma 3.1 we have

$${}^{x_0}x_1 \in A_j(\mathcal{L}).$$

On the other hand,

$$\alpha({}^{x_0}x_1) =^{x_0} \alpha(x_1) +^{\beta(x_0)} x_1 \;\Rightarrow\; {}^{x_0}\alpha(x_1) = \alpha({}^{x_0}x_1) -^{\beta(x_0)} x_1.$$

By given inductive assumption, it is clear that $\alpha({}^{x_0}x_1) \in A_{j-1}(\mathcal{L})$. Also, since $\beta \in ID^*(L_0)$ then there exists $y_0, z_0 \in L_0$ such that $\beta(x_0) = [y_0, z_0]$. Moreover, using the Lemma 3.1, it is easily seen that ${}^{\beta(x_0)}x_1 \in A_{j-1}(\mathcal{L})$. Hence, ${}^{x_0}\alpha(x_1) \in A_{j-1}(\mathcal{L})$, and using the Lemma 3.1, $\alpha(x_1) \in A_j(\mathcal{L})$. Consequently,

$$[A_{j+1}(\mathcal{L}) \cap ID^*(L_1)] \subseteq A_j(\mathcal{L}). \tag{10}$$

On the other hand, since $x_0 \in B_{j+1}(\mathcal{L}) \cap Z_{j+1}(L_0)$ then for all $l_0 \in L_0$, using the Lemma 3.1

$$[x_0, l_0] \in B_j(\mathcal{L}) \cap Z_j(L_0).$$

Now, using a similar method, we can easily conclude that

$$\beta(x_0) \in B_j(\mathcal{L}) \cap Z_j(L_0).$$

Thus,

$$[B_{j+1}(\mathcal{L}) \cap Z_{j+1}(L_0), ID^*(L_0)] \subseteq B_j(\mathcal{L}) \cap Z_j(L_0). \tag{11}$$

By using (9), (10) and (11), we have

$$[Z_{j+1}(\mathcal{L}), ID^*\mathrm{Act}(\mathcal{L})] \subseteq Z_j(\mathcal{L}).$$

Then for $i = 1$, (8) holds.
In the following, assume that for i, (8) holds. Hence, we have

$$[B_j(\mathcal{L}) \cap Z_j(L_0), D^i_{ID^*(\mathcal{L})}(ID^*(L_0, L_1))] + [A_j(\mathcal{L}), {ID^*}^i(L_1)] \subseteq A_{j-i}(\mathcal{L}),$$

$$[B_j(\mathcal{L}) \cap Z_j(L_0), {ID^*}^i(L_0)] \subseteq B_{j-i}(\mathcal{L}) \cap Z_{j-i}(L_0).$$

Let $\delta \in D^{i+1}_{ID^*(\mathcal{L})}(ID^*(L_0, L_1))$ and $x_0 \in B_j(\mathcal{L}) \cap Z_j(L_0)$, thus, there exist $\delta_1 \in D^i_{ID^*(\mathcal{L})}(ID^*(L_0, L_1))$ and $(\alpha, \beta) \in ID^*(\mathcal{L})$ such that $\delta = {}^{(\alpha,\beta)} \delta_1$. Moreover,

$$\delta(x_0) = {}^{(\alpha,\beta)} \delta_1(x_0) = \alpha\delta_1(x_0) - \delta_1\beta(x_0).$$

By given inductive assumption and (10), then, we have $\alpha(\delta_1(x_0)) \in A_{j-i-1}(\mathcal{L})$. Also, again by inductive assumption and (11), we get $\delta_1(\beta(x_0)) \in A_{j-i-1}(\mathcal{L})$. Consequently,

$$\delta(x_0) \in A_{j-i-1}(\mathcal{L}). \tag{12}$$

Let $(\alpha, \beta) \in {ID^*}^{i+1}(\mathcal{L})$ and $x_1 \in A_j(\mathcal{L})$, thus, there exist $\alpha_1 \in {ID^*}^i(L_1)$ and $\alpha_2 \in ID^*(L_1)$ such that $\alpha = [\alpha_1, \alpha_2]$. Moreover,

$$\alpha(x_1) = [\alpha_1, \alpha_2](x_1) = (\alpha_1\alpha_2 - \alpha_2\alpha_1)(x_1) = \alpha_1\alpha_2(x_1) - \alpha_2\alpha_1(x_1).$$

By given inductive assumption and (10), we have $\alpha_2(\alpha_1(x_1)), \alpha_1(\alpha_2(x_1)) \in A_{j-i-1}(\mathcal{L})$. Consequently,

$$\alpha(x_1) \in A_{j-i-1}(\mathcal{L}). \tag{13}$$

Using the same way, let $x_0 \in B_j(\mathcal{L}) \cap Z_j(L_0)$. Since $\beta \in {ID^*}^{i+1}(L_0)$, thus, there exist $\beta_1 \in {ID^*}^i(L_0)$ and $\beta_2 \in ID^*(L_0)$ such that $\beta = [\beta_1, \beta_2]$. Moreover,

$$\beta(x_0) = [\beta_1, \beta_2](x_0) = (\beta_1\beta_2 - \beta_2\beta_1)(x_0) = \beta_1\beta_2(x_0) - \beta_2\beta_1(x_0).$$

By given inductive assumption and (11), we have $\beta_2\beta_1(x_0), \beta_1\beta_2(x_0) \in B_{j-i-1}(\mathcal{L}) \cap Z_{j-i-1}(L_0)$. Consequently,

$$\beta(x_0) \in B_{j-i-1}(\mathcal{L}) \cap Z_{j-i-1}(L_0). \tag{14}$$

Now, by using (12), (13) and (14), we get

$$[Z_j(\mathcal{L}), (ID^*\mathrm{Act}(\mathcal{L}))^{i+1}] \subseteq Z_{j-i-1}(\mathcal{L}).$$

□

Lemma 4.4. *Let $\mathcal{L} : (L_1, L_0, d)$ be a Lie algebra crossed module and $\mathcal{H} : (H_1, H_0, \Delta_|)$ a subcrossed module of* $\mathrm{Act}(\mathcal{L})$ *such that $\mathcal{H}$ be a subcrossed module of* $\mathrm{ID}^*\mathrm{Act}(\mathcal{L})$ *contains* $\mathrm{InnAct}(\mathcal{L})$. *Then*

$$\mathcal{H} \cap \mathrm{Act}^{Z_j(\mathcal{L})}(\mathcal{L}) = Z_j(\mathcal{H}). \tag{15}$$

Proof. We prove (15) by induction on j. First, by Lemma 3.5 (15) holds for $j = 1$. Now, assume that for j, (15) holds. Hence, we have

$$H_1 \cap \mathrm{Der}^{Z_j(\mathcal{L})}(L_0, L_1) = A_j(\mathcal{H}),$$
$$H_0 \cap \mathrm{Der}^{Z_j(\mathcal{L})}(\mathcal{L}) = B_j(\mathcal{H}) \cap Z_j(H_0).$$

Let $\delta \in H_1 \cap \mathrm{Der}^{Z_{j+1}(\mathcal{L})}(L_0, L_1)$ and $(\alpha, \beta) \in H_0$ are arbitrary. We have

$$^{(\alpha,\beta)}\delta(l_0) = \alpha(\delta(l_0)) - \delta(\beta(l_0)) \quad \forall l_0 \in L_0.$$

Since $\delta(l_0) \in A_{j+1}(\mathcal{L})$ and $\alpha \in \mathrm{ID}^*(L_1)$, using the Lemma 4.3 $\alpha(\delta(l_0)) \in A_j(\mathcal{L})$. Moreover, since $\beta \in \mathrm{ID}^*(L_0)$, then there exist $x_0, y_0 \in L_0$ such that $\beta(l_0) = [x_0, y_0]$. Thus,

$$\delta(\beta(l_0)) = \delta([x_0, y_0]) =^{x_0} \delta(y_0) -^{y_0} \delta(x_0).$$

Now, since $\delta(x_0), \delta(y_0) \in A_{j+1}(\mathcal{L})$, then by Lemma 3.1 we have $\delta(\beta(l_0)) \in A_j(\mathcal{L})$. Consequently,

$$^{(\alpha,\beta)}\delta \in H_1 \cap \mathrm{Der}^{Z_j(\mathcal{L})}(L_0, L_1).$$

Thus, $^{(\alpha,\beta)}\delta \in A_j(\mathcal{H})$, and using the Lemma 3.1, $\delta \in A_{j+1}(\mathcal{H})$. Hence, we conclude that

$$H_1 \cap \mathrm{Der}^{Z_{j+1}(\mathcal{L})}(L_0, L_1) \subseteq A_{j+1}(\mathcal{H}). \tag{16}$$

Conversely, suppose $\delta \in A_{j+1}(\mathcal{H})$. It is clear that $\delta \in H_1$. It is enough to show $\delta \in \mathrm{Der}^{Z_{j+1}(\mathcal{L})}(L_0, L_1)$. Since $\delta \in A_{j+1}(\mathcal{H})$, by the Lemma 3.1, for all $(\alpha, \beta) \in H_0$, $^{(\alpha,\beta)}\delta \in A_j(\mathcal{H})$.

Consider $(\alpha_{l_0}, \beta_{l_0}) \in H_0$, then

$$^{(\alpha_{l_0},\beta_{l_0})}\delta \in A_j(\mathcal{H}) = H_1 \cap \mathrm{Der}^{Z_j(\mathcal{L})}(L_0, L_1) \;\Rightarrow\; ^{(\alpha_{l_0},\beta_{l_0})}\delta(x_0) \in A_j(\mathcal{L}), \quad \forall x_0 \in L_0.$$

Therefore, we have

$$\begin{aligned} \alpha_{l_0}\delta(x_0) - \delta\beta_{l_0}(x_0) &=^{l_0} \delta(x_0) - \delta([l_0, x_0]) \\ &=^{l_0} \delta(x_0) -^{l_0} \delta(x_0) +^{x_0} \delta(l_0) \\ &=^{x_0} \delta(l_0) \in A_j(\mathcal{L}), \quad \forall l_0 \in L_0. \end{aligned}$$

Now, by the Lemma 3.1 $\delta(l_0) \in A_{j+1}(\mathcal{L})$. Thus, $\delta \in \mathrm{Der}^{Z_{j+1}(\mathcal{L})}(L_0, L_1)$. Consequently,

$$A_{j+1}(\mathcal{H}) \subseteq H_1 \cap \mathrm{Der}^{Z_{j+1}(\mathcal{L})}(L_0, L_1). \tag{17}$$

Using (16) and (17)

$$H_1 \cap \mathrm{Der}^{Z_{j+1}(\mathcal{L})}(L_0, L_1) = A_{j+1}(\mathcal{H}).$$

Also, assume $(\alpha, \beta) \in H_0 \cap \mathrm{Der}^{Z_{j+1}(\mathcal{L})}(\mathcal{L})$. We show that $(\alpha, \beta) \in B_{j+1}(\mathcal{H}) \cap Z_{j+1}(H_0)$. To this end, for all $(\alpha', \beta') \in H_0$

$$[(\alpha, \beta), (\alpha', \beta')] = ([\alpha, \alpha'], [\beta, \beta']) = (\alpha\alpha' - \alpha'\alpha, \beta\beta' - \beta'\beta).$$

Consider $x_1 \in L_1$ be arbitrary, then

$$(\alpha\alpha' - \alpha'\alpha)(x_1) = \alpha\alpha'(x_1) - \alpha'\alpha(x_1).$$

Now, since $\alpha'(x_1) \in D_{L_0}(L_1)$, using the Lemma 4.2, $\alpha(\alpha'(x_1)) \in A_j(\mathcal{L})$. On the other hand, by given the assumption, $\alpha(x_1) \in A_{j+1}(\mathcal{L})$, and using the Lemma 4.3, $\alpha'(\alpha(x_1)) \in A_j(\mathcal{L})$. Therefore, for all $x_1 \in L_1$

$$[\alpha, \alpha'](x_1) \in A_j(\mathcal{L}). \tag{18}$$

Also, if $x_0 \in L_0$ be arbitrary, using a similar method, we have

$$[\beta, \beta'](x_0) \in B_j(\mathcal{L}) \cap Z_j(L_0). \tag{19}$$

Using (18) and (19)

$$[(\alpha, \beta), (\alpha', \beta')] \in H_0 \cap \mathrm{Der}^{Z_j(\mathcal{L})}(\mathcal{L}) = B_j(\mathcal{H}) \cap Z_j(H_0).$$

Now, by the Lemma 3.1

$$(\alpha, \beta) \in B_{j+1}(\mathcal{H}) \cap Z_{j+1}(H_0).$$

Conversely, suppose $(\alpha, \beta) \in B_{j+1}(\mathcal{H}) \cap Z_{j+1}(H_0)$. We show that $(\alpha, \beta) \in H_0 \cap \mathrm{Der}^{Z_{j+1}(\mathcal{L})}(\mathcal{L})$. It is clear that $(\alpha, \beta) \in H_0$. It is enough to show $(\alpha, \beta) \in \mathrm{Der}^{Z_{j+1}(\mathcal{L})}(\mathcal{L})$. Let $(\alpha'_{l_0}, \beta'_{l_0}) \in H_0$ be arbitrary, then using the Lemma 3.1 and inductive assumption, we have

$$[(\alpha, \beta), (\alpha'_{l_0}, \beta'_{l_0})] = ([\alpha, \alpha'_{l_0}], [\beta, \beta'_{l_0}]) \in B_j(\mathcal{H}) \cap Z_j(H_0) = H_0 \cap \mathrm{Der}^{Z_j(\mathcal{L})}(\mathcal{L}).$$

Moreover, using the Lemma 3.3, Proposition 2.2 and the above statement, we obtain

$$[\beta, \beta'_{l_0}](x_0) = \beta'_{\beta(l_0)}(x_0) = [\beta(l_0), x_0] \in B_j(\mathcal{L}) \cap Z_j(L_0) \quad \forall x_0 \in L_0.$$

Now, using the Lemma 3.1 $\beta(l_0) \in B_{j+1}(\mathcal{L}) \cap Z_{j+1}(L_0)$. Similarly, it can be shown for all $l_1 \in L_1$, $\alpha(l_1) \in A_{j+1}(\mathcal{L})$. Thus, $(\alpha, \beta) \in \mathrm{Der}^{Z_{j+1}(\mathcal{L})}(\mathcal{L})$, and Consequently,

$$(\alpha, \beta) \in H_0 \cap \mathrm{Der}^{Z_{j+1}(\mathcal{L})}(\mathcal{L}).$$

□

Corollary 4.5. *Let $\mathcal{L} : (L_1, L_0, d)$ be a Lie algebra crossed module. Then*

$$\mathrm{Act}_{pi}(\mathcal{L}) \cap \mathrm{Act}^{Z_j(\mathcal{L})}(\mathcal{L}) = Z_j(\mathrm{Act}_{pi}(\mathcal{L})).$$

Proof. Using the Lemma 4.4, it is clear. □

Theorem 4.6. *Let $\mathcal{L}$ be a Lie algebra crossed module and $\mathrm{Act}_{pi}(\frac{\mathcal{L}}{Z_j(\mathcal{L})})/\mathrm{InnAct}(\frac{\mathcal{L}}{Z_j(\mathcal{L})})$ the nilpotent of class k, then $\mathrm{Act}_{pi}(\mathcal{L})/\mathrm{InnAct}(\mathcal{L})$ is the nilpotent of the maximum class $k+j$. Moreover, if $\mathrm{Act}_{pi}(\frac{\mathcal{L}}{Z_j(\mathcal{L})})/\mathrm{InnAct}(\frac{\mathcal{L}}{Z_j(\mathcal{L})})$ be an obvious crossed module, then $\mathrm{Act}_{pi}(\mathcal{L})/\mathrm{InnAct}(\mathcal{L})$ is the nilpotent of the maximum class j.*

Proof. Since $\mathrm{Act}_{pi}(\frac{\mathcal{L}}{Z_j(\mathcal{L})})/\mathrm{InnAct}(\frac{\mathcal{L}}{Z_j(\mathcal{L})})$ is the nilpotent of the class k, so

$$\mathrm{Act}^{k+1}_{pi}(\frac{\mathcal{L}}{Z_j(\mathcal{L})}) \subseteq \mathrm{InnAct}(\frac{\mathcal{L}}{Z_j(\mathcal{L})}).$$

By given the Lemma 4.1, we have

$$\mathrm{Act}^{k+1}_{pi}(\mathcal{L}) \subseteq \mathrm{Act}_{pi}(\mathcal{L}) \cap \mathrm{Act}^{Z_j(\mathcal{L})}(\mathcal{L}) + \mathrm{InnAct}(\mathcal{L}),$$

and using the Corollary 4.5

$$\mathrm{Act}^{k+1}_{pi}(\mathcal{L}) \subseteq Z_j(\mathrm{Act}_{pi}(\mathcal{L})) + \mathrm{InnAct}(\mathcal{L}).$$

Therefore,

$$\mathrm{Act}^{j+k+1}_{pi}(\mathcal{L}) \subseteq \mathrm{Inn}^{j+1}\mathrm{Act}(\mathcal{L}).$$

Thus, we conclude that $\mathrm{Act}_{pi}(\mathcal{L})/\mathrm{InnAct}(\mathcal{L})$ is the nilpotent of the maximum class $k+j$. □

Definition. Let $\mathcal{L} : (L_1, L_0, d)$ be a Lie algebra crossed module. we define the dimension of $\mathcal{L}$ as follows:

$$\dim \mathcal{L} = (\dim L_1, \dim L_0).$$

Corollary 4.7. *Let $\mathcal{L} : (L_1, L_0, d)$ be a non-abelian Lie algebra crossed module such that $\dim(\mathcal{L}^i/(\mathcal{L}^i \cap Z_j(\mathcal{L}))) \leqslant (1,1)$, then $\mathrm{Act}_{pi}(\mathcal{L})/\mathrm{InnAct}(\mathcal{L})$ is the nilpotent of the maximum class $i+j$.*

Proof. It is proved by considering $\mathcal{L}^i/(\mathcal{L}^i \cap Z_j(\mathcal{L})) \cong (\mathcal{L}/(Z_j(\mathcal{L})))^i$, using Theorem 4.6 and Theorem 3.10 [7]. □

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