

## Weighted Sequence Spaces and Cyclicity

**J. Doroodgar**

Islamic Azad University-Shiraz Branch  
Shiraz Farzanegan Pre-University

**B. Yousefi**

Shiraz University

**Abstract:** In this paper we investigate the cyclicity of the multiplication operator  $M_z$  acting on the weighted Hardy spaces of formal Laurent series.

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### 1. Introduction

Suppose that  $1 < p < \infty$  and  $\{\beta(n)\}_{n=-\infty}^{\infty}$  denotes a sequence of positive numbers with  $\beta(0) = 1$ . For a sequence  $f = \{\hat{f}(n)\}_{n=-\infty}^{\infty}$ , we define

$$\|f\| = \|f\|_p = \left( \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^p |\beta(n)|^p \right)^{\frac{1}{p}}.$$

Furthermore, we shall use the notation  $f(z) = \sum_{n=-\infty}^{\infty} \hat{f}(n)z^n$  regardless whether the series converges for any complex value of  $z$ . Throughout

this article, by the space  $L^p(\beta)$  we mean

$$L^p(\beta) = \{f : f(z) = \sum_{n=-\infty}^{\infty} \hat{f}(n)z^n, \|f\|_p < \infty\}$$

which is called a weighted Hardy space of formal Laurent series (note that when  $n$  ranges on  $\mathbb{N} \cup \{0\}$ , it is called a weighted Hardy space of formal power series and is denoted by  $H^p(\beta)$ ). These are reflexive Banach spaces with the norm  $\|\cdot\|_\beta$ . Let  $\hat{f}_k(n) = \delta_k(n)$ . So  $f_k(z) = z^k$  and then  $\{f_k\}_{k=-\infty}^{\infty}$  is a basis for  $L^p(\beta)$  such that  $\|f_k\| = \beta(k)$ . Now consider  $M_z$ , the operator of multiplication by  $z$  on  $L^p(\beta)$ :

$$(M_z f)(z) = \sum_{n=-\infty}^{\infty} \hat{f}(n)z^{n+1}$$

where

$$f(z) = \sum_{n=-\infty}^{\infty} \hat{f}(n)z^n \in L^p(\beta).$$

In other words  $(\widehat{M_z f})(n) = \hat{f}(n-1)$  for all  $n \in \mathbb{Z}$ . Clearly  $M_z$  shifts the basis  $\{f_k\}_k$ . The operator  $M_z$  is bounded if and only if  $\{\beta(k+1)/\beta(k)\}_k$  is bounded and in this case

$$\|M_z^n\| = \sup_k [\beta(k+n)/\beta(k)]$$

for all  $n \in \mathbb{N} \cup \{0\}$ .

By the same method used in [3] we can see that  $L^p(\beta)^* = L^q(\beta^{\frac{p}{q}})$ ,

where  $\frac{1}{p} + \frac{1}{q} = 1$ . Also if

$$f(z) = \sum_{n=-\infty}^{\infty} \hat{f}(n)z^n \in L^p(\beta)$$

and

$$g(z) = \sum_{n=-\infty}^{\infty} \hat{g}(n)z^n \in L^q(\beta^{\frac{p}{q}}),$$

then clearly

$$\langle f, g \rangle = \sum_{n=-\infty}^{\infty} \hat{f}(n)\overline{\hat{g}(n)}\beta(n)^p$$

and

$$\begin{aligned} \|g\|_q^q &= \sum_{n=-\infty}^{\infty} |\hat{g}(n)|^q (\beta(n)^{\frac{p}{q}})^q \\ &= \sum_{n=-\infty}^{\infty} |\hat{g}(n)|^q \beta(n)^p \end{aligned}$$

(see [3]). Here for simplicity we used  $\|g\|_q$  instead of  $\|g\|_{L^q(\beta^{\frac{p}{q}})}$ . For some topics on these spaces see [2–16].

Let  $X$  be a Banach space. We denote by  $B(X)$ , the set of bounded operators on the Banach space  $X$ . Let  $A \in B(X)$  and  $x \in X$ . We say that  $x$  is a cyclic vector of  $A$  if  $X$  is equal to the closed linear span of the set

$$\{A^n x : n = 0, 1, 2, \dots\}.$$

An operator  $A \in B(X)$  is called cyclic if it has a cyclic vector.

If  $X$  is a Banach space, it is convenient and helpful to introduce the notation  $(x, x^*)$  to stand for  $x^*(x)$ , for  $x \in X$  and  $x^* \in X^*$ .

In [3] and [5] we studied the cyclicity of the multiplication operator  $M_z$  on  $H^p(\beta)$  and here we want to investigate the cyclicity of the multiplication operator  $M_z$  on the both spaces  $H^p(\beta)$  and  $L^p(\beta)$ .

## 2. Main Results

First we note that the multiplication operator  $M_z$  on  $L^p(\beta)$  ( $H^p(\beta)$ ) is unitarily equivalent to an injective bilateral (unilateral) weighted shift and conversely, every injective bilateral (unilateral) weighted shift is unitarily equivalent to  $M_z$  acting on  $L^p(\beta)$  ( $H^p(\beta)$ ) for a suitable choice of  $\beta$  (the proof is similar to the case  $p=2$  that was proved in [2]).

We will use the following notations:

$$r_0 = \overline{\lim} \beta(-n)^{-1/n},$$

$$r_1 = \underline{\lim} \beta(n)^{1/n},$$

$$\Omega_0 = \{z \in \mathbf{C} : |z| > r_0\},$$

$$\Omega_1 = \{z \in \mathbf{C} : |z| < r_1\},$$

$$\Omega = \Omega_0 \cap \Omega_1.$$

From now on we consider that  $M_z$  is bounded on  $L^p(\beta)$ .

**Theorem 1.** *Let  $0 < r_0 < r_1 = 1$  and  $\frac{1}{p} + \frac{1}{q} = 1$ . If*

$$\sum_{n<0} \frac{r_0^{nq}}{\beta(n)^q} < \infty \quad ; \quad \sum_{n \geq 0} \frac{1}{\beta(n)^q} < \infty,$$

*then  $M_z$  has no cyclic vector on  $L^p(\beta)$ .*

**Proof.** Note that  $\Omega$  is an annulus with the unit disc as an outer boundary. Now for any function

$$f = \sum_{n=-\infty}^{\infty} \hat{f}(n) f_n$$

in  $L^p(\beta)$ , by the Holder inequality we have

$$\begin{aligned} \sum_{n=-\infty}^{\infty} |\hat{f}(n)| |z|^n &\leq (\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^p \beta(n)^p)^{1/p} (\sum_{n=-\infty}^{\infty} \frac{|z|^{nq}}{\beta(n)^q})^{1/q} \\ &= \|f\|_p [(\sum_{n<0} \frac{|z|^{nq}}{\beta(n)^q})^{1/q} + (\sum_{n \geq 0} \frac{|z|^{nq}}{\beta(n)^q})^{1/q}] \\ &\leq \|f\|_p [(\sum_{n<0} \frac{r_0^{nq}}{\beta(n)^q})^{1/q} + (\sum_{n \geq 0} \frac{1}{\beta(n)^q})^{1/q}] \end{aligned}$$

for all  $z$  in  $\Omega$ . Since

$$\sum_{n<0} \frac{r_0^{nq}}{\beta(n)^q} < \infty \quad ; \quad \sum_{n \geq 0} \frac{1}{\beta(n)^q} < \infty,$$

by a similar method used in the proof of Theorem 3 in [3] and Theorem 1 in [7], we get  $H^p(\beta) \subset H(\Omega) \cap C(T)$  where  $H(\Omega)$  is the set of analytic

functions on  $\Omega$  and  $C(T)$  is the set of continuous functions on the unit circle  $T$ .

Now define the operator

$$L : L^p(\beta) \longrightarrow C(T) ; \quad L(f) = f|_T.$$

Clearly  $L$  maps the set of all Laurent polynomials onto the set of all polynomials in  $z$  and  $\bar{z}$  which is dense in  $C(T)$  by the Stone-Weierstrass theorem. Thus  $L$  has dense range. Now if  $g$  is a cyclic vector for  $M_z$  as an operator on  $L^p(\beta)$ , then  $g|_T$  is a cyclic vector for  $M_z$  as an operator on  $C(T)$ . Thus  $g$  has no zero on  $T$  and this implies that the operator  $M_g$  is invertible on  $C(T)$ . Let  $V\{\cdot\}$  denotes the uniform closed linear span of the set  $\{\cdot\}$  in  $C(T)$ . Clearly  $V\{M_z^n g|_T : n \geq 0\}$  is equal to the uniform closure of the set

$$\{pg|_T : p \text{ is an analytic polynomial}\}.$$

Thus, we get

$$V\{M_z^n g|_T : n \geq 0\} = M_g A$$

where  $A$  is the disc algebra of analytic functions in  $C(T)$ . Indeed

$$A = \text{uniform-closure } \{p|_T : p \text{ is an analytic polynomial}\}$$

(see [1]). But  $A$  is a proper closed subspace of  $C(T)$ , so  $M_g A$  is also a proper closed subspace of  $C(T)$ , since  $M_g$  is invertible on  $C(T)$ . This says that  $g|_T$  can not be a cyclic vector for  $M_z$  as an operator on  $C(T)$ , hence  $g$  can not be a cyclic vector for  $M_z$  as an operator on  $L^p(\beta)$  that is a contradiction. Now the proof is complete.  $\square$

**Theorem 2.** *i) If a function  $f$  in  $H^p(\beta)$  is cyclic, then the zeros of  $f$  can not belong to  $\Omega_1$ .*

*ii) If the zeros of a polynomial  $P$  are not belong to  $\Omega_1$ , then  $P$  is a cyclic vector for  $M_z$ .*

**Proof.** See [3].  $\square$

By the same method we can have a similar result for the spaces  $L^p(\beta)$  and in this case we should use  $\Omega$  instead of  $\Omega_1$ .

**Theorem 3.** *Let  $1 \leq p < \infty$ . Suppose that  $\beta(n)$  is in the form  $\beta(n) = \alpha(n)\gamma(n)$  where  $\{\alpha(n)\}$  and  $\{\gamma(n)\}$  satisfies:*

*i) There exists a positive number  $M$ , such that*

$$\sup\left\{ \left| \frac{\gamma(n+i)}{\gamma(n)\gamma(i)} \right| : i, n = 0, 1, 2, \dots \right\} \leq M$$

*ii) There exists a positive integer  $m_0$  such that:*

$$L_{m_0} = \sup\left\{ \left| \frac{\alpha(n+i)\alpha(m_0)}{\alpha(n+m_0)\alpha(i)} \right| : n > 0, i \geq m_0 \right\} < \infty$$

and

$$\left\{ \frac{\alpha(n+m_0)}{\alpha(n)} \right\}_n \in \ell^q,$$

where  $\frac{1}{p} + \frac{1}{q} = 1$ .

If  $x = \sum_m x_m f_m$  belongs to  $H^p(\beta)$  and  $x_0 \neq 0$ , then  $x$  is a cyclic vector of  $M_z$  as an operator on  $H^p(\beta)$ .

**Proof.** See [5].  $\square$

**Corollary 4.** Let  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $M_z$  be power bounded and  $f = \sum_{m=0}^{\infty} \hat{f}(m) z^m \in H^p(\beta)$  be such that  $\hat{f}(0) \neq 0$ . If we have

$$\left\{ \frac{\beta(n+j_0)}{\beta(n)} \right\}_n \in \ell^q$$

for some  $j_0 \in \mathbb{N}$  and  $\beta(n) > 0$  for all  $n$ , then  $f$  is a cyclic vector of  $M_z$  on  $H^p(\beta)$ .

By a similar method the above results can be extended from the formal power sequence spaces  $H^p(\beta)$  to the formal Laurent sequence spaces  $L^p(\beta)$ .

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**Jinalo Doroodgar**

Deptartment of Mathematics  
Islamic Azad University-Shiraz Branch  
Shiraz, Iran

Shiraz Farzanegan Pre-University  
Shiraz Training-Education  
Shiraz, Iran  
E-mail: jinalollo\_dorodgar@yahoo.com

**Bahman Yousefi**

Deptartment of Mathematics  
College of Sciences  
Shiraz University  
Shiraz 71454, Iran  
E-mail: byousefi@shirazu.ac.ir