Limit Points of Trigonometric Sequences

M. Faghih Ahmadi¹
Islamic Azad University - Sepidan Branch

K. Hedayatian
Shiraz University

Abstract: In this article, we find the set of all limit points of sequences of polynomials with real coefficients, in \( \cos n, \ n = 1, 2, 3, \ldots \) with degree less than or equal to three. Also, when the degree is four, the mentioned set is found in some special cases.

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1. Introduction

Finding the limit points of a sequences or, at least, finding some topological properties of the limit points of a sequence is one of the remarkable problems in analysis. For instance, in [2], the authors have found some necessary and sufficient conditions for the connectedness of the set of all limit points of a sequence in a metric space. Some other results on the limit points of certain sequences is obtained, for example, in [3] and [4].

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Our claim in this article is to find the set of all limit points of sequence of polynomials with real coefficients, \( \cos n, n = 1, 2, 3, \ldots \) with degree less than or equal to three. Also, when the degree is four, the mentioned set is found in some cases.

2. Main Results

**Theorem 1.** Suppose \( f \) is a real valued continuous, periodic function on the real numbers \( \mathbb{R} \) and its period is an irrational number \( \alpha \). Then the set of all limit points of the sequence \( \{f(n)\}_{n=1}^{\infty} \) is the closed interval \( [m, M] \) where \( m = \text{Min}\{f(x) : x \in \mathbb{R}\} \) and \( M = \text{Max}\{f(x) : x \in \mathbb{R}\} \).

**Proof.** Since \( f \) is continuous and periodic, it is uniformly continuous. So for \( \varepsilon > 0 \), there exists a \( \delta > 0 \) such that for every \( x, y \) in \( \mathbb{R} \), if \( |x - y| < \delta \) then \( |f(x) - f(y)| < \varepsilon \). But the set \( \mathbb{Z} + \alpha \mathbb{Z} = \{m + \alpha n : m, n \in \mathbb{Z}\} \) is a countable dense subset of \( \mathbb{R} \) where \( \mathbb{Z} \) denotes the set of all integers. Therefore, for each \( x \in \mathbb{R} \), integers \( m \) and \( n \) can be found so that

\[
|m - (n\alpha + x)| < \delta,
\]

and consequently, \( |f(m) - f(x)| < \varepsilon \). Now, considering the fact that \( f(\mathbb{R}) \) is a connected subset of \( \mathbb{R} \), the result follows. \( \square \)

We remark that for an irrational number \( \alpha \), \( \mathbb{N} + \alpha \mathbb{Z} \) is not dense
in \( \mathbb{R} \) where \( N \) denotes the natural numbers and so this proof can not be used when replacing \( \{f(n)\}_{-\infty}^{+\infty} \) by \( \{f(n)\}_{n=1}^{\infty} \). Nevertheless, a direct conclusion of the above theorem runs as follows:

**Theorem 2.** Let the function \( f \) satisfy the hypotheses of the preceding theorem. Suppose, furthermore, that \( f \) is an even function. Then the set of limit points of the sequence \( \{f(n)\}_{n=1}^{\infty} \) is the range of \( f \).

In all that follows, for a sequence \( \{p(n)\}_{n=1}^{\infty} \) let \( L_p \) be the set of all limit points of this sequence.

**Theorem 3.** Let \( q(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \), where \( a_3 \neq 0 \), and take \( p(n) = q(\cos n) \). If \( a_2^2 - 3a_1a_3 < 0 \) then \( L_p = [m, M] \) where \( m \) and \( M \) are, respectively, the minimum and maximum of the set \( \{q(1), q(-1)\} \).

If \( a_2^2 - 3a_1a_3 \geq 0 \) then \( L_p = [m, M] \) where \( m \) and \( M \) are, respectively, the minimum and maximum of the set

\[
\{q(1), q(-1), q\left(\frac{-a_2 + \sqrt{a_2^2 - 3a_1a_3}}{3a_3}\right), q\left(\frac{-a_2 - \sqrt{a_2^2 - 3a_1a_3}}{3a_3}\right)\}
\]

**Proof.** Consider the function \( p \) defined on \([0, 2\pi]\) by \( p(x) = q(\cos x) \).

Then \( p(x) \) is clearly even and periodic, allowing us to use Theorem 2; it remains to find the range of \( p \). If \( a_2^2 - 3a_1a_3 < 0 \) then \( p'(x) = 0 \) implies
that $x = 0, \pi, 2\pi$, and so the only values that $\cos x$ can take are 1 and -1. On the other hand, when $a_2^2 - 3a_1a_3 \geq 0$, an easy argument shows that if $p'(x) = 0$ then $\cos x$ can be

$$1, -1, \frac{-a_2 + \sqrt{a_2^2 - 3a_1a_3}}{3a_3}, \text{ or } \frac{-a_2 - \sqrt{a_2^2 - 3a_1a_3}}{3a_3}.$$  □

**Theorem 4.** Let $q(x) = a_0 + a_1 x + a_2 x^2$ for $a_2 \neq 0$, and let $p(n) = q(\cos n)$. Then $L_p = [m, M]$ where $m$ and $M$ are, respectively, the minimum and maximum of the set

$$\{a_0 + a_1 + a_2, a_0 - a_1 + a_2, a_0 - \frac{a_2^2}{4a_2}\}.$$  

**Proof.** Considering $p(x) = a_0 + a_1 \cos x + a_2 \cos^2 x$, $x \in [0, 2\pi]$; it is sufficient to find $x$ in the interval $[0, 2\pi]$ such that $p'(x) = 0$. Then apply Theorem 2. □

**Theorem 5.** Let

$$q(x) = a_0 + a_1 a_3 x + \frac{a_2 a_3}{2} x^2 + \frac{a_1 a_4}{3} x^3 + \frac{a_2 a_4}{4} x^4,$$

where $a_2 a_4 \neq 0$; and for $n \in \mathbb{N}$, take $p(n) = q(\cos n)$. If $a_3 a_4 \leq 0$ then $L_p = [m, M]$ where $m$ and $M$ are, respectively, the minimum and
maximum of the set
\[ \{ q(1), q(-1), q(\pm \sqrt{-\frac{a_3}{a_4}}), q(-\frac{a_1}{a_2}) \} \]
and if $a_3a_4 > 0$ then we use the set $\{ q(1), q(-1), q(-\frac{a_1}{a_2}) \}$.

**Proof.** If $\frac{d}{dx}(q(cos x)) = 0$ then $\sin x = 0$ or
\[
(a_4 \cos^2 x + a_3)(a_2 \cos x + a_1) = a_1a_3 + a_2a_3 \cos x + a_4 \cos^2 x + a_2a_4 \cos^3 x = 0.
\]
Consequently, if the inequality $a_3a_4 \leq 0$ holds, we get $\sin x = 0$ or $\cos x = \pm \sqrt{-a_3/a_4}$ or $\cos x = -a_1/a_2$. Also, whenever $a_3a_4 > 0$ we get $\sin x = 0$ or $\cos x = -a_1/a_2$. In each case, the result holds from Theorem 2. □

**Remark 1.** An immediate consequence of Theorem 2, is that the set of limit points of the sequence $\{ \cos n \}_{n=1}^{\infty}$ is $[-1, 1]$. This fact has been proved before, using more complicated techniques. For instance, one can see [1, Problem 3.15, p.14].

**Remark 2.** In Theorems 3, 4 and 5, substituting $\cos n$ by $\sin n$, one can show that the same results hold for the sequence $\{ \sin n \}_{-\infty}^{+\infty}$. 
References


Masoumeh Faghih Ahmadi  
Islamic Azad University - Sepidan Branch  
Sepidan, Iran  
E-mail: faghiha@shirazu.ac.ir  
E-mail: m_faghih_a@yahoo.com

Karim Hedayatian  
Department of Mathematics  
College of Sciences  
Shiraz University  
Shiraz 71454, Iran  
E-mail: hedayatian@susc.ac.ir