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On the Structure of Supercyclic Operators on the Operator Algebra

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Abstract. For a separable Banach space X, let $\mathcal{L}$ denotes the algebra of bounded operators on X endowed with the strong operator topology. In this note we provide a sufficient condition,in terms of open set in the strong operator topology, for a bounded linear mapping acting on $\mathcal{L}$ to be supercyclic with this topology. The main result has some useful consequences and make easy the proof of some related results.

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1. Introduction

Let X be a separable infinite dimensional topological vector space and $L : X \to X$ be a continuous linear map. L is called supercyclic provided there exists some vector $f \in X$ such that the set $\{\lambda L^n f : n \in \mathbb{N} \text{ and } \lambda \in \mathbb{C}\}$ is dense in X. In this case the vector f is called a supercyclic vector for L. When the orbit itself is dense, without the help of the scalar multiples, the operator as well as the vector are called hypercyclic. Note that the separability of the underlying space is needed for both cases. The notation of supercyclicity was invented by Hilden and Wallen [5]. They showed that all unilateral weighted backward shifts are supercyclic, but not a vector is supercyclic for all unilateral weighted backward shifts. Good sources of background information on hypercyclic and supercyclic operators include [3,4].

By the Baire category theorem, the structure of supercyclic operators acting on a Frechet space can be stated in terms of open subsets.

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Theorem 1.1. *Let X be a separable Frechet space and $T \in L(X)$. The following are equivalent:*

(i) T is supercyclic;

(ii) For each pair of non-empty open sets (U, V) of X, there exist $n \in \mathbb{N}$ and $\lambda \in \mathbb{C}$ such that $\lambda T^n(U) \cap V \neq \emptyset$.

Salas [8] gave a sufficient condition for supercyclicity in Frechet spaces which is called the Supercyclicity Criterion and in the following we give a version for normed spaces.

Supercyclicity Criterion. Let T be a bounded operator on a separable Banach space X. If there exist an increasing sequence of integers (n_k), two dense sets $E_1, E_2 \subseteq X$ and a sequence of maps $S_k : D_2 \to X$ such that:

(1) $\|T^{n_k}x\|\|S_k y\| \to 0$ for any $x \in E_1$ and any $y \in E_2$;

(2) $T^{n_k}S_k(y) \to y$ for each $y \in E_2$.

then T is supercyclic.

Suppose that X denotes an infinite dimensional separable real or complex Banach space, and when X is a Hilbert space we use the symbol H. Hypercyclic phenomena in the operator algebra $\mathcal{L}(X)$ of bounded linear operators have been studied in [1,2]. In fact, Chan initiated the study in [1] by investigating hypercyclicity in $\mathcal{L}(H)$ of left multiplication operators $L_T(S) = TS, T \in \mathcal{L}(H)$, and pursued his work in [2] to the Banach space setting together with Taylor. Due to the fact that $\mathcal{L}(X)$ is in general not separable for the operator norm topology, they worked instead with the coarser strong operator topology (SOT), i.e., the topology of pointwise convergence. It follows that $\mathcal{L}(X)$ is always separable for the SOT [2, Corollary 3].

The surprising fact is that no Baire argument is available in $\mathcal{L}(X)$ endowed with this topology as it is far from being a Frechet space. As an extension of hypercyclicity, supercyclicity, on the operator algebra was considered in [7,9]. It was shown that the left multiplication operator L_T on $\mathcal{L}(X)$ with the SOT are supercyclic provided that T satisfies the supercyclicity Criterion [7,9].

In the present paper, applying the idea in [8], we provide a sufficient condition, in terms of SOT-open sets in the unit ball of $\mathcal{L}(X)$, for a bounded operator on $\mathcal{L}(X)$ to be supercyclic in SOT. Using this, we provide a simple proof for supercyclicity of the left multiplication operator L_T, different from what has been proven in [7,9]. Also, it establishes a general criterion for supercyclicity of bounded operators on $\mathcal{L}(X)$ as well.

2. Main Results

In all that follows let $\mathbb{C}_0$ be the complex plane without zero, X will be a separable Banach space, and $\mathcal{L}(X)$ denote the algebra of all bounded operator on X. We use $\mathcal{B}_i$ for denote the open i-ball of $\mathcal{L}(X)$, that is the set of all operators $T \in \mathcal{L}(X)$ with $\|T\| < i$ and $\bar{\mathcal{B}}_i$ used for the close i-ball as well. We agree that only those topological terms with the prefix "SOT" refer to the strong operator topology; otherwise, they refer to the operator norm topology.

Theorem 2.1. *Let $L : \mathcal{L}(X) \to \mathcal{L}(X)$ be a bounded linear mapping. Assume that for each non-empty norm open set $\mathcal{U}$ and non-empty SOT-open set $\mathcal{V}$ in $\mathcal{L}(X)$ that $\mathcal{U} \cap \mathcal{B}_1 \neq \emptyset$ and $\mathcal{V} \cap \mathcal{B}_1 \neq \emptyset$, there exist some integer $r > 0$ and $\lambda \in \mathbb{C}_0$ such that*

$$\lambda L^r(\mathcal{U} \cap \mathcal{B}_1) \cap \mathcal{V} \cap \mathcal{B}_1 \neq \emptyset \mathcal{B}(1)$$

then L is SOT-supercyclic.

Proof. Since X is separable, the bounded subsets of $\mathcal{L}(X)$ are SOT-metrizable and so are first countable. Let $\mathcal{D}$ be a countable SOT-dense subset of $\mathcal{L}(X)$. For each $T \in \mathcal{D}$, let

$$\{\mathcal{V}_{i,j}(T) \cap \bar{\mathcal{B}}_i\}_{j \geqslant 1}$$

be a countable basis of SOT-neighborhoods of T in $\bar{\mathcal{B}}_i$; where $\mathcal{V}_{i,j}(T)$ is a SOT-open subset of $\mathcal{L}(X)$ containing T. Consider the non-empty index set

$$I_{i,j}(T) = \{(r, s) \in \mathbb{N} \times \mathbb{S} : sL^{-r}\mathcal{V}_{i,j}(T) \cap \mathcal{B}_1 \neq \emptyset\}$$

where $\mathbb{S}$ is a countable dense subset of $\mathbb{C}_0$. Then each of the sets

$$G_T(i, j, r, s) = \bigcup_{(n,\lambda) \in \mathbb{N} \times \mathbb{C}_0} \lambda L^{-n}(sL^{-r}\mathcal{V}_{i,j}(T) \cap \mathcal{B}_1)$$

is open in $\mathcal{L}(X)$ and dense in $\bar{\mathcal{B}}_1$ for $(r, s) \in I_{i,j}(T)$. To prove the density, let $T \in \mathcal{D}$, $i, j \in \mathbb{N}$, $(r, s) \in I_{i,j}(T)$ and let $\mathcal{U} \cap \bar{\mathcal{B}}_1$ be a non-empty open set in $\bar{\mathcal{B}}_1$. Then

$$\mathcal{U} \cap \mathcal{B}_1 \neq \emptyset, sL^{-r}\mathcal{V}_{i,j}(T) \cap \mathcal{B}_1 \neq \emptyset \text{ (because } (r, s) \in I_{i,j}(T))$$

and $L^{-r}\mathcal{V}_{i,j}(T)$ is SOT-open set in $\mathcal{L}(X)$. By the assumption,

$$\lambda L^{-n}(sL^{-r}\mathcal{V}_{i,j}(T) \cap \mathcal{B}_1) \cap \mathcal{U} \cap \mathcal{B}_1 \neq \emptyset$$

for some $n \geqslant 1$ and $\lambda \in \mathbb{C}_0$. This implies that $G_T(i, j, r, s)$ is dense in $\bar{\mathcal{B}}_1$. By the Bair Category Theorem for $\bar{\mathcal{B}}_1$, the set

$$H_{\text{sot}}(L) = \bigcap_{T \in \mathcal{D}} \bigcap_{i,j \in \mathbb{N}} \bigcap_{(r,s) \in I_{i,j}(T)} G_T(i, j, r)$$

is also dense in $\bar{\mathcal{B}}_1$. Now, we claim every element of $H_{\text{sot}}(L)$ is SOT-supercyclic vector for L. For this, let $S \in H_{\text{sot}}(L)$ and $\mathcal{V}$ be an arbitrary SOT-open set in $\mathcal{L}(X)$. Then there is a SOT-neighborhood $\mathcal{V}(T)$ of some $T \in \mathcal{D}$ contained in $\mathcal{V}$. Pick a scalar α such that $\alpha V(T) \cap \mathcal{B}_1 \neq \emptyset$. Using (1) for $\mathcal{U} = X$ and $\mathcal{V} = \alpha\mathcal{V}(T)$, we get $\lambda L^r \mathcal{B}_1 \cap \alpha\mathcal{V}(T) \neq \emptyset$ for some integer $r \geqslant 1$ and scalar $\lambda \in \mathbb{C}_0$. Thus,

$$\beta L^r \mathcal{B}_1 \cap \mathcal{V}(T) \neq \emptyset$$

where $\beta = \lambda\alpha^{-1}$. In the sequel, choose a positive integer $i > \|L\|^r$ in such a way that the set $\beta L^r \mathcal{B}_1 \cap \mathcal{V}(T) \cap \bar{\mathcal{B}}_i$ and so $\beta^{-1} L^{-r}(\mathcal{V}(T) \cap \bar{\mathcal{B}}_i) \cap \mathcal{B}_1$ to be nonempty. The first countability of $\bar{\mathcal{B}}_i$ with SOT and the density of $\mathbb{S}$ in $\mathbb{C}_0$ allow us to select an integer $j \in \mathbb{N}$ and $s \in \mathbb{S}$ such that $sL^{-r}(\mathcal{V}_{i,j}(T) \cap \bar{\mathcal{B}}_i) \cap \mathcal{B}_1$ is non-empty. This yields that $sL^{-r}\mathcal{V}_{i,j}(T) \cap \mathcal{B}_1$ is also non-empty and hence $(r, s) \in I_{i,j}(T)$. Since $S \in G_T(i, j, r, s)$, there are $n \in \mathbb{N}$ and $\lambda \in \mathbb{C}_0$, such that

$$s^{-1}\lambda^{-1} L^{n+r} S \in \mathcal{V}_{i,j}(T) \text{ and } \lambda^{-1} L^n S \in \mathcal{B}_1.$$

But $\|\lambda^{-1} L^{r+n} S\| \leqslant \|L\|^r < i$, so $s^{-1}\lambda^{-1} L^{r+n} S \in \mathcal{V}_{i,j}(T) \cap s^{-1}\bar{\mathcal{B}}_i$ which is a subset of $\mathcal{V}(T) \cap s^{-1}\bar{\mathcal{B}}_i$ and so $s^{-1}\lambda^{-1} L^{r+n} S \in \mathcal{V}$. Therefore, S is a SOT-supercyclic vector for L and L is SOT-supercyclic. $\square$

Theorem 2.2. *The left multiplication operator $L_T : \mathcal{L}(X) \to \mathcal{L}(X)$ is SOT-supercyclic if and only if $T \oplus T$ is supercyclic on $X \oplus X$.*

Proof. Let $T \oplus T$ be supercyclic on $X \oplus X$. Fixing a vector $a \in X$ and define $\Lambda : X \oplus X \to \mathcal{L}(X)$ by

$$\Lambda(x \oplus y) = (x + y) \otimes a \quad (x, y \in X).$$

Then Λ is a norm continuous linear map on $X \oplus X$. Let $\mathcal{U}$ and $\mathcal{V}$ be two SOT-open sets in $\mathcal{L}(X)$ such that $\mathcal{U} \cap \mathcal{B}_1 \neq \emptyset$ and $\mathcal{V} \cap \mathcal{B}_1 \neq \emptyset$. Then for $n \in \mathbb{N}$ and $\lambda \in \mathbb{C}_0$,

$$\begin{aligned}
\Lambda^{-1}(\mathcal{U} \cap \mathcal{B}_1 \cap \lambda^{-1} L_T^{-n}(\mathcal{V} \cap \mathcal{B}_1)) &= \Lambda^{-1}(\mathcal{U} \cap \mathcal{B}_1) \cap (\lambda L_T^n \Lambda)^{-1}(\mathcal{V} \cap \mathcal{B}_1) \\
&= \Lambda^{-1}(\mathcal{U} \cap \mathcal{B}_1) \cap (\Lambda(\lambda T \oplus T)^n)^{-1}(\mathcal{V} \cap \mathcal{B}_1) \\
&= \Lambda^{-1}(\mathcal{U} \cap \mathcal{B}_1) \cap \lambda^{-1}(T \oplus T)^{-n}(\Lambda^{-1}(\mathcal{V} \cap \mathcal{B}_1)).
\end{aligned}$$

Since $T \oplus T$ is hypercyclic, the last term of the above relation is non-empty for some $n \in \mathbb{N}$ and $\lambda \in \mathbb{C}_0$. Consequently, the left side is also non-empty and by the pervious theorem L_T is SOT-supercyclic.

For the converse, let L_T be SOT-supercyclic on $\mathcal{L}(X)$ and S be a SOT-supercyclic vector for L_T. Define the linear mapping $\Lambda : \mathcal{L}(X) \to X \oplus X$ by

$$\Lambda(S) = Sa \oplus Sb \quad (S \in \mathcal{L}(X))$$

Hence, Λ is bounded, onto and $\Lambda L_T = (T \oplus T)\Lambda$. Then

$$\begin{aligned}\Lambda(\mathbb{C}\mathrm{orb}(T \oplus T, \Lambda(S))) &= \{\lambda(T \oplus T)^n \Lambda(S) : n \in \mathbb{N}, \lambda \in \mathbb{C}\} \\ &= \{\lambda \Lambda L_T^n(S) : n \in \mathbb{N}, \lambda \in \mathbb{C}\} = \Lambda(\mathbb{C}\mathrm{orb}(L_T, S))\end{aligned}$$

Since S is a SOT-supercyclic vector for L_T and Λ is bounded, $\Lambda(S)$ is a supercyclic vector for $T \oplus T$ and hence it is supercyclic on $X \oplus X$. $\square$
We next give a Supercyclicity Criterion on the operator algebra $\mathcal{L}(X)$

Proposition 2.3. *Let $L : \mathcal{L}(X) \to \mathcal{L}(X)$ be a bounded linear mapping. Suppose that there exist two SOT-dense subsets Y and Z in $\mathcal{B}_1$ such that for all $T \in Y \cap \mathcal{B}_1$ and $S \in Z \cap \mathcal{B}_1$, there exist a sequence $(n_k) \subseteq \mathbb{N}$ and functions $R_k : Z \to \mathcal{L}(X)$ for which*

(i) $\|L^{n_k}T\|\|R_kS\| \to 0$

(ii) $L^{n_k}R_kS \to S$

Then L is SOT-supercyclic.

Proof. Let U and V be two SOT-open set in $B(X)$ which have nonempty intersections with $B_1(X)$. Pick $T \in U \cap Y \cap \mathcal{B}_1$ and $S \in V \cap Z \cap \mathcal{B}_1$. Put $\lambda_k = \|R_k(S)\|$, by condition (ii) it can be assumed $\lambda_k \neq 0$ for sufficiently large k. Let $T_k = T + \lambda_k^{-1}R_k(S)$. Then $\lambda_k\|L^{n_k}T_k\| \to S$ and since $\|S\| < 1$, $\lambda_k\|L^{n_k}(T)\| < 1$ for sufficiently large k. It gives that for a certain k,

$$\lambda_k L^{n_k}(U \cap \mathcal{B}_1) \cap V \cap \mathcal{B}_1 \neq \emptyset.$$

Now from Theorem 2.1, we conclude that L is SOT-hypercyclic. $\square$
Remember that the ideal of finite rank operators in $\mathcal{L}(X)$ is SOT-dense in $\mathcal{L}(X)$ [1,2]. Consider the non-empty SOT-neighborhood

$$U = \{S \in \mathcal{L}(X) : \|(S-T)(x_i)\| < \varepsilon_i, \text{ for } 1 \leqslant i \leqslant m\}$$

of $T \in \mathcal{B}_1$. Since $\mathrm{span}\{x_1, ...x_m\}$ is finite-dimensional, it is complemented in X. Therefore, we can take a projection P from X onto $\mathrm{span}\{x_1, ..., x_m\}$. The finite rank operator TP belongs to U since $TPx_i = Tx_i$ for $1 \leqslant i \leqslant m$. Also, $\|TP\| \leqslant \|T\| < 1$. These observations show that, the set of finite rank operators is also SOT-dense in $\mathcal{B}_1$. Moreover, if E is a dense subset of X, then the subset $D(E)$ of $\mathcal{L}(X)$, consisting of only finite rank operators whose range is contained in the span of E, is a SOT-dense in $\mathcal{B}_1$. In fact, each finite rank operator in $\mathcal{L}(X)$ can be approximated arbitrarily close, in the operator norm, by an operator in $D(E)$. Note that any operator $T \in D(E)$ can be represented as $T = \sum_{k=1}^{n} f_i \otimes x_i$ where $\{x_1, ..., x_n\}$ is a basis in $\mathrm{ran}(T) \subseteq E$, $f_1, f_2, ...f_n$ are bounded functionals on X and $f_i \otimes x_i(x) = f_i(x)x_i$. We conclude this paper by an application of Proposition 2.3.

Theorem 2.4. *Assume that T satisfies the Supercyclicity Theorem on X, then L_T is SOT-supercyclic on $\mathcal{L}(X)$.*

Proof. Let T satisfies the hypothesis of Supercyclicity Criterion for two dense subsets E_1 and E_2, positive integer sequence (n_k) and sequence of maps $S_k : E_2 \to X$ satisfying conditions (1) and (2) of criterion. For $T_1 \in D(E_1)$ with $T_1 = \sum_{i=1}^{m} f_i \otimes x_i$, we have $L_T^n(T_1) = \sum_{i=1}^{m} f_i \otimes T^n x_i$ and for $T_2 \in D(E_2)$ with $T_1 = \sum_{i=1}^{m} g_i \otimes y_i$. we define $R_k(T_2) = \sum_{i=1}^{m} g_i \otimes S_k y_i$. Now, $D(E_1)$ and $D(E_2)$ are SOT-dense in $\mathcal{B}_1$ and it is easy to check that, the bounded linear mapping $L = L_T$ and sequence of maps R_k satisfy the hypothesis of Proposition 2.3 and hence L_T is SOT-supercyclic on $\mathcal{L}(X)$. $\square$

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