A novel inverse DEA-R model for inputs/output estimation

Abstract
In this paper, we propose inverse data envelopment analysis (DEA) models in the presence of ratio data. We present the inputs/output estimation process based on ratio based DEA (DEA-R) models. We first present a multiple objective linear programming (MOLP) model to determine the level of inputs based on the perturbed outputs, assuming that the relative efficiency of the under evaluation decision making unit (DMU) preserve. We also present the relationship between the Pareto solutions of the proposed MOLP model and the optimal level of inputs and outputs of the new DMU. We presented criterion models to determine the efficiency of the new DMU in the inputs/output estimation process based on inverse DEA-R models in the presence of ratio data. We showed that in the presence of ratio data the selection of criterion model can be important, in order to we provide a new criterion model in the inputs/output estimation process in the presence of ratio data, and so on the amount of calculations is reduced. We have shown that the results for the new criterion model are the same as the existing criterion model presented in the paper. In order to show the validity of the proposed approach in the inputs/output estimation process based on the inverse DEA-R models, we provide an application of our models in a real life for a set of data regarding to medical centers in Taiwan and finally we present the research results.
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Keywords and Phrases: Data envelopment analysis, Multiple objective linear programming, Input/output estimation, Rati data; Criterion model, Inverse DEA.

1. Introduction

Traditional DEA models determine the efficiency of DMUs based on their inputs and outputs. However, in inverse DEA, the efficiency of the DMU is predetermined by the decision-maker (DM), and based on this score of efficiency, the optimal level of inputs or outputs are determined. This amount of efficiency that is predetermined by the DM is called the target efficiency. The concept of inverse DEA was first present by Wei et al. [39] and then by Yan et al. [38] developed on the issue of resource allocation. Hadi-Vencheh and Foroughi [24] proposed a generalized inverse DEA mode based on the model of Wei et al. [39]. They showed that some special cases of the inverse DEA model proposed by Wei et al. [39] may fail in some situations and then they revised these failures. Lertworasirikul et al. [28] considered the issue of inverse DEA by considering two different strategies. In the first strategy, by determining the specific level of efficiency for each unit under evaluation DMU, they determined the best possible level of inputs corresponding to a given level of outputs. In the second strategy, again considering a specific level for the efficiency for the under evaluation DMU, they determined the best possible level of outputs corresponding to a level of given inputs and presented their models as resource allocation models. But the early models that they presented were nonlinear models. Due to the problems in solving nonlinear models, they presented new their inverse DEA model in the form of MOLP model. In the following, Ghiyasi [20] points out the drawbacks of Lertworasirikul et al. [28] and then revised the use of MOLP in the proposed inverse DEA model considering the variable return to scale technology (VRS) (Banker, Charnes and Cooper [9]).

Gattouf et al. [18] presented a new model of inverse DEA on mergers and acquisitions to estimate the optimal level of inputs and outputs for the merged entity for a given target efficiency value. Amin et al. [4] presented a general model on mergers and acquisitions. They presented a generalized firm restructuring in two scenarios in the form of consolidation or a split. They considering a set of DMUs that called pre-restructuring DMUs, they produced a set of new DMUs that called post-restructuring DMUs, and the level of inputs and outputs from post-restructuring DMUs are determined based on the level of inputs and outputs of pre-restructuring DMUs also the efficiency scores of post-restructuring DMUs are predetermined by the DM as target efficiency scores. Emrouznejad et al. [15] proposed a new application of inverse DEA in environmental efficiency to determine the optimal allocation of CO2 emissions reduction by Chinese manufacturing industries. Wegener and Amin [35] suggested an inverse DEA model for minimizing greenhouse gas emissions with an application in oil and gas. Other applications of inverse DEA including an application in resource allocation (Ghiyasi [21, 23]), new product target setting given expected changes of production frontier (Lim [29]), inter-temporal dependence (Jahanshahloo et al. [26]), revenue setting problems of chain stores, inverse DEA models based on cost and revenue efciency (Ghiyasi [22]), application of the inverse DEA to sensitivity analysis of DMUs (Eyni et al. [16]). Amin and Al-Muharrami [2] addresses the model of inverse DEA in the mergers and acquisitions of firms with negative data. Amin et al. [3] suggested a combined inverse DEA and goal programming approach for target setting of mergers as allows DM to incorporate their preferences. Emrouznejad and Yang [14] presented a literature review of DEA and inverse DEA.

Amin and Ibn Boamah [6] proposed a new model of inverse DEA for estimating potential merger gains based on cost efficiency and used the proposed approach in the Canadian banking industry. Amin and Ibn Boamah [7] presented an inverse DEA approach for the two-stage network in the US banking sector.

In the real world, there are many cases in which data are ratio and the ratio of input data to output data or vice versa is important to the DM or input/output data is presented in the form of ratio or percentage data. Traditional DEA models can no longer be used to evaluate the efficiency of DMUs if the ratio of input to output or vice versa is important to the DM, or if the input and output data are ratio data. We need to develop DEA models and in this situation we use DEA-R models. In general, we divide ratio data into three categories as follows.

The first category includes ratio data in which the input and output data of ratio numbers are in the form of a fraction and the numerator and denominator corresponding to these fractions are known, but the DM can use this ratio data in the form of decimal numbers in the model. In this case, the data are used in both absolute and ratio forms in the efficiency evaluation model. In the presence of ratio data, the principle of convexity in underlying assumptions of the production possibility set (PPS) is not established in DEA. Among the articles that have been presented in this category to deal with ratio data in DEA models, the following articles can be mentioned. Emrouznejad and Amin [13], Hatami-Marbini and Toloo [25], Khoshnevis and Teirlinck [27].

The above articles modified DEA models to evaluate efficiency in the presence of ratio data. In this category, the numerator and denominator corresponding to these fractions corresponding to the ratio data are known, but the nature of the data is ratio.

The second category includes ratio data in which the ratio data are in the form of a fraction and the numerator and denominator corresponding to these fractional numbers may not be available and we have ratio numbers only available as decimal numbers or percentages. It is known and we must use these decimal numbers as ratio data in the model. From a series of articles that modified DEA models and change the underlying assumptions of the PPS in constant and variable return to scale technologies in the presence of ratio data. These papers provide new DEA models to calculate efficiency in the presence of absolute and ratio data. Among the articles that have been presented in this category to deal with ratio data in DEA models, the following articles can be mentioned. Olesen, Petersen and Podinovski [33, 34].

The third category includes ratio data in which the ratio data are in the form of a fraction and the numerator and denominator corresponding to these fractional numbers are important for the DM and the DM cannot use these fractions as decimal numbers in the model. These ratio data are in the form of the ratio of components input to components output or vice versa. These models were initially presented as ratio analysis models (Fernandez-Castro and Smith [17]). In these models, we must use the ratio of inputs to outputs and vice versa in the model. Among the articles that have been presented in this category to deal with ratio data in DEA models, the following articles can be mentioned.

Fernandez-Castro and Smith [17], Despic, Despic, and Paradi [12], Wei et al. [36, 37, 38], Mozaffari et al. [32], Mozaffari, Gerami and Jablonsky [31], Gerami et al. [19, 20], Mozaffari et al. [21].

Despic, Despic and Paradi [12] presented DEA-R models by combining ratio analysis and DEA models. They proposed DEA-R models in the output orientation to calculate the efficiency of DMUs in the presence of ratio data as the ratio of components output to components input.

Wei et al. [36, 37, 38] examined DEA-R models in the input orientation. They showed that by using DEA-R models in the input orientation, we can avoid the available problems in of traditional DEA models such as efficiency underestimation and pseudo-inefficiency. They showed that DEA-R models in the input orientation have higher efficiency scores than their corresponding scores from CCR models in the input orientation. Mozaffari et al. [32] used DEA-R models to evaluate cost and revenue efficiency. Gerami et al. [20] used DEA-R models to evaluate the efficiency of the hospital supply chain in the presence of ratio data. Gerami, Mozaffari, and Wanke [21] proposed DEA-R models to evaluate the efficiency of two-stage network structures in the presence of ratio data.

In the first category of ratio data, we can refer to the ratio of the number of research projects presented by some professors in a course to the total number of professors in a university, and these ratios are important for the DM. DM used of the numerator and denominator corresponding ratio data in the model. In the second category of ratio data can be referred to the percentage of successful operations performed to the total number of operations performed in a hospital during a treatment period. But the DM only uses the decimal form of this data and this data is only available in the form of decimal numbers. In the third category of data, we can refer to some concepts in economics such as immediate and current profit, or the number of patients treated to the total number of patients admitted to a hospital during a treatment period. It should be noted that in the third category of data, we use the input and output data of each of the DMUs directly in the model and put this data as the form of the ratio of components input to components output or vice versa in the model, but what is important is that we assume that the input and output data are definite and their ratio is important for the DM.

In this paper, we use the ratio data in the third category and assume that the input and output data are definite numbers and their ratio is important for the DM, and we put this data as the form of the ratio of components input to components output in the models and do not use the fractional or decimal form of these numbers.

It can be said that the main contribution of the article is as follows. In this paper, we examine one of the most important issues in DAE, namely inverse DEA, and estimate inputs and outputs if some of the input and output components change and the DM wants create a new DMU with a relative efficiency score that predetermined and is equal to relative efficiency score of the initial unit. In the process of estimating the level of inputs and outputs, we can choose two different strategies in inverse DEA models in the presence of ratio data. In the first strategy, by determining the specific level of efficiency for each unit under evaluation DMU, they determined the best possible level of inputs corresponding to a given level of outputs. In the second strategy, again considering a specific level for the efficiency for the under evaluation DMU, they determined the best possible level of outputs corresponding to a level of given inputs. That is, if we want the efficiency of the DMU to remain unchanged, we determine the optimal level of input or output based on DEA-R models. We obtain the necessary and sufficient conditions for inverse DEA-R in the input orientation models. As we know, one of the important issues in inverse DEA is the selection of a criterion model for comparing the efficiency scores of DMU before and after the process of estimating inputs and outputs. In this paper, we first develop inverse DEA models in the presence of ratio data, and by providing a suitable criterion model in the presence of ratio data, we show that we can significantly reduce the computations and thus show that the new criterion model presented have the same results as the previous criterion models. Finally, we provide a case study to examine the validity of the proposed models.

The remainder of the paper unfolds as follows. In the Section 2, we examine DEA-R models in the input and output orientations and present the relationship between these models and traditional DEA models. Section 3 proposes the inverse DEA-R models and the inputs/output estimation process based on inverse DEA-R models in the presence of ratio data, in following, we present the criterion models for evaluating the efficiency of the new units created. Section 4 provides a numerical example, in this way, we illustrate the inputs/output estimation process based on inverse DEA-R models in the presence of ratio data. Section 4 provides a real world data empirical investigation and shows the applicability and potential use of the proposed models, we present an application of the proposed approach related to medical centers in Taiwan and at the end, we present the results of the research.

2. Ratio-based DEA models.

Suppose we have n decision units as $DMU_j = \left(x_j, y_j\right),\ j = 1, \ldots, n$. The input and output vectors corresponding to $DMU_j,\ j = 1, \ldots, n$ as $x_j = \left(x_{1j}, \ldots, x_{mj}\right)$ and $y_j = \left(y_{1j}, \ldots, y_{sj}\right)$. We suppose that $x_{ij} > 0, y_{rj} > 0,\ i = 1, \ldots, m,\ r = 1, \ldots, s,\ \ j = 1, \ldots, n$. Suppose the ratios $\frac{x_j}{y_j},\ j = 1, \ldots, n$, in the input orientation and the ratios $\frac{y_j}{x_j}, j = 1, \ldots, n$, in the output orientation are defined.

Suppose, we consider the multiples corresponding to the ratios $\frac{x_{ij}}{y_{rj}},\ j = 1, \ldots, n,\ i = 1, \ldots, m,$

$r = 1, \dots, s$, as w_{ir}. Fernandez-Castro and Smith [17] proposed ratio analysis model in the input orientation as follows.

$$\begin{aligned} \min\ & \sum_{r=1}^{s}\sum_{i=1}^{m} w_{ir}\left(\frac{x_{io}}{y_{ro}}\right) \\ s.t.\ & \sum_{r=1}^{s}\sum_{i=1}^{m} w_{ir}\left(\frac{x_{ij}}{y_{rj}}\right) \geq 1,\quad j = 1, \dots, n, \\ & w_{ir} \geq 0,\ i = 1, \dots, m,\ \ r = 1, \dots, s. \end{aligned} \tag{1}$$

We consider the variable corresponding to first constraint in model (1) as $\hat{\mu}_j$, $j = 1, \dots, n$. The dual model (1) is as follows.

$$\begin{aligned} max\ & \sum_{j=1}^{n} \hat{\mu}_j \\ s.t.\ & \sum_{j=1}^{n} \hat{\mu}_j \left(\frac{x_{ij}}{y_{rj}}\right) \leq \left(\frac{x_{io}}{y_{ro}}\right),\quad i = 1, \dots, m,\ r = 1, \dots, s \\ & \hat{\mu}_j \geq 0,\ j = 1, \dots, n. \end{aligned} \tag{2}$$

By considering $\sum_{j=1}^{n} \hat{\mu}_j = t$, $\mu_j = \frac{\hat{\mu}_j}{t}$ and placing $\theta_R = \frac{1}{t}$ from the optimization point of the model (2) is converted as follows.

$$\begin{aligned} \theta_R^I = & Min\ \theta_R \\ s.t.\ & \sum_{j=1}^{n} \mu_j \left(\frac{x_{ij}}{y_{rj}}\right) \leq \theta_R\left(\frac{x_{io}}{y_{ro}}\right),\quad i = 1, \dots, m,\ r = 1, \dots, s \\ & \sum_{j=1}^{n} \mu_j = 1,\quad \mu_j \geq 0,\ \ j = 1, \dots, n. \end{aligned} \tag{3}$$

Model (3) is called the DEA-R model in the input orientation in the envelopment form. Model (3) by Wei et al. [36, 37, 38] and Mozaffari et al. [31] were also studied. We now examine the relationship between the above model (1) to traditional DEA model.

Theorem 1. Model (1) is equivalent to the CCR multiplier model in the input orientation.

Proof: If we define $\widehat{DMU}_j = \left(\frac{x_{ij}}{y_{rj}}, 1\right)$, $j = 1, \dots, n$, $i = 1, \dots, m$, $r = 1, \dots, s$. Then we have n DMUs with one output and $m + s$ input. Considering the multiples corresponding to the input components of the new units as $w_{ir} \geq 0$, $i = 1, \dots, m$, $r = 1, \dots, s$, and the multiples corresponding to the output component of the new units as u_1. Then model (1) is converted as follows.

$$\begin{aligned} \min\ & \sum_{r=1}^{s}\sum_{i=1}^{m} w_{ir}\left(\frac{x_{io}}{y_{ro}}\right) \\ s.t.\ & \sum_{r=1}^{s}\sum_{i=1}^{m} w_{ir}\left(\frac{x_{ij}}{y_{rj}}\right) - 1u_1 \geq 1,\quad j = 1, \dots, n, \\ & 1u_1 = 1,\ \ u_1 \geq 0,\ w_{ir} \geq 0,\ i = 1, \dots, m,\ r = 1, \dots, s. \end{aligned} \tag{4}$$

That model (4) is CCR multiplier model in the input orientation (Charnes, Cooper and Rhodes [10]) in evaluation $\widehat{DMU}_o = \left(\frac{x_{io}}{y_{ro}}, 1\right)$, $j = 1, \dots, n$, $i = 1, \dots, m$, $r = 1, \dots, s$. If the set of DMUs be as $\widehat{DMU}_j = \left(\frac{x_{ij}}{y_{rj}}, 1\right)$, $j = 1, \dots, n$, $i = 1, \dots, m$, $r = 1, \dots, s$, which completes the proof.■

Similarly, suppose, we consider the multiples corresponding to the ratios $\frac{y_{rj}}{x_{ij}},\ j=1,\dots,n,\ i=1,\dots,m,\ r=1,\dots,s$, as w_{ir}. Fernandez-Castro and Smith [17] proposed ratio analysis model in the output orientation as follows.

$$\begin{aligned} &\max\ \sum_{r=1}^{s}\sum_{i=1}^{m} u_{ir}\left(\frac{y_{ro}}{x_{io}}\right) \\ &s.t.\ \ \sum_{r=1}^{s}\sum_{i=1}^{m} u_{ir}\left(\frac{y_{rj}}{x_{ij}}\right) \le 1, \quad j=1,\dots,n, \\ &\qquad\quad u_{ir} \ge 0,\ i=1,\dots,m, r=1,\dots,s. \end{aligned} \tag{5}$$

We consider the variable corresponding to first constraint in model (5) as $\hat{\lambda}_j$, $j=1,\dots,n$. The dual model (5) is as follows.

$$\begin{aligned} &min\ \sum_{j=1}^{n}\hat{\lambda}_j \\ &s.t.\ \ \sum_{j=1}^{n}\hat{\lambda}_j\left(\frac{y_{rj}}{x_{ij}}\right) \le \left(\frac{y_{ro}}{x_{io}}\right), \quad i=1,\dots,m,\ r=1,\dots,s \\ &\qquad \hat{\lambda}_j \ge 0, \quad j=1,\dots,n. \end{aligned} \tag{6}$$

By considering $\sum_{j=1}^{n}\hat{\lambda}=t,\ \ \lambda_j=\frac{\hat{\lambda}_j}{t}$, and placing $\varphi_R=\frac{1}{t}$, from the optimization point, the model (5) is converted as follows.

$$\begin{aligned} &\varphi_R^O = max\ \varphi_R \\ &s.t.\ \ \sum_{j=1}^{n}\lambda_j\left(\frac{y_{rj}}{x_{ij}}\right) \le \varphi_R\left(\frac{y_{ro}}{x_{io}}\right), \quad i=1,\dots,m,\ r=1,\dots,s \\ &\qquad \lambda_j \ge 0,\ \ j=1,\dots,n. \end{aligned} \tag{7}$$

Model (7) is called the DEA-R model in the output orientation in the envelopment form. Model (7) by Despic, Despic and Paradi [12] were also studied. We now examine the relationship between the above model (5) to traditional DEA model.

Theorem 2. Model (5) is equivalent to the CCR multiplier model in the output orientation.

Proof: If we define $\widetilde{DMU}_j=\left(1,\frac{y_{rj}}{x_{ij}}\right),\ \ j=1,\dots,n,\ \ \ i=1,\dots,m,\ \ \ r=1,\dots,s$. Then we have n DMUs with one input and $m+s$ output. Considering the multiples corresponding to the output components of the new units as $u_{ir}\ge 0,\ i=1,\dots,m,r=1,\dots,s$, and the multiples corresponding to the input component of the new units as v_1. Then model (5) is converted as follows.

$$\begin{aligned} &\max\ \sum_{r=1}^{s}\sum_{i=1}^{m} u_{ir}\left(\frac{y_{ro}}{x_{io}}\right) \\ &s.t.\ \ \sum_{r=1}^{s}\sum_{i=1}^{m} u_{ir}\left(\frac{y_{rj}}{x_{ij}}\right) - 1v_1 \le 1, \quad j=1,\dots,n, \\ &\qquad\quad u_{ir} \ge 0,\ i=1,\dots,m, r=1,\dots,s. \end{aligned} \tag{8}$$

That model (8) is CCR multiplier model in the output orientation (Charnes, Cooper and Rhodes [10]) in evaluation $\widetilde{DMU}_o=\left(1,\frac{y_{ro}}{x_{io}}\right),\ \ j=1,\dots,n,\ \ \ i=1,\dots,m,\ \ \ r=1,\dots,s$,

if we consider the set of DMUs as $\widetilde{DMU}_j=\left(1,\frac{y_{rj}}{x_{ij}}\right),\ \ j=1,\dots,n,\ \ \ i=1,\dots,m,\ \ r=1,\dots,s$, which completes the proof.■

3. The inputs/output estimation process based on inverse DEA-R.

In this section, we present inverse DEA-R models in the presence of ratio data in the input orientation. In this regard, we present inputs/output estimation process based on inverse DEA-R models. We provide criterion models to evaluate the efficiency of new units. In other words, we need to find the new input level of under evaluation DMU that guarantees unchanged relative efficiency for this DMU.

Suppose we have n DMUs as $DMU_j = (x_j, y_j),\ j = 1, \dots, n$ that each DMU consume input vector $x_j = (x_{1j}, \dots, x_{mj})$ to product output vector $y_j = (y_{1j}, \dots, y_{sj})$. We suppose that $x_{ij} > 0$, $y_{rj} > 0,\ i = 1, \dots, m,\ r = 1, \dots, s,\ \ j = 1, \dots, n$. Suppose the ratios $\frac{x_{ij}}{y_{rj}},\ j = 1, \dots, n$, are the ratio ith input component to rth output component of $DMU_j = (x_j, y_j),\ j = 1, \dots, n$. We show under evaluation DMU as $DMU_o = (x_o, y_o)$, also, assume DMU_o perturbs its output level into $\eta^o = y_o + \Delta y_o,\ \Delta y_o \geq -y_o, \Delta y_o \in R$. Now we want to know how much we need to change the input level of $DMU_o = (x_o, y_o)$, so that the relative efficiency of this unit remains unchanged. In other words, we first perturbs the output level of $DMU_o = (x_o, y_o)$ to a certain extent, and then, we must determine the input level of the new DMU namely $\gamma^o = x_o + \Delta x_o,\ \Delta x_o \geq -x_o, \Delta x_o \in R$, in such a way that the relative efficiency of the new unit is equal to the relative efficiency of $DMU_o = (x_o, y_o)$.

We proposed the following MOLP model in the inverse DEA-R and in the presence of ratio data to determine $\gamma^o = x_o + \Delta x_o,\ \Delta x_o \geq -x_o, \Delta x_o \in R$, as follows.

$$\begin{aligned} &\min\ (\gamma_1, \gamma_2, \dots, \gamma_m) \\ &s.t.\quad \sum_{j=1}^{n} \mu_j \left(\frac{x_{ij}}{y_{rj}}\right) \leq \theta_R^I \left(\frac{\gamma_i}{\eta_r^o}\right), i = 1, \dots, m,\ r = 1, \dots, s, \\ &\qquad \sum_{j=1}^{n} \mu_j = 1,\ \mu_j \geq 0, j = 1, \dots, n. \end{aligned} \tag{9}$$

Definition 1. $DMU_o = (x_o, y_o)$ is called a weak efficient solution in evaluation with model (3) if the optimal value of model (3) is equal to one.

Definition 2. Suppose (μ, γ) that $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_m)$ and $\mu = (\mu_1, \dots, \mu_n)$ are a feasible solution of model (9). If there does not exist a feasible solution $(\bar{\mu}, \bar{\gamma})$ of model (9) such that $\bar{\gamma} < \gamma$ then (μ, γ) will be a weakly efficient solution of model (9).

A weak efficient solution of model (9) are as new input values from $DMU_o = (x_o, y_o)$ for a disturbed output level $\eta^o = y_o + \Delta y_o, \Delta y_o \geq -y_o, \Delta y_o \in R$, to preserve relative efficiency of $DMU_o = (x_o, y_o)$ after the output changes. At first, to check the relative efficiency of the new unit namely $DMU_o^{new} = (x_o + \Delta x_o, y_o + \Delta y_o) = (\gamma^o, \eta^o)$, we present the following criterion model.

$$\begin{aligned} &\theta_R^+ = Min\ \theta_R \\ &s.t.\quad \sum_{j=1}^{n} \mu_j \left(\frac{x_{ij}}{y_{rj}}\right) + \mu_{n+1} \left(\frac{\gamma_i^o}{\eta_r^o}\right) \leq \theta_R \left(\frac{\gamma_i^o}{\eta_r^o}\right), \quad i = 1, \dots, m,\ r = 1, \dots, s, \\ &\qquad \sum_{j=1}^{n} \mu_j + \mu_{n+1} = 1,\ \mu_j \geq 0,\ j = 1, \dots, n,\ \mu_{n+1} \geq 0. \end{aligned} \tag{10}$$

Theorem 3. Suppose that $DMU_o = (x_o, y_o)$ perturbs its output from y_o to $\eta^o = y_o + \Delta y_o, \Delta y_o \geq -y_o, \Delta y_o \in R$. Then (γ^o, μ) is a weak efficient solution of MOLP model (9) if and only if $\theta_R^+ = \theta_R^I$.

Proof: First assume that (γ^o, μ) is a weak efficient solution of MOLP model (9). We show that the efficiency score of $DMU_o = (x_o, y_o)$ and $DMU_o^{new} = (x_o + \Delta x_o, y_o + \Delta y_o) = (\gamma^o, \eta^o)$ are equal, i.e. $\theta_R^+ = \theta_R^I$. Put $\mu' = (\mu, 0)^T$, it is easily seen that (θ_R^I, μ') is a feasible solution for model (10), so we will have $\theta_R^+ \leq \theta_R^I$. Now suppose $\theta_R^+ < \theta_R^I$, let that (θ_R^+, μ^+) is an optimal solution of model (10), so according to the constraints of model (10), we will have

$$\sum_{j=1}^{n} \mu_j^+ \left(\frac{x_{ij}}{y_{rj}}\right) + \mu_{n+1}^+ \left(\frac{\gamma_i^o}{\eta_r^o}\right) \leq \theta_R^+ \left(\frac{\gamma_i^o}{\eta_r^o}\right), \quad i = 1, \dots, m, \ r = 1, \dots, s, \qquad (11)$$

$$\sum_{j=1}^{n} \mu_j^+ + \mu_{n+1}^+ = 1, \quad , \mu_{n+1}^+ \geq 0, \mu_j^+ \geq 0, \ j = 1, \dots, n.$$

Given that $\theta_R^I \leq 1$ and (γ^o, μ) is a weak efficient solution of MOLP model (9), so we will have

$$\sum_{j=1}^{n} \mu_j \left(\frac{x_{ij}}{y_{rj}}\right) \leq \theta_R^I \left(\frac{\gamma_i^o}{\eta_r^o}\right) \leq \frac{\gamma_i^o}{\eta_r^o}, \ i = 1, \dots, m, \ r = 1, \dots, s, \qquad (12)$$

$$\sum_{j=1}^{n} \mu_j = 1, \ \mu_j \geq 0, \ j = 1, \dots, n.$$

By comparing relations (11) and (12) we will have

$$\sum_{j=1}^{n} \mu_j^+ \left(\frac{x_{ij}}{y_{rj}}\right) + \mu_{n+1}^+ \left(\frac{\gamma_i^o}{\eta_r^o}\right) \geq \sum_{j=1}^{n} \mu_j^+ \left(\frac{x_{ij}}{y_{rj}}\right) + \mu_{n+1}^+ \left(\sum_{j=1}^{n} \mu_j \left(\frac{x_{ij}}{y_{rj}}\right)\right) = \sum_{j=1}^{n} \Big(\mu_j^+ + \mu_{n+1}^+\big(\sum_{j=1}^{n} \mu_j\big)\Big) \left(\frac{x_{ij}}{y_{rj}}\right), \quad i = 1, \dots, m, \ r = 1, \dots, s, \qquad (13)$$

Now, we put

$$\tilde{\mu}_j = \mu_j^+ + \mu_{n+1}^+\big(\sum_{j=1}^{n} \mu_j\big), \ j = 1, \dots, n.$$

And so according to relation (13), we will have

$$\sum_{j=1}^{n} \tilde{\mu}_j \left(\frac{x_{ij}}{y_{rj}}\right) \leq \theta_R^+ \left(\frac{\gamma_i^o}{\eta_r^o}\right), i = 1, \dots, m, \ r = 1, \dots s, \qquad (14)$$

Given that $\sum_{j=1}^{n} \mu_j = 1$ and $\sum_{j=1}^{n} \mu_j^+ + \mu_{n+1}^+ = 1, \quad \mu_{n+1}^+ \geq 0, \mu_j^+ \geq 0, \ j = 1, \dots, n$. Then

$$\sum_{j=1}^{n} \tilde{\mu}_j = 1, \ \tilde{\mu}_j \geq 0, \ j = 1, \dots, n.$$

Therefore, we have

$$\sum_{j=1}^{n} \tilde{\mu}_j \left(\frac{x_{ij}}{y_{rj}}\right) \leq \theta_R^+ \left(\frac{\gamma_i^o}{\eta_r^o}\right) < \theta_R^I \left(\frac{\gamma_i^o}{\eta_r^o}\right), i = 1, \dots, m, \ r = 1, \dots, s, \qquad (15)$$

$$\sum_{j=1}^{n} \tilde{\mu}_j = 1, \ \tilde{\mu}_j \geq 0, \ j = 1, \dots, n.$$

Therefore $(\gamma^o, \tilde{\mu})$ that $\tilde{\mu} = (\tilde{\mu}_1, \dots, \tilde{\mu}_n)$ will be a feasible solution for model (9). According to relation (15) there exists a $0 < t < 1$ such that

$$\sum_{j=1}^{n} \tilde{\mu}_j \left(\frac{x_{ij}}{y_{rj}}\right) \leq \theta_R^+ \left(\frac{\gamma_i^o}{\eta_r^o}\right) \leq t\theta_R^I \left(\frac{\gamma_i^o}{\eta_r^o}\right), \ i = 1, \dots, m, \ r = 1, \dots, s, \qquad (16)$$

$$\sum_{j=1}^{n} \tilde{\mu}_j = 1, \ \tilde{\mu}_j \geq 0, \ j = 1, \dots, n.$$

Therefore, according to relation (16), $(t\gamma^o, \tilde{\mu})$ is a feasible solution of model (9), which $t\gamma^o < \gamma^o, \ 0 < t < 1$ and but this is impossible because γ^o is a weak efficient solution of model (9). Therefore, the contradiction assumption is invalid and we will have $\theta_R^+ = \theta_R^I$.

Conversely, let $\theta_R^+ = \theta_R^I$, we show that (γ^o, μ) a feasible solution of model (9). By contradiction assume (γ^o, μ) is not a weakly efficient solution of model (9). Therefore, a feasible solution of model (9) will exist as $(\hat{\gamma}, \hat{\mu})$ such that $\hat{\gamma} < \gamma^o$. Given that $(\hat{\gamma}, \hat{\mu})$ is a feasible solution of model (9), so we will have

$$\sum_{j=1}^{n} \hat{\mu}_j \left(\frac{x_{ij}}{y_{rj}}\right) \leq \theta_R^I \left(\frac{\hat{\gamma}_i}{\eta_r^o}\right) < \theta_R^I \left(\frac{\gamma_i^o}{\eta_r^o}\right), \ i = 1, \dots, m, \ r = 1, \dots, s, \qquad (17)$$

$$\sum_{j=1}^{n} \hat{\mu}_j = 1, \ \hat{\mu}_j \geq 0, \ j = 1, \dots, n.$$

Then there exists a $0 < t < 1$ such that

$$\sum_{j=1}^{n} \hat{\mu}_j \left(\frac{x_{ij}}{y_{rj}}\right) \leq \theta_R^I \left(\frac{\hat{\gamma}_i}{\eta_r^o}\right) \leq t\theta_R^I \left(\frac{\gamma_i^o}{\eta_r^o}\right), \ i = 1, \dots, m, \ r = 1, \dots, s, \qquad (18)$$

$$\sum_{j=1}^{n} \hat{\mu}_j = 1, \ \hat{\mu}_j \geq 0, \ j = 1, \dots, n.$$

Let $\mu^{+} = (\hat{\mu}, 0)^{T}$, according to relation (18), we have
$\sum_{j=1}^{n} \mu_j^{+} = 1,\ \mu_j^{+} \geq 0,\ j = 1, \dots, n.$
Then, $(\hat{\mu}, t\theta_R^{I})$ is a feasible solution of model (10), so we will have $t\theta_R^{I} < \theta_R^{+}$. But this is against the assumption that θ_R^{I} is the optimal value of model (10). Therefore, the contradiction assumption is invalid and we will have (γ^{o}, μ) is not a weakly efficient solution of model (9) and the proof is complete. ■
Suppose we have n decision units as $DMU_j = \left(x_j, y_j\right),\ j = 1, \dots, n$. Each DM uses the input vector $x_j = \left(x_{1j}, \dots, x_{mj}\right)$ to product the output vector $y_j = \left(y_{1j}, \dots, y_{sj}\right)$. Then we define the set T_{DEA-R} as follows.

$$T_{DEA-R} = \left\{ F \,\middle|\, \sum_{j=1}^{n} \lambda_j \left(\frac{x_j}{y_j}\right) \leq F\ , \sum_{j=1}^{n} \lambda_j = 1, \lambda_j \geq 0 \right\}. \tag{19}$$

We define a division data set, which are m × s dimension vectors as follows.

$$\left\{\frac{x}{y} = \left(\frac{x_1}{y_1}, \dots, \frac{x_m}{y_1}, \frac{x_1}{y_2}, \dots, \frac{x_m}{y_2}, \dots, \frac{x_1}{y_s}, \dots, \frac{x_m}{y_s}\right)\right\} \text{ with } y = (y_1, \dots, y_s), x = (x_1, \dots, x_m).$$
(20)

The technology set T_{DEA-R} has the properties inclusion of observations, a free-disposal and convexity.
Now to obtain the efficiency score based on the concept of radial efficiency, we obtain the value θ_R in such a way that the unit under evaluation i.e. $DMU_o = (x_o, y_o)$ in the form $F^{o} = \left(\frac{x^{o}}{y^{o}}\right)$ be on the efficiency frontier of the set T_{DEA-R}. Therefore, we solve model (21) as follows.

$$\begin{aligned} & min\ \ \theta_R \\ & s.t.\ \ \theta_R\left(\frac{x^{o}}{y^{o}}\right) \in T_{DEA-R}. \end{aligned} \tag{21}$$

By considering multiplier corresponding to the ratio input to output of $DMU_j = \left(x_j, y_j\right)$ as μ_j, model (21) is equivalent to the following model.

$$\begin{aligned} & min\ \theta_R \\ & s.t.\ \sum_{j=1}^{n} \mu_j \left(\frac{x_{ij}}{y_{rj}}\right) \leq\ \theta_R\left(\frac{x_{io}}{y_{ro}}\right), \quad i = 1, \dots, m,\ r = 1, \dots, s, \\ & \qquad \sum_{j=1}^{n} \mu_j = 1, \quad \mu_j \geq 0, \quad j = 1, \dots, n. \end{aligned} \tag{22}$$

The model (22) is the identical to the DEA-R in input orientation namely model (3) which was introduced in the second section.
We now present a new criterion model compared to model (10). If the created new unit means $DMU_o^{new} = (x_o + \Delta x_o, y_o + \Delta y_o) = (\gamma^{o}, \eta^{o})$ belong to the set T_{DEA-R}, that is, the created new unit is an internal point or a point on the efficient frontier of the set T_{DEA-R}.
In this case, we can present the criterion model (23) to check whether the relative efficiency of the unit under evaluation changes after perturbation of its inputs and outputs or not.

$$\begin{aligned} & \theta_R^{II} = Min\ \theta_R \\ & s.t.\ \ \sum_{j=1}^{n} \mu_j \left(\frac{x_{ij}}{y_{rj}}\right) \leq \theta_R \left(\frac{\gamma_i^{o}}{\eta_r^{o}}\right), \quad i = 1, \dots, m,\ r = 1, \dots, s, \\ & \qquad \sum_{j=1}^{n} \mu_j = 1, \quad \mu_j \geq 0,\ j = 1, \dots, n. \end{aligned} \tag{23}$$

In model (23), θ_R^{II} is the relative efficiency of $DMU_o^{new} = (x_o + \Delta x_o, y_o + \Delta y_o) = (\gamma^o, \eta^o)$. Model (23) compared to model (10) has one variable less.

Theorem 4. Suppose that $DMU_o = (x_o, y_o)$ perturbs its output from y_o to $\eta^o = y_o + \Delta y_o$, $\Delta y_o \geq -y_o$, $\Delta y_o \in R$. Then (γ^o, μ^o) is a weak efficient solution of MOLP model (9) if and only if $\theta_R^{II} = \theta_R^I$.

Proof: First assume that $\theta_R^{II} = \theta_R^I$, we show that (γ^o, μ^o) is a weak efficient solution of MOLP model (9). Assume that (γ^o, μ^o) is not a weak efficient solution of MOLP model (9). Thus there is a feasible solution $(\hat{\gamma}, \hat{\mu})$ of model (9) such that $\hat{\gamma} < \gamma^o$. So there exists a $0 < t < 1$ such that $\hat{\gamma} \leq t\gamma^o$, Given that $(\hat{\gamma}, \hat{\mu})$ is a feasible solution of model (9), so we will have

$$\sum_{j=1}^{n} \hat{\mu}_j \begin{pmatrix} x_{ij} \\ y_{rj} \end{pmatrix} \leq \theta_R^I \begin{pmatrix} \hat{\gamma}_i \\ \eta_r^o \end{pmatrix} < \theta_R^I \begin{pmatrix} \gamma_i^o \\ \eta_r^o \end{pmatrix},\ i = 1, \dots, m,\ r = 1, \dots, s, \qquad (24)$$

$$\sum_{j=1}^{n} \hat{\mu}_j = 1,\ \hat{\mu}_j \geq 0,\ j = 1, \dots, n.$$

Assuming that $\theta_R^{II} = \theta_R^I$, therefore

$$\sum_{j=1}^{n} \hat{\mu}_j \begin{pmatrix} x_{ij} \\ y_{rj} \end{pmatrix} \leq \theta_R^{II} \begin{pmatrix} \hat{\gamma}_i \\ \eta_r^o \end{pmatrix} \leq t\theta_R^{II} \begin{pmatrix} \gamma_i^o \\ \eta_r^o \end{pmatrix},\ i = 1, \dots, m,\ r = 1, \dots, s, \qquad (25)$$

$$\sum_{j=1}^{n} \hat{\mu}_j = 1,\ \hat{\mu}_j \geq 0,\ j = 1, \dots, n.$$

Therefore $(\hat{\mu}, t\theta_R^{II})$ is a feasible solution for model (23) and we will have $t\theta_R^{II} < \theta_R^{II}$.
This contradicts with the optimality of θ_R^{II} in model (23) since $t\theta_R^{II} < \theta_R^{II}$. Therefore, the contradiction assumption is invalid and then (γ^o, μ^o) is a weak efficient solution of MOLP model (9).

Conversely, assume that (γ^o, μ^o) is a weak efficient solution of MOLP model (9). We show that $\theta_R^{II} = \theta_R^I$. Given that (γ^o, μ^o) is a weak efficient solution of MOLP model (9), so we have

$$\sum_{j=1}^{n} \mu_j^o \begin{pmatrix} x_{ij} \\ y_{rj} \end{pmatrix} \leq \theta_R^I \begin{pmatrix} \gamma_i^o \\ \eta_r^o \end{pmatrix},\ \ i = 1, \dots, m,\ r = 1, \dots, s, \qquad (26)$$

$$\sum_{j=1}^{n} \mu_j^o = 1,\ \mu_j^o \geq 0, j = 1, \dots, n.$$

This set of constraints in (26) will be the same as the set of constraints in model (23). Therefore (μ^o, θ_R^I) is a feasible solution of the model (23) and according to the optimality θ_R^{II}, we will have $\theta_R^{II} \leq \theta_R^I$. Now suppose that $\theta_R^{II} < \theta_R^I$, thus there exists a $0 < t < 1$ such that $\theta_R^{II} \leq t\theta_R^I$. Given that θ_R^{II} is the optimal value of the model (23), assume that the optimal solution corresponding to this optimal value is (μ', θ_R^{II}). Therefore, the set of constraints from model (23) will be as following.

$$\sum_{j=1}^{n} \mu_j' \begin{pmatrix} x_{ij} \\ y_{rj} \end{pmatrix} \leq \theta_R^{II} \begin{pmatrix} \gamma_i^o \\ \eta_r^o \end{pmatrix}, \quad i = 1, \dots, m,\ r = 1, \dots, s, \qquad (27)$$

$$\sum_{j=1}^{n} \mu_j' = 1, \quad \mu_j' \geq 0,\ j = 1, \dots, n.$$

Given that $\theta_R^{II} \leq t\theta_R^I$ therefore

$$\sum_{j=1}^{n} \mu_j' \begin{pmatrix} x_{ij} \\ y_{rj} \end{pmatrix} \leq \theta_R^{II} \begin{pmatrix} \gamma_i^o \\ \eta_r^o \end{pmatrix} \leq t\theta_R^I \begin{pmatrix} \gamma_i^o \\ \eta_r^o \end{pmatrix}, \quad i = 1, \dots, m,\ r = 1, \dots, s, \qquad (28)$$

$$\sum_{j=1}^{n} \mu_j' = 1, \quad \mu_j' \geq 0,\ j = 1, \dots, n.$$

Therefore $(t\gamma^o, \mu')$ is a feasible solution of model (9) and $\gamma_i^o \neq 0, i = 1, \dots, m,$, because otherwise if there exist a $1 \leq i_p \leq m$, such that $\gamma_{i_p}^o = 0$, then

$$\sum_{j=1}^{n} \mu_j' \begin{pmatrix} x_{i_p j} \\ y_{rj} \end{pmatrix} = 0, r = 1, \dots, s,$$

The relation (29) concludes that $\mu_j' = 0,\ j = 1, \dots, n$, which is inconsistent with that $\sum_{j=1}^{n} \mu_j' = 1$. Therefore $\gamma_i^o \neq 0,\ i = 1, \dots, m$. Therefore, given that $t\gamma^o < \gamma^o$, which is a contradiction with the fact that (γ^o, μ^o) is a weak efficient solution of MOLP model (9).

Therefore, the contradiction assumption is invalid and we will have $\theta_R^{II} = \theta_R^{I}$ and the proof is complete. ■

As you know, in proposed inverse DEA-R models in this paper, we determine the level of inputs based on the perturbed outputs, assuming that the relative efficiency of the under evaluation DMU i.e. $DMU_o = (x_o, y_o)$ preserve. We replace the under evaluation DMU i.e. $DMU_o = (x_o, y_o)$ with a new unit as $DMU_o^{new} = (x_o + \Delta x_o, y_o + \Delta y_o) = (\gamma^o, \eta^o)$. So that the efficiency of these units are equal. We are now looking to introduce a new criterion model that has less number of variables than previous criterion models, and with fewer calculations we can compare the efficiency of the created new DMU with the efficiency of the original DMU i.e. $DMU_o = (x_o, y_o)$. In the new model, we remove the unit under evaluation, i.e. $DMU_o = (x_o, y_o)$ from between all DMUs, and the we put new unit, $DMU_o^{new} = (x_o + \Delta x_o, y_o + \Delta y_o) = (\gamma^o, \eta^o)$ instead of the primitive DMU i.e. $DMU_o = (x_o, y_o)$ in a set of DMUs. So the difference between models (10) and (23) with the new created model is that in models (10) and (23) we evaluate the new unit in the presence of all DMUs, but in the proposed new criterion model, we evaluate a new unit, $DMU_o^{new} = (x_o + \Delta x_o, y_o + \Delta y_o) = (\gamma^o, \eta^o)$ in the presence of all units and new unit , with the exception of the under evaluation unit i.e. $DMU_o = (x_o, y_o)$. In the new model $DMU_o = (x_o, y_o)$ does not exist among DMUs, that is, we remove $DMU_o = (x_o, y_o)$ from the set T_{DEA-R}. Now, we proposed the new criterion model for evaluating the efficiency of the new unit, i.e. $DMU_o^{new} = (x_o + \Delta x_o, y_o + \Delta y_o) = (\gamma^o, \eta^o)$ in the absence of the unit under evaluation, i.e. $DMU_o = (x_o, y_o)$ as follows.

$$\begin{aligned} \theta_R^{new} &= Min\ \theta_R \\ s.t.\ \ & \sum_{\substack{j=1 \\ j\neq o}}^{n} \mu_j \left(\frac{x_{ij}}{y_{rj}}\right) + \mu_{new}\left(\frac{\gamma_i^o}{\eta_r^o}\right) \le \theta_R \left(\frac{\gamma_i^o}{\eta_r^o}\right),\ \ i = 1, \dots, m,\ r = 1, \dots, s, \qquad (30) \\ & \sum_{\substack{j=1 \\ j\neq o}}^{n} \mu_j + \mu_{new} = 1,\ \ \mu_j \ge 0,\ j = 1, \dots, n,\ j \neq o,\ \mu_{new} \ge 0. \end{aligned}$$

Theorem 5. Suppose that $DMU_o = (x_o, y_o)$ perturbs its output from y_o to $\eta^o = y_o + \Delta y_o$, $\Delta y_o \ge -y_o$, $\Delta y_o \in R$. Then (γ^o, μ^o) is a weak efficient solution of MOLP model (9) if and only if $\theta_R^{new} = \theta_R^{I}$.

Proof: We now consider two cases. In the first case, suppose that $DMU_o = (x_o, y_o)$ is efficient in evaluation with model (3), i.e. $\theta_R^I = 1$. According to Theorem (3), we have $\theta_R^+ = \theta_R^I$, i.e. the optimal value of model (10) is also equal to one. Now we show that the optimal value obtained from model (30) is also equal to one, i.e. $\theta_R^{new} = 1$. Assuming that (μ'', θ_R^{new}) is an optimal solution for model (30), we know that $\theta_R^{new} \le 1$. We show that $\theta_R^{new} = 1$. Suppose that $\theta_R^{new} < 1$. For this purpose, we put $\mu_{n+1}''' = \mu_{new}''$ and

$$\mu_j''' = \begin{cases} 0 & j = o \\ \mu_j'' & j \neq o \end{cases}$$

Given that (μ'', θ_R^{new}) is an optimal solution for model (30), so it is easy to see that $(\mu'', \mu_{n+1}''', \theta_R^{new})$ is a feasible solution for model (10) which results in $\theta_R^+ < 1$, which cannot be true because the value of the optimal value of model (10) is equal to one. Therefore, the contradiction assumption is invalid and we have $\theta_R^{new} = 1$, then $\theta_R^{new} = \theta_R^+ = \theta_R^I = 1$.

In the second case, suppose that $DMU_o = (x_o, y_o)$ is inefficient in evaluating with model (3). Therefore $DMU_o = (x_o, y_o)$ is an internal point of the set T_{DEA-R} and adding a new unit does not change the set T_{DEA-R} and therefore the new unit is $DMU_o^{new} = (x_o + \Delta x_o, y_o + \Delta y_o) =$

(γ^o, η^o) is also an interior point of T_{DEA-R} and is an inefficient unit, and removing it will not change the set T_{DEA-R}. So the new unit means $DMU_o^{new} = (x_o + \Delta x_o, y_o + \Delta y_o) = (\gamma^o, \eta^o)$ will be evaluated in terms of units $DMU_j = (x_j, y_j),\ \ j = 1, \dots, n,\ j \neq o$. Therefore, the solutions of models (10) and (30) are the same, i.e. in this case, too, $\theta_R^{new} = \theta_R^+ = \theta_R^I$ and the proof is complete.■

It should be noted that they are different in criterion models (10) and (30). Model (10) has $n + 2$ variables and $m + s + 1$ constraints and model (30) has $n + 1$ variables and $m + s + 1$constraints. Therefore, the number of calculations related to model (30) is significantly reduced compared to model (10).

The DEA-R models presented in this paper are in the input orientation based on model (3) and we have proposed the approach presented in this paper in the output orientation based on model (7) and we consider the ratio of output components to input components, which is beyond the scope of this paper and is suggested as future work.

4. Numerical example

In this section we use the data from the paper of Ali, Lerme, and Seiford [1] and Chen and Ali [11] to illustrate the validity of the proposed models. Suppose we have 11 DMU that use two inputs to generate two outputs. Table (1) shows the input and output data.

Table 1. Input and output data of eleven DMUs.

DMU	I1	I2	O1	O2	Efficiency scores (model 3)
DMU1	40	30	160	100	1
DMU2	30	60	180	70	1
DMU3	93	40	170	60	0.729
DMU4	50	70	190	130	1
DMU5	80	30	180	120	1
DMU6	35	45	140	82	0.94
DMU7	105	75	120	90	0.356
DMU8	97	67	100	82	0.361
DMU9	100	50	140	40	0.494
DMU10	90	60	140	105	0.512
DMU11	98	65	140	50	0.397

First, we use model (3) to evaluate the efficiency of DMUs based on the ratio of input components to output components. As can be seen in the last column of Table (1), units, 1, 2, 4, and 5 are efficient units, and other units are inefficient in evaluated by model (3).

Suppose that the amount of changes in the components of the first and second outputs of the unit under evaluation, i.e. $DMU_o = (x_o, y_o)$ is denoted by Δy_{1o} and Δy_{2o}, respectively, and also the value of the components of the first and second outputs of the new unit i.e. $DMU_o^{new} = (x_o + \Delta x_o, y_o + \Delta y_o) = (\gamma^o, \eta^o)$ are denoted by η_1^o and η_2^o, respectively. Assume that the amount of changes of the components of the first and second inputs corresponding to the unit under evaluation i.e. $DMU_o = (x_o, y_o)$ from model (9) is indicated by Δx_{1o}^* and Δx_{2o}^*, respectively, and also the value of the components of the first and second inputs of the new unit i.e. $DMU_o^{new} = (x_o + \Delta x_o, y_o + \Delta y_o) = (\gamma^o, \eta^o)$ are denoted by γ_1^o and γ_2^o, respectively that theses values determine from model (9).

To illustrate the inputs/output estimation process based on inverse DEA-R models in the presence of ratio data, first consider the inefficient unit 3. As can be seen in the last column of

Table (1), the efficiency score of unit 3 is equal to 0.729. Suppose this unit increases the value of its first and second outputs by 10 and 50 units, respectively. Then we have

$(\eta_1^3, \eta_2^3) = (y_{13} + \Delta y_{13}, y_{23} + \Delta y_{23}) = (170 + 10, 60 + 50) = (180, 110)$

In this case, if we want its efficiency does not change and its be equal to 0.729. According to model (9), the minimum input level of this DMU is determined as follows.

$(\gamma_1^3, \gamma_2^3) = (x_{13} + \Delta x_{13}^*, x_{23} + \Delta x_{23}^*) = (93 - 31.272, 40 + 6.296) = (61.728, 46.296\ ).$

As can be seen, the amount of the first and second inputs decreases and increases by 31.272 and 6.296, respectively. The efficiency score of the new unit means $DMU_3^{new} = (x_3 + \Delta x_3, y_3 + \Delta y_3) = (\gamma^3, \eta^3) = (61.728, 46.296, 180, 110)$, based on the criterion models (10), (23), and (30) are equal to $\theta_R^+ = 0.729$, $\theta_R^{II} = 0.729$, $\theta_R^{new} = 0.729$, respectively.

As can be seen, all three criterion models obtain the efficiency score of the new unit equal to 0.729 and this shows that the solution proposed by model (9) have the relative efficiency score equal to the efficiency score of the unit under evaluation.

Now consider efficiency unit 5. As can be seen in the last column of Table (1), the efficiency value of the unit 3 is equal to one and this unit is an efficient unit. Suppose this unit increases the value of its first output by 20 units and does not change its second output. Then we have

$(\eta_1^5, \eta_2^5) = (y_{15} + \Delta y_{15}, y_{25} + \Delta y_{25}) = (180 + 20, 120 + 0) = (200, 120).$

In this case, if we want its efficiency does not change and its be is equal to one. According to model (9), the minimum input level of this DMU is determined as follows.

$(\gamma_1^5, \gamma_2^5) = (x_{15} + \Delta x_{15}^*, x_{25} + \Delta x_{25}^*) = (80 - 30, 30 + 7.5) = (50, 37.5).$

As can be seen, the amount of the first and second inputs decreases and increases by 30 and 7.5, respectively. The efficiency score of the new unit means $DMU_5^{new} = (x_5 + \Delta x_5, y_5 + \Delta y_5) = (\gamma^5, \eta^5) = (200, 120, 50, 37.5)$, based on the criterion models (10), (23), and (30) are equal $\theta_R^+ = 1$, $\theta_R^{II} = 1$, $\theta_R^{new} = 1$, respectively.

As can be seen, all three criterion models obtain the efficiency score of the new unit equal to one and this shows that the solution proposed by model (9) have the relative efficiency score equal to the efficiency score of the unit under evaluation.

In order to the sensitivity analysis of the results related to the proposed approach in this paper, we also used the inputs/output estimation process based on inverse DEA-R models in the presence of ratio data for units 6 and 9, the results are shown in Table (2).

Table 2. The results corresponding to different DMU for inputs/outputs estimation based on proposed approach.

DMU	DMU3	DMU6	DMU5	DMU9	DMU9
η_1^o	180	180	200	175	165
η_2^o	110	120	120	80	65
Δy_{1o}	10	40	20	35	25
Δy_{2o}	50	38	0	40	25
γ_1^o	61.728	51.064	50	88.563	83.502
γ_2^o	46.296	38.298	37.5	66.422	62.627
Δx_{1o}^*	−31.272	16.064	-30	-11	−16.498
Δx_{2o}^*	6.296	−6.702	7.5	16.422	12.627
θ_R^+	0.729	0.94	1	0.494	0.494
θ_R^{II}	0.729	0.94	1	0.494	0.494
θ_R^{new}	0.729	0.94	1	0.494	0.494

5. Case study

In this section, we apply the approach presented in this paper to the real-world data set. For this purpose, we apply the inputs/output estimation process based on inverse DEA-R models in the presence of ratio data that proposed in this paper for 21 medical centers in Taiwan. These medical centers are included private and public health centers in 2005. Also, this data has been used in the article Wei et al. [37]. Input and output data sets including two inputs and outputs are listed in Table (3).

Table 3.
The input and output variables of Taiwan medical centers in 2005. (Wei et al. [37]).

DMU	Sickbed	Physician	Out-patient	In-patient	Surgeries	Efficiency scores (model 3)
DMU1	2618	1106	2,029,864	680,136	38,714	0.814
DMU2	1212	473	1,003,707	297,719	18,575	0.792
DMU3	1721	531	1,592,960	408,556	36,658	0.843
DMU4	2902	973	2,596,143	855,467	75,348	1
DMU5	1389	447	1,116,161	337,523	23,803	0.842
DMU6	1500	547	1,476,282	378,658	22,503	0.842
DMU7	340	145	1,300,016	55,003	5,614	1
DMU8	571	305	1,052,992	199,780	26,026	1
DMU9	1168	369	1,849,711	326,109	30,967	1
DMU10	921	372	1,089,975	209,323	23,847	0.746
DMU11	920	316	334,090	268,723	15,130	0.981
DMU12	3236	1023	1,954,775	920,215	56,167	0.98
DMU13	495	130	332,741	136,351	23,423	1
DMU14	1759	491	1,465,374	430,407	35,599	0.908
DMU15	1357	390	1,277,752	368,174	36,006	0.986
DMU16	2468	675	1,825,332	668,467	32,275	0.98
DMU17	962	316	550,700	247,961	15,618	0.878
DMU18	745	272	1,277,899	217,371	11,671	1
DMU19	1662	590	1,916,888	418,205	21,551	0.855
DMU20	898	275	698,945	209,134	11,748	0.822

In this paper we select all medical centers (21) as evaluation subjects, including seven public hospitals (33%) and private hospitals (67%). Two inputs and three outputs were selected. Note that the total inputs and outputs were less than half of all DMUs in conformity with empirical rules. The inputs include: sickbeds and physicians, outputs include: out-patients, in-patients, and surgeries. For example, consider DMU 4, this DMU serviced 2,596,143 out-patients, and 855,467 in-patients, and conducted 75,348 surgeries in 2005, with 2902 sickbeds and 973 physicians.

Due to the nature of the data, we can use DEA-R models in the input orientation, i.e. model (3) to evaluate the efficiency of these centers. In the input orientation models presented in the paper, we use the ratio of input components to output components as in Table (4) and these ratios are defined and are important for the DM and management.

Table 4.
The ratios of inputs to outputs in order to using in the input orientation.

Number of sickbeds / Number of out-patients	Number of sickbeds / Number of in-patients	Number of sickbeds / Number of surgeries
Number of physicians / Number of out-patients	Number of physicians / Number of in-patients	Number of physicians / Number of surgeries

For example, the ratio of the total number of sickbeds admitted to the hospital to the number of out-patients is important for hospital management, because whatever decreases the ratio of total sickbeds or increases the number of out-patients is important for management and the treatment system. Also, the goal is to determine efficient medical centers that provide more out-patients with the least number of sickbeds. This increases hospital services, because if the numerator and denominator corresponding to these fractional numbers decreases and increases respectively, then the number of treated patients increases to the total number of patients admitted to the hospital, and this issue is important for the hospital management and consequently the cost and revenue of the hospital decreases and increases respectively. Or consider another ratio, for example consider the ratio of the number of sickbeds to the number of successful surgeries, this ratio should be a good ratio for hospital management to offer more number of successful surgeries compared to the smaller number of out-patients. Then the medical centers are introduced as successful and efficient that offer a higher number of successful surgeries with a smaller number of sickbeds and in this case, this ratio is a suitable ratio.
Or consider the ratio of the total number of physicians to the number of successful surgeries. If this ratio decreases, then the number of unsuccessful surgeries increases compared to the number of physicians, which means that the hospital performs more successful surgeries for a lower fee, including fees paid to physicians and staff and other costs. The hospital management perspective is important to reduce this ratio, because by reducing this ratio, the costs paid to the treatment staff will decrease, and in contrast, with the increase in the number of surgeries or successful operations in the hospital, the amount of services received by patients will increase and the income received from these patients will increase that is suitable from the point of view of optimization. For other ratios in Table (4) we can provide similar interpretations.
The last column in Table (3) shows the efficiency scores obtained from Model (3) in the evaluation of medical centers. As can be seen, units 4, 7, 8, 9, 13, and 18 are introduced as efficient medical centers and other centers are inefficient. It should be noted that the technology used in this paper is constant returns to scales technology.
Now, in order to examine the results of the proposed approach presented in this paper, we we apply the inputs/output estimation process based on inverse DEA-R models in the presence of ratio data that proposed in this paper. Table (5) shows these results.
At first, consider the inefficient unit 10. As can be seen in the last column of Table (3), the efficiency value of unit 10 is equal to 0.746. Suppose this unit increases the amount of its first, second, and third outputs by 120,000, 35,000, and 80, respectively. Then, we have
$(\eta_1^{10}, \eta_2^{10}, \eta_3^{10}) = (y_{110} + \Delta y_{110}, y_{210} + \Delta y_{210},, y_{310} + \Delta y_{310}) = (1089975 + 120000, 209323 + 35000, 23847 + 80) = (1209975, 244323, 23927).$

In this case, according to model (9), the minimum input level of this DMU is determined as follows, if we want its efficiency does not change and its be equal to 0.746. $(\gamma_1^{10}, \gamma_2^{10}) = (x_{110} + \Delta x_{110}^*, x_{210} + \Delta x_{210}^*) = (921 + 15.073, 372 + 130) = (936.073, 502)$.
As can be seen, the first and second inputs increase to 15.073 and 130, respectively. The efficiency of the new unit means $DMU_{10}^{new} = (x_{10} + \Delta x_{10}, y_{10} + \Delta y_{10}) = (\gamma^{10}, \eta^{10}) = (1209975, 244323, 23927, 936.073, 502\ )$. Based on the criterion models (10), (23), and (30) are equal to $\theta_R^+ = 0.746$, $\theta_R^{II} = 0.746$, $\theta_R^{new} = 0.746$.
As it was observed, all three criterion models obtain the efficiency score of the new unit equal to 0.746 and this shows that the solution proposed by model (9) have the relative efficiency score equal to the efficiency score of the unit under evaluation.
Now, consider efficient unit 8. As shown in the last column of Table (3), the efficiency score of unit 10 is equal to one. Assume that this unit increases the value of its first, second, and third outputs by 80,000, 43,000, and 50 units, respectively. Then we have
$(\eta_1^8, \eta_2^8, \eta_3^8) = (y_{18} + \Delta y_{18},\ y_{28} + \Delta y_{28}, y_{38} + \Delta y_{38}) = (1052992 + 80000, 199780 + 43000, 26026 + 50) = (1132992, 242780, 26076)$.
In this case, the minimum input level of a new unit corresponding to DMU 10 according to model (9) is determined as follows, if we want efficiency score of new unit does not change and its be equal to one.
$(\gamma_1^8, \gamma_2^8) = (x_{18} + \Delta x_{18}^*, x_{28} + \Delta x_{28}^*) = (571 + 120, 305 + 65.647) = (691, 370.647)$.
As can be seen, the first and second inputs increase to 120 and 65.6470, respectively. The efficiency of the new unit means
$DMU_8^{new} = (x_8 + \Delta x_8, y_8 + \Delta y_8) = (\gamma^8, \eta^8) = (1132992, 242780, 26076, 691, 370.647)$.
Based on the criterion models (10), (23), and (30) are equal to $\theta_R^+ = 1$, $\theta_R^{II} = 1$, $\theta_R^{new} = 1$.
As can be seen, all three criterion models obtain the efficiency score of the new unit equal to one and this shows that the solution proposed by model (9) have the relative efficiency score equal to the efficiency score of the unit under evaluation.
In order to the sensitivity analysis of the results related to the proposed approach in this paper, we also used the inputs/output estimation process based on inverse DEA-R models in the presence of ratio data for units 4 and 17, the results are shown in Table (5).

Table 5. The results corresponding to different medical centers for inputs/outputs estimation based on proposed approach.

DMU	DMU10	DMU8	DMU17	DMU4
η_1^o	1209975	1132992	630700	2819143
η_2^o	244323	242780	269961	919467
η_3^o	23927	26076	15639	75643
Δy_{1o}	120000	80000	80000	223000
Δy_{2o}	35000	43000	22000	64000
Δy_{3o}	80	50	21	295
γ_1^o	936.073	691	880	2602
γ_2^o	502	370.647	470	1403
Δx_{1o}^*	15.073	120	−82	−300
Δx_{2o}^*	130	65.647	154	430
θ_R^+	0.746	1	0.878	1
θ_R^{II}	0.746	1	0.878	1
θ_R^{new}	0.746	1	0.878	1

In this analysis, we used GAMS software to analyze the results and solve the proposed models. According to the constraints of the criterion model (30) compared the constraints of the criterion models (10) and (23), this model has a smaller number of variables and we expect that the computational rate based on the criterion model (30) compared to the criterion models (10), (23) is less and we can use this model as a criterion model in the inputs/output estimation process based on inverse DEA-R models in the presence of ratio to reduce the time and number of calculations.

6. Conclusion

This paper presents inverse DEA-R models in the presence of ratio data. In this paper, we used the input orientation DEA-R models. We presented the inputs/output estimation process based on ratio based DEA (DEA-R) models. We showed that by determining the specific level of efficiency for each unit under evaluation DMU, we can determine the best possible level of inputs corresponding to a given level of outputs in the inputs/output estimation process based on inverse DEA-R models in the presence of ratio input components to output components. Next, we examined the criterion models in the inputs/output estimation process based on ratio based DEA (DEA-R) models. In this way, in order to reduce calculations, we presented a new criterion model based on DEA-R model, we have shown that by using this new criterion model, we can reduce the amount of computation outputs in the inverse DEA-R in order to compare the amount of efficiency of the unit under evaluation and the new unit created. We can easily use the proposed approach given that the models presented are linear and always feasible. As future work, we can examine inverse DEA-R models in the presence of ratio output components to input components, and we can develop the above models in other technologies such as VRS technology or non-convex technology. We can also develop the proposed models in this paper based on cost and revenue efficiency concepts, or develop the proposed models in this paper for other data structures in DEA, such as fuzzy data or network data.

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