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Weighted Sequence Spaces and Cyclicity

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Abstract: In this paper we investigate the cyclicity of the multiplication operator M_z acting on the weighted Hardy spaces of formal Laurent series.

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1. Introduction

Suppose that $1 < p < \infty$ and $\{\beta(n)\}_{n=-\infty}^{\infty}$ denotes a sequence of positive numbers with $\beta(0) = 1$. For a sequence $f = \{\hat{f}(n)\}_{n=-\infty}^{\infty}$, we define

$$||f|| = ||f||_p = (\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^p |\beta(n)|^p)^{\frac{1}{p}}.$$

Furthermore, we shall use the notation $f(z) = \sum_{n=-\infty}^{\infty} \hat{f}(n) z^n$ regardless whether the series converges for any complex value of z. Throughout this article, by the space $L^p(\beta)$ we mean

$$L^p(\beta) = \{f : f(z) = \sum_{n=-\infty}^{\infty} \hat{f}(n)z^n, ||f||_p < \infty\}$$

which is called a weighted Hardy space of formal Laurent series (note that when n ranges on $\mathbb{N} \cup \{0\}$, it is called a weighted Hardy space of formal power series and is denoted by $H^p(\beta)$). These are reflexive Banach spaces with the norm $\| \cdot \|_\beta$. Let $\hat{f}_k(n) = \delta_k(n)$. So $f_k(z) = z^k$ and then $\{f_k\}_{k=-\infty}^{\infty}$ is a basis for $L^p(\beta)$ such that $\|f_k\| = \beta(k)$. Now consider M_z, the operator of multiplication by z on $L^p(\beta)$:

$$(M_z f)(z) = \sum_{n=-\infty}^{\infty} \hat{f}(n)z^{n+1}$$

where

$$f(z) = \sum_{n=-\infty}^{\infty} \hat{f}(n)z^n \in L^p(\beta).$$

In other words $(\widehat{M_z f})(n) = \hat{f}(n-1)$ for all $n \in \mathbf{Z}$. Clearly M_z shifts the basis $\{f_k\}_k$. The operator M_z is bounded if and only if $\{\beta(k+1)/\beta(k)\}_k$ is bounded and in this case

$$\|M_z^n\| = \sup_k [\beta(k+n)/\beta(k)]$$

for all $n \in \mathbb{N} \cup \{0\}$.

By the same method used in [3] we can see that $L^p(\beta)^* = L^q(\beta^{\frac{p}{q}})$, where $\frac{1}{p} + \frac{1}{q} = 1$. Also if

$$f(z) = \sum_{n=-\infty}^{\infty} \hat{f}(n) z^n \in L^p(\beta)$$

and

$$g(z) = \sum_{n=-\infty}^{\infty} \hat{g}(n) z^n \in L^q(\beta^{\frac{p}{q}}),$$

then clearly

$$< f, g >= \sum_{n=-\infty}^{\infty} \hat{f}(n) \overline{\hat{g}(n)} \beta(n)^p$$

and

$$\begin{aligned} \|g\|_q^q &= \sum_{n=-\infty}^{\infty} |\hat{g}(n)|^q (\beta(n)^{\frac{p}{q}})^q \\ &= \sum_{n=-\infty}^{\infty} |\hat{g}(n)|^q \beta(n)^p \end{aligned}$$

(see [3]). Here for simplicity we used $\|g\|_q$ instead of $\|g\|_{L^q(\beta^{\frac{p}{q}})}$. For some topics on these spaces see [2–16].

Let X be a Banach space. We denote by $B(X)$, the set of bounded operators on the Banach space X. Let $A \in B(X)$ and $x \in X$. We say that x is a cyclic vector of A if X is equal to the closed linear span of the set

$$\{A^n x : n = 0, 1, 2, \cdots\}.$$

An operator $A \in B(X)$ is called cyclic if it has a cyclic vector.

If X is a Banach space, it is convenient and helpful to introduce the notation (x, x^*) to stand for $x^*(x)$, for $x \in X$ and $x^* \in X^*$.

In [3] and [5] we studied the cyclicity of the multiplication operator M_z on $H^p(\beta)$ and here we want to investigate the cyclicity of the multiplication operator M_z on the both spaces $H^p(\beta)$ and $L^p(\beta)$.

2. Main Results

First we note that the multiplication operator M_z on $L^p(\beta)$ ($H^p(\beta)$) is unitarily equivalent to an injective bilateral (unilateral) weighted shift and conversely, every injective bilateral (unilateral) weighted shift is unitarily equivalent to M_z acting on $L^p(\beta)$ ($H^p(\beta)$) for a suitable choice of β (the proof is similar to the case p=2 that was proved in [2]).

We will use the following notations:

$$\begin{aligned} r_0 &= \overline{\lim}\beta(-n)^{-1/n}, \\ r_1 &= \underline{\lim}\beta(n)^{1/n}, \\ \Omega_0 &= \{z \in \mathbf{C} : |\mathbf{z}| > \mathbf{r_0}\}, \\ \Omega_1 &= \{z \in \mathbf{C} : |\mathbf{z}| < \mathbf{r_1}\}, \\ \Omega &= \Omega_0 \cap \Omega_1. \end{aligned}$$

From now on we consider that M_z is bounded on $L^p(\beta)$.

Theorem 1. *Let* $0 < r_0 < r_1 = 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$. *If*

$$\sum_{n<0} \frac{{r_0}^{nq}}{\beta(n)^q} < \infty \qquad ; \qquad \sum_{n\geqslant 0} \frac{1}{\beta(n)^q} < \infty,$$

then M_z *has no cyclic vector on* $L^p(\beta)$.

Proof. Note that Ω is an annulus with the unit disc as an outer boundary. Now for any function

$$f = \sum_{n=-\infty}^{\infty} \hat{f}(n) f_n$$

in $L^p(\beta)$, by the Holder inequality we have

$$\begin{aligned} \sum_{n=-\infty}^{\infty} |\hat{f}(n)||z|^n &\leqslant \Big(\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^p \beta(n)^p\Big)^{1/p} \Big(\sum_{n=-\infty}^{\infty} \frac{|z|^{nq}}{\beta(n)^q}\Big)^{1/q} \\ &= ||f||_p\Big[\Big(\sum_{n<0} \frac{|z|^{nq}}{\beta(n)^q}\Big)^{1/q} + \Big(\sum_{n\geqslant 0} \frac{|z|^{nq}}{\beta(n)^q}\Big)^{1/q}\Big] \\ &\leqslant ||f||_p\Big[\Big(\sum_{n<0} \frac{{r_0}^{nq}}{\beta(n)^q}\Big)^{1/q} + \Big(\sum_{n\geqslant 0} \frac{1}{\beta(n)^q}\Big)^{1/q}\Big] \end{aligned}$$

for all z in Ω. Since

$$\sum_{n<0} \frac{{r_0}^{nq}}{\beta(n)^q} < \infty \qquad ; \qquad \sum_{n\geqslant 0} \frac{1}{\beta(n)^q} < \infty,$$

by a similar method used in the proof of Theorem 3 in [3] and Theorem 1 in [7], we get $H^p(\beta) \subset H(\Omega) \cap C(T)$ where $H(\Omega)$ is the set of analytic functions on Ω and $C(T)$ is the set of continuous functions on the unit circle T.

Now define the operator

$$L : L^p(\beta) \longrightarrow C(T) \qquad ; \qquad L(f) = f|_T.$$

Clearly L maps the set of all Laurent polynomials onto the set of all polynomials in z and $\bar{z}$ which is dense in $C(T)$ by the Stone-Weierstrass theorem. Thus L has dense range. Now if g is a cyclic vector for M_z as an operator on $L^p(\beta)$, then $g|_T$ is a cyclic vector for M_z as an operator on $C(T)$. Thus g has no zero on T and this implies that the operator M_g is invertible on $C(T)$. Let $V\{.\}$ denotes the uniform closed linear span of the set $\{.\}$ in $C(T)$. Clearly $V\{M_z^n g|_T : n \geqslant 0\}$ is equal to the uniform closure of the set

$$\{pg|_T : p \text{ is an analytic polynomial}\}.$$

Thus, we get

$$V\{M_z^n g|_T : n \geqslant 0\} = M_g A$$

where A is the disc algebra of analytic functions in $C(T)$. Indeed

$$A = \text{uniform-closure } \{p|_T : p \text{ is an analytic polynomial}\}$$

(see [1]). But A is a proper closed subspace of $C(T)$, so M_gA is also a proper closed subspace of $C(T)$, since M_g is invertible on $C(T)$. This says that $g|_T$ can not be a cyclic vector for M_z as an operator on $C(T)$, hence g can not be a cyclic vector for M_z as an operator on $L^p(\beta)$ that is a contradiction. Now the proof is complete.□

Theorem 2. *i) If a function f in $H^p(\beta)$ is cyclic, then the zeros of f can not belong to Ω_1.*

ii) If the zeros of a polynomial P are not belong to Ω_1, then P is a cyclic vector for M_z.

Proof. See [3].□

By the same method we can have a similar result for the spaces $L^p(\beta)$ and in this case we should use Ω instead of Ω_1.

Theorem 3. *Let $1 \leqslant p < \infty$. Suppose that $\beta(n)$ is in the form $\beta(n) = \alpha(n)\gamma(n)$ where $\{\alpha(n)\}$ and $\{\gamma(n)\}$ satisfies:*

i) There exists a positive number M, such that

$$sup\{|\frac{\gamma(n+i)}{\gamma(n)\gamma(i)}| : i, n = 0, 1, 2, \cdots\} \leqslant M$$

ii) There exists a positive integer m_0 such that:

$$L_{m_0} = sup\{|\frac{\alpha(n+i)\alpha(m_0)}{\alpha(n+m_0)\alpha(i)}| : n > 0, i \geqslant m_0\} < \infty$$

and

$$\{\frac{\alpha(n+m_0)}{\alpha(n)}\}_n \in \ell^q,$$

where $\frac{1}{p}+\frac{1}{q}=1$.

If $x=\sum_m x_m f_m$ belongs to $H^p(\beta)$ and $x_0 \neq 0$, then x is a cyclic vector of M_z as an operator on $H^p(\beta)$.

Proof. See [5].□

Corollary 4. *Let* $\frac{1}{p}+\frac{1}{q}=1$, M_z *be power bounded and* $f=\sum_{m=0}^{\infty}\hat{f}(m)z^m \in H^p(\beta)$ *be such that* $\hat{f}(0)\neq 0$. *If we have*

$$\{\frac{\beta(n+j_0)}{\beta(n)}\}_n \in \ell^q$$

for some $j_0 \in \mathbb{N}$ *and* $\beta(n)>0$ *for all* n, *then* f *is a cyclic vector of* M_z *on* $H^p(\beta)$.

By a similar method the above results can be extended from the formal power sequence spaces $H^p(\beta)$ to the formal Laurent sequence spaces $L^p(\beta)$.

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