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On the Universal Interpolating Sequences on $H^2(\beta)$

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Abstract. In this paper we investigate the relation between universal interpolating sequence and the approximate point spectrum of the adjoint multiplication operator acting on the Hilbert spaces of formal power series.

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1. Introduction

Let $\{\beta(n)\}$ be a sequence of positive numbers with $\beta(0) = 1$. We consider the space of sequences $f = \{\hat{f}(n)\}_{n=0}^{\infty}$ such that

$$\|f\|^2 = \|f\|_\beta^2 = \sum_{n=0}^{\infty} |\hat{f}(n)|^2 \beta(n)^2 < \infty.$$

The notation

$$f(z)=\sum_{n=0}^{\infty}\hat{f}(n)z^n$$

shall be used whether or not the series converges for any value of z. These are called formal power series. Let $H^2(\beta)$ denote the space of such formal power series. These are Hilbert spaces with the inner product

$$<f,g>=\sum_{n=0}^{\infty}\hat{f}(n)\overline{\hat{g}(n)}\beta(n)^2,$$

where

$$f(z)=\sum_{n=0}^{\infty}\hat{f}(n)z^n\in H^2(\beta)$$

and

$$g(z)=\sum_{n=0}^{\infty}\hat{g}(n)z^n\in H^2(\beta).$$

Let $\hat{f}_k(n)=\delta_k(n)$. So $f_k(z)=z^k$ and then $\{f_k\}_k$ is a basis such that $\|f_k\|=\beta(k)$. Now consider M_z, the operator of multiplication by z on $H^2(\beta)$ defined by

$$(M_zf)(z)=\sum_{n=0}^{\infty}\hat{f}(n)z^{n+1},$$

where

$$f(z)=\sum_{n=0}^{\infty}\hat{f}(n)z^n\in H^2(\beta).$$

In other words,

$$(M_z\hat{f})(n)=\begin{cases}\hat{f}(n-1) & n\geqslant 1\\ 0 & n=0\end{cases}.$$

Clearly M_z shifts the basis $\{f_k\}_k$. The operator M_z is bounded if and only if $\{\beta(k+1)/\beta(k)\}_k$ is bounded and in this case we have

$$\|M_z^n\| = \sup_k \frac{\beta(n+k)}{\beta(k)} \quad , \quad n = 0, 1, 2, \dots \ .$$

Also note that if

$$\sup_n \sum_{i=1}^{n} \left(\frac{\beta(n)}{\beta(i)\beta(n-i)}\right)^2 < \infty,$$

then clearly by the Holder inequality one can see that $H^2(\beta)$ is an algebra. For a good source in formal series, we refer the reader to [4, 9, 11, 12, 14, 15, 18, 19, 21].

We say that a complex number λ is a bounded point evaluation on $H^2(\beta)$ if the functional $e_\lambda : H^2(\beta) \longrightarrow \mathbb{C}$ defined by $e_\lambda(f) = f(\lambda)$ is bounded. If the point evaluation is continuous at λ, then the Riesz representation theorem implies that there is a unique function $k_\lambda \in H^2(\beta)$ such that

$$e_\lambda(f) = f(\lambda) =< f, k_\lambda >, \qquad f \in H^2(\beta).$$

The function k_λ is called the reproducing kernel for the point λ.

Let E be a Banach space. The set of bounded linear operators on E is denoted by $B(E)$. If $A \in B(E)$, the point spectrum of A, $\sigma_p(A)$, is defined by

$$\sigma_p(A) = \{\lambda \in \mathbb{C} : \ \ker(A - \lambda) \neq (0)\}.$$

Also, the approximate Point spectrum of A, $\sigma_{ap}(A)$, is defined by

$$\sigma_{ap}(A) = \{\lambda \in \mathbb{C} \quad : \quad \text{there is a sequence } \{x_n\} \text{ in } E \text{ such that}$$
$$\|x_n\| \quad = \quad 1 \text{ for all } n \text{ and } \|(A-\lambda)x_n\| \to 0\}.$$

Note that $\sigma_p(A) \subset \sigma_{ap}(A)$.

Let X be a separable reflexive Banach space whose elements are analytic functions on a complex domain Ω. A complex valued function φ on Ω for which $\varphi f \in X$ for every $f \in X$ is called a multiplier of X and the collection of all these multipliers is denoted by $M(X)$.

2. Main Theorem

Let X be a separable reflexive Banach space whose elements are analytic functions on a complex domain Ω. From [16] we note that a sequence $\{w_n\}_{n=1}^{\infty}$ of points of Ω is said an interpolating sequence for X if there exists a positive weight sequence $\{k_n\}_{n=1}^{\infty}$ such that the sequence $\{f(w_n)k_n\}_{n=1}^{\infty}$ is in ℓ^{∞} for all f in X and conversely every sequence in ℓ^{∞} can be written in that form. Also a sequence $\{w_n\}_{n=1}^{\infty}$ of points of Ω is said an interpolating sequence for $M(X)$ if for each bounded sequence $\{a_n\}_{n=1}^{\infty} \subset \mathbb{C}$, there exists $\varphi \in M(X)$ such that $\phi(w_n) = a_n$ for all $n \in \mathbb{N}$.

In [3] Carleson proved a necessary and sufficient condition for a sequence to be interpolating for H^{∞}. In [8] Carleson's result was generalized to the Hardy space H^p. Also, Berndtsson, Chang, and Lin in [2] studied the analogue of Carleson's condition for the polydisk. The multiplier space is a proper subspace of bounded analytic functions on a plane domain Ω, so that it is harder for a sequence to be interpolating for $M(X)$ than for $H^\infty(\Omega)$. Not every interpolating sequence for $H^\infty(\Omega)$ is interpolating for $M(X)$, as can be seen from the study by Sundberg and Wolff of interpolating sequences for spaces that properly lie between $M(X)$ and H^∞ ([10]). Interpolating sequences for the set of multipliers of the Dirichlet space has been studied in [1] by Axler. When H is a Hilbert space of analytic functions on a plane domain, the interpolating sequence for $M(H)$ was studied in [6] and it's extension for $M(X)$ where X is a Banach space of analytic functions on a special plane domain was studied in [16]. Also the interpolating sequence for a Banach space of analytic functions, on a special plane domain, was studied in [13].

From now on we suppose that $\lim_n \frac{\beta(n+1)}{\beta(n)} = 1$ or $\liminf_n \beta(n)^{\frac{1}{n}} = 1$. Then $H^2(\beta)$ consists of functions analytic on the open unit disc $\mathbb{D}$ and each point of the open unit disk $\mathbb{D}$ is a bounded point evaluation on $H^2(\beta)$ ([11]).

Now following the interpolation theory for the Hardy space H^2 in [8] and for certain Banach spaces of analytic functions in [13] we give the following definition.

Definition. *Suppose that* $\{w_n\}_{n\in N}$ *is a sequence of distinct points in* $\mathbb{D}$ *and consider the linear transformation* $T : H^2(\beta) \to \ell^2$ *defined by*

$$Tf = \{\frac{f(w_n)}{||e_{w_n}||}\}_{n\in N}.$$

The sequence $\{w_n\}_{n\in N}$ *is called a universal interpolating sequence for* $H^2(\beta)$ *if* T *maps* $H^2(\beta)$ *onto* ℓ^2.

For some sources on the interpolating topics one can see [5, 7, 8, 13, 16, 17, 20].

In this section we will investigate the relation between a universal interpolating sequence and the approximate point spectrum of the adjoint multiplication operator acting on $H^2(\beta)$.

Theorem. *Suppose that* $\{w_n\}_{n=1}^{\infty}$ *is a universal interpolating sequence for* $H^2(\beta)$*. If* $\mathcal{F} = \bigcap_{n\in\mathbb{N}} \ker e_{w_n}$*, then*

$$(\sigma_p(M_z^*|_{\mathcal{F}^\perp}) \cap \mathbb{D})^{-} = (\sigma_{ap}(M_z^*|_{\mathcal{F}^\perp}) \cap \mathbb{D})^{-} = \{\overline{w}_n\}^{-}_{n\in N}.$$

Proof. First note that if $\mathcal{F} \in Lat(M_z)$, then $\mathcal{F}^\perp \in Lat(M_z^*)$. Now let $\{d_n\}_{n\in N}$ be the canonical basis for ℓ^2. If $f \in H^2(\beta)$ and $n \in \mathbb{N}$, then we get

$$\begin{aligned} < f, T^*d_n > &= < Tf, d_n >=< \{\frac{f(w_k)}{||e_{w_k}||}\}, d_n > \\ &= \frac{f(w_n)}{||e_{w_n}||} =< f, \frac{e_{w_n}}{||e_{w_n}||} > . \end{aligned}$$

This implies that for $a = \{a_n\}_{n\in N} \in \ell^2$ we have

$$T^*a = \sum_{n\in N} a_n \frac{e_{w_n}}{||e_{w_n}||}.$$

If $f \in H^2(\beta)$, then

$$\begin{aligned} |< f, \sum_{n\in N} a_n \frac{e_{w_n}}{||e_{w_n}||} >| &\leqslant \sum_{n\in N} |a_n| \frac{|f(w_n)|}{||e_{w_n}||} \\ &\leqslant ||a||_2 ||Tf|| < \infty. \end{aligned}$$

Note that for all f in $\mathcal{F}$ we have $e_{w_n}(f) = f(w_n) = 0$. Thus $e_{w_n} \in \mathcal{F}^\perp$ for all $n \in \mathbb{N}$. Also if $f \in H^2(\beta)$, then we have

$$\begin{aligned} < f, M_z^* e_{w_n} > &= < M_z f, e_{w_n} >= w_n f(w_n) \\ &= w_n < f, e_{\overline{w}_n} >=< f, w_n e_{w_n} > . \end{aligned}$$

Therefore

$$M_z^* e_{w_n} = \overline{w}_n e_{w_n}.$$

But

$$(M_z - w_n)^* = M_z^* - \overline{w}_n,$$

so indeed

$$\begin{aligned} \{\overline{w}_n\}_{n\in N} &\subseteq \sigma_p(M_z^*|_{\mathcal{F}^\perp}) \cap \mathbb{D} \\ &\subseteq \sigma_{ap}(M_z^*|_{\mathcal{F}^\perp}) \cap \mathbb{D}. \end{aligned}$$

Thus to complete the proof it is sufficient to show that

$$(\sigma_{ap}(M_z^*|_{\mathcal{F}^\perp}) \cap \mathbb{D})^{\bar{}} \subseteq \{\overline{w}_n\}^{\bar{}}_{n\in N}.$$

Let $\overline{w} \in \mathbb{D} \backslash \overline{\{\overline{w}_n\}}_{n \in N}$. So there exists $r > 0$ such that $|w - w_n| > r$ for all $n \in \mathbb{N}$. Now if

$$\overline{w} \in \sigma_{ap}(M_z^*|_{\mathcal{F}^\perp}),$$

then there exists a sequence $\{h_n\}_{n \in N}$ of unit vectors in $\mathcal{F}^\perp$ such that $||(M_z - w)^* h_n|| \to 0$. Since

$$(M_z - w) = \ker\ e_w,$$

the subspace $(M_z - w)$ is closed and so $(M_z - w)^*$ is also closed. But

$$\ker(M_z - w)^* = (\,(M_z - w))^\perp = [k_w].$$

Thus $(M_z - w)^*$ is injective on $[k_w]^\perp$ and hence $(M_z - w)^*$ is bounded below on $[k_w]^\perp$. So $k_w \in \mathcal{F}^\perp$. Clearly we have $T^* = \mathcal{F}^\perp$, hence there exists a nonzero sequence $a = \{a_n\}_{n \in N} \in \ell^2$ such that $T^* a = k_w$. Now we have

$$\begin{aligned} ||(M_z - w)^* k_w|| &= ||(M_z - w)^* T^* a|| \\ &= ||\sum_{n \in N} \frac{a_n}{||k_{w_n}||}(M_z^* - \overline{w}) k_{w_n}|| \\ &= ||\sum_{n \in N} a_n(\overline{w}_n - \overline{w}) \frac{k_{w_n}}{||k_{w_n}||}|| \\ &= ||T^* d||, \end{aligned}$$

where

$$d = \{a_n(\,\overline{w_n - w}\,)\}_{n \in N} \in \ell^2$$

and $||d||_2 \geqslant r||a||_2$. On the other hand since $\{w_n\}_{n\in N}$ is a universal interpolating sequence, the operator $T : H^2(\beta) \to \ell^2$ is onto and so $T^* : \ell^2 \to H^2(\beta)$ is bounded below. So there exists $\alpha > 0$ such that $||T^*b|| \geqslant \alpha||b||_2$ for all $b \in \ell^2$. Also, since $(M_z - w)^* k_w = 0$, we obtain

$$0 = ||T^*d|| \geqslant \alpha||d||_2 \geqslant \alpha\ r||a||_2 > 0$$

which is a contradiction. Hence

$$\overline{w} \notin \sigma_{ap}(M_z^*|_{\mathcal{F}^\perp})$$

and we have proved that

$$\mathbb{D}\backslash\overline{\{\overline{w}_n\}}_{n\in N} \subseteq \mathbb{C}\backslash\ \sigma_{ap}(M_z^*|_{\mathcal{F}^\perp}).$$

Now clearly we get

$$\overline{(\sigma_{ap}(M_z^*|_{\mathcal{F}^\perp}) \cap \mathbb{D})} \subseteq \overline{\{\overline{w}_n\}}_{n\in N}.$$

This completes the proof. □

Remark. *The above theorem has been extended for the Banach spaces of analytic functions on a plane domain ([22]).*

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