

## Notes on the Hypercyclic Operator

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**Abstract:** In this paper by using a nice criterion, we show that the perturbation of identity operators by some multiples of the standard backward shift is hypercyclic. This gives a new proof for Salas Theorem in ([10], Theorem 3.3).

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### 1. Introduction

Suppose that  $X$  is a separable topological vector space and  $T$  is a continuous linear mapping on  $X$ . If  $x \in X$ , then the orbit of  $x$  under  $T$  is denoted by  $Orb(T, x)$  and is defined by  $Orb(T, x) = \{x, Tx, T^2x, \dots\}$ . An operator  $T$  is called a hypercyclic if there is a vector  $x$  such that  $Orb(T, x)$  is dense in  $X$  and in this case  $x$  is called a hypercyclic vector for  $T$ .

It is interesting that many continuous linear mapping can actually be hypercyclic. The first example was constructed by Rolewicz in 1969

[9]. He showed that if  $B$  is the backward shift on  $\ell^2(\mathbb{N})$ , then  $\lambda B$  is hypercyclic if and only if  $|\lambda| > 1$ .

A nice criterion namely the Hypercyclicity Criterion, was developed independently by Kitai [8] and, Gethner and Shapiro [6]. This criterion has been used to show that hypercyclic operators arise within the classes of composition operators [4], weighted shifts [10], adjoints of multiplication operators [5], and adjoints of subnormal and hyponormal operators [3].

**The Hypercyclicity Criterion.** *Suppose  $X$  is a separable Banach space and  $T$  is a continuous linear mapping on  $X$ . If there exists two dense subsets  $Y$  and  $Z$  in  $X$  and a sequence  $\{n_k\}$  such that:*

1.  $T^{n_k}y \rightarrow 0$  for every  $y \in Y$ , and
2. *There exists functions  $S_{n_k} : Z \rightarrow X$  such that for every  $z \in Z$ ,  $S_{n_k}z \rightarrow 0$ , and  $T^{n_k}S_{n_k}z \rightarrow z$ , then  $T$  is hypercyclic.*

The above formulation of the Hypercyclicity Criterion was given by J.Bes in the Ph.D thesis [1] (see also [2]).

In the present paper we give a nice criterion that reduce the question of hypercyclicity to a study of the eigenvectors of the operators. This is a specially interesting when the eigenvectors are easy compute.

## 2. Main Results

We will denote by  $H$  an infinite-dimensional separable complex Hilbert space. For any  $T \in B(H)$ , let  $N_+(T)$  be the vector space spanned by the kernels  $\ker(T - \lambda I)$  with  $|\lambda| > 1$ , and  $N_-(T)$  be the space spanned by the kernels  $\ker(T - \lambda I)$  with  $|\lambda| < 1$ .

The following result is used by Godetroy and Shapiro in their paper [7], but is not stated explicitly there. For this reason, we sketch a proof of it.

**Theorem 2.1.** *For any bounded operator  $T$  on  $H$ , if  $N_+(T)$  and  $N_-(T)$  are dense subspace of  $H$ , then  $T$  is hypercyclic. Moreover  $T$  satisfies the Hypercyclicity Criterion.*

**Proof.** Put  $X = N_-(T)$  and  $Y = N_+(T)$ . Let  $(x_i)_{i \in I}$  be an algebraic basis of  $X$  such that for every  $i \in I$ , there exist an eigenvalue  $\lambda_i$  with  $|\lambda_i| < 1$  such that  $Tx_i = \lambda_i x_i$ . Every vector  $x$  in  $X$  can be written as a finite sum  $x = \sum a_i x_i$  and for every  $k \geq 0$ ,  $T^k x = \sum a_i \lambda_i^k x_i$ , which obviously converge to zero as  $k \rightarrow +\infty$ . Also we can choose the algebraic basis  $(y_j)_{j \in J}$  of  $Y$  such that for every  $j \in J$ , there exists an eigenvalue  $\beta_j$  with  $|\beta_j| > 1$  such that  $Ty_j = \beta_j y_j$ . Define the sequence of mapping

$\{S_k\}$  on  $Y$  by

$$S_k(y_j) = \frac{1}{(b_j)^k} y_j$$

for every  $j \in J$ . Like as above,  $S_k y \rightarrow 0$  for every  $y \in Y$ , and if  $y = \sum b_j y_j$  is any vector of  $Y$ , then  $T^k S_k y = T(\sum \frac{b_j}{\beta_j} y_j) = y$ .  $\square$

H. Salas in ([10], Theorem 3.3) shows that, the perturbation of identity operators by weighted backward shift is hypercyclic. In the following Theorem, we give a simple proof for a special case.

**Theorem 2.2.** *Let  $(e_i)_{i \geq 1}$  be an orthonormal basis of  $H$ , and  $B$  be a the backward shift defined by  $Be_1 = 0$  for  $i \geq 2, Be_i = e_{i-1}$ , then the operator  $I + wB$  is hypercyclic when  $|w| > 1$ .*

**Proof.** Let  $x = \sum_{i \geq 1} x_i e_i$  be a vector of  $H$ . If  $\lambda$  is any complex number, then  $x \in \ker(I + wB - \lambda I)$  if and only if for every  $i \geq 0$ ,  $(1 - \lambda)x_i + wx_{i+1} = 0$ , which means that  $x_{i+1} = \frac{\lambda-1}{w}x_i$ . If  $\frac{|\lambda|+1}{|w|} < 1$ , then  $\lambda$  is an eigenvector of  $I + wB$  and the eigenspace  $\ker(I + wB - \lambda I)$  is spanned by the vector

$$x_\lambda = e_1 + \sum_{n \geq 2} \left(\frac{\lambda-1}{w}\right)^{n-1} e_n.$$

The disc  $B(0, R)$ , where  $R = |w| - 1$ , entirely consists of the eigenvalues of  $I + wB$ . If  $y = \sum_{i \geq 1} y_i e_i$  belongs to the orthogonal complement of

$N_+(I + wB)$ , then the function

$$\varphi(\lambda) = \bar{y}_1 + \sum_{n=2}^{+\infty} \bar{y}_n \left( \frac{\lambda - 1}{w} \right)^{n-1}$$

vanishes on the annulus  $1 < |\lambda| < R$ . Because for every  $\lambda$  with  $1 < |\lambda| < R$ ,  $x_\lambda \in N_+(I + wB)$  and

$$\varphi(\lambda) = \langle x_\lambda, y \rangle = 0.$$

Since the sequence  $(\bar{y}_i)_{i \geq 1}$  is bounded, this function is analytic in the open disc  $B(0, R)$ , and this implies that  $\varphi$  is identically zero on the whole disc  $B(0, R)$ . Note that  $R > 1$ , because  $R = |w| - 1$  and  $|w| > 2$ . So  $\varphi(d_1) = 0$  and  $\bar{y}_1 = 0$ . This implies that  $\varphi$  can be written as

$$\varphi(\lambda) = (\lambda - d_1)(\bar{y}_2 + \sum_{n=2}^{+\infty} \bar{y}_n \left( \frac{\lambda - 1}{w} \right)^{n-1}) = (\lambda - d_1)\varphi_1(\lambda),$$

where  $\varphi_1$  is analytic on the disk of radius  $R$ . Now we repeat the above process for  $\varphi_1$  instead of  $\varphi$ , so the result is  $\varphi_1(d_2) = \bar{y}_2 = 0$ . Continuing this way, we see that all coordinates  $\bar{y}_n = 0$  for every positive integer  $n$ . This implies that  $y = 0$  and  $N_-t(I + wB)$  is dense in  $H$  and by a similar way,  $N_+t(I + wB)$  is also dense in  $H$ . Thus indeed  $I + wB$  is hypercyclic.  $\square$

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