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The First Survey for Abilities of Wavelets in Solving Optimal Control Problems by Embedding Methods

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Abstract. By a brief review on the applications of wavelets in solving optimal control problems, a multiresolution analysis for two dimensional Sobolev spaces and the square spline wavelets are considered. Regarding the density and approximation properties of these wavelets, for the first time, they are employed for solving optimal control problems by embedding method. Existence and the determination way for the solution are also discussed. Finally, the abilities of the new approach are explained by a numerical example and some comparisons.

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1. Introduction

As a mathematical point of view, generally, an orthonormal basis with some suitable approximation properties is called a wavelet ([5]). Indeed the first serious study in this area was done by Haar in 1910 by obtaining a basis set of functions in $L^2(\mathbb{R})$. But the basic research on wavelets for application was started about 30 years ago. In 1980's analysis and solution of the optimal control problem of dynamical systems via applying the orthogonal functions are considered by researchers like Chen and Hsiao (1975), Balanisami and Batachara (1981), and Livoshio (1981) (see [10]). Donoho in 1993 showed that wavelets could be closed to optimal solution for a class of problems (see [6]). Hsiao also in [14] presented a method for determining the control function for dynamical systems by the approximation properties of the Haar orthogonal function in $[0, 1)$ in 1997. The main advantages of this method is the transfer of the control problem into a simpler optimization one. In fact, strong properties of wavelets in solving differential equations (see [13] and [14]) recently have caused to consider them for solving optimal control problems more than before (see [16] and [17]). Moreover these works also show how wavelets are important for approximation targets (see also [1], [2], [3], [4] and [7]).

2. Dense Wavelets in $C'(\mathbb{R}^2)$

Definition. *A Multiresolution Analysis (MRA) for the Sobolev space of order k,* $H^k(\Omega)$ *(or similarly* $L^2(\mathbb{R})$*), is a sequence* $\{v_m\}_{m\in\mathbb{Z}}$ *of closed* *and linear subspaces of $H^k(\Omega)$ that satisfies the following conditions:*

- *(MRA1)* $\forall m \in \mathbb{Z}, v_m \subset v_{m+1}$,
- *(MRA2)* $\bigcap_{m\in\mathbb{Z}} v_m = \{0\}$ *and* $\overline{\bigcup_{m\in\mathbb{Z}} v_m} = H^k(\Omega)$ *(i.e.* $\bigcup_{m\in\mathbb{Z}} v_m$ *is dense in* $H^k(\Omega)$*),*
- *(MRA3)* $\forall m \in \mathbb{Z} : f(x) \in v_m \Leftrightarrow f(2x) \in v_{m+1}$,
- *(MRA4)* $\forall n \in \mathbb{Z}, f(x) \in v_0 \Leftrightarrow f(x-n) \in v_0$,
- *(MRA5) there exist a function* $\varphi \in v_0$ *such that* $\{\varphi(x-n)|n \in \mathbb{Z}\}$ *is an orthogonal basis for* v_0*;* φ *is called scaling function (or father wavelet). Moreover* $\{v_m\}_{m\in\mathbb{Z}}$*is called the MRA generated by* φ.

We note (MRA2) means that for every $f \in H^k(\Omega)$ there exists a sequence $\{f_n\}_{n\in\mathbb{N}}$ in $\bigcup_{m\in\mathbb{Z}} v_m$ which converges to f.

Definition. *A wavelet (or mother wavelet) is a function* $\psi \in L^2(\mathbb{R})$ *in which for* $\psi_{m,n}(x) = \sqrt{2^m}\psi(2^m x - n)$ *the set* $\{\psi_{m,n}(x)|m, n \in \mathbb{Z}\}$ *be an orthogonal basis for* $L^2(\mathbb{R})$*. We also define* $v_m = \overline{span\{\varphi_{m,n}(x)|n \in \mathbb{Z}\}}$ *and* $w_m = \overline{span\{\psi_{m,n}(x)|n \in \mathbb{Z}\}}$.

When we consider $\{v_0, w_0, w_1, \ldots\}$ as a basis, a function $f \in L^2(\mathbb{R})$ could be shown with respect to the above wavelet as follows ([19]):

$$f =< f, \varphi(x) > \varphi + \sum_{m=0}^{\infty} \sum_{n\in\mathbb{Z}} < f, \psi_{m,n}(x) > \psi_{m,n}(x) \tag{1}$$

in which $< f, g >= \int f(x)\overline{g(x)}dx$; this fact presents one of the powerful ability of wavelets in approximation schemes. The following example can explain it.

Example 2.1. *By applying the above equation, the known function* sin *is approximated for* $n = 0, 1, \ldots, 7$ *and two cases of* $m = 0, 1, 2, 3$ *and* $n = 0, 1, \ldots, 8$. *The following figures show how these approximations are closed.*

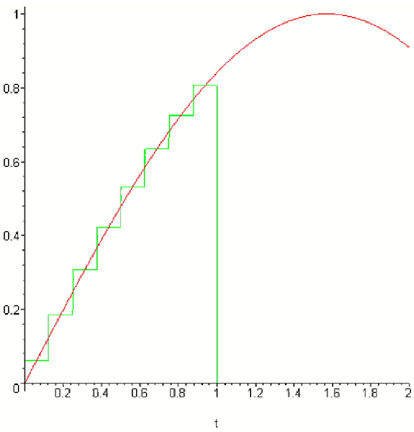

Figure 1: *Approximating* sin *function for* $m = 0, 1, 2, 3$.

Definition. *For given* $a > 0$ *and* $n \in \mathbb{Z}$, *a spline wavelet with nods in* $a\mathbb{Z}$ *is a function* $f : \mathbb{R} \to \mathbb{R}$ *in which* $f \in C^{n-1}(\mathbb{R})$ *and* $\forall j \in \mathbb{Z}, f|_{[ja,(j+1)a]}$ *is a polynomial with maximum degree* n.

Definition. *Let* $N_0(x) = \chi_{[0,1)}(x)$ *(a characteristic function); for* $n =$

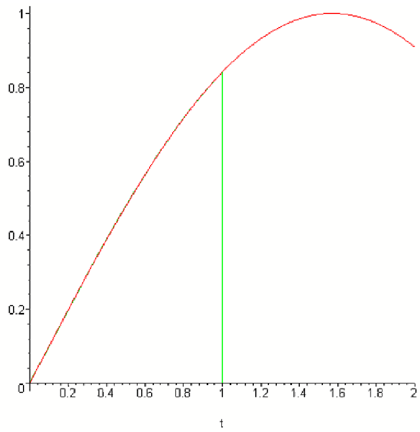

Figure 2: *Approximating* sin *function for* $m = 0, 1, \dots, 8$.

$1, 2, \dots$ *we define recursively* $N_n = N_{n-1} * N_0$ *in which* $(f * g)(x) = \int_{\mathbb{R}} f(x-y)g(y)dy$. N_n *is called a (uniform) B-spline of degree* n.

Example 2.2. *By use of Maple 8,* N_2 *(a square B-spline) is calculated as:*

$$N_2 = \begin{cases} 0 & x < 0 \\ \frac{1}{2}x^2 & 0 \leqslant x < 1 \\ -x^2 + 3x - \frac{3}{2} & 1 \leqslant x < 2 \\ \frac{1}{2}x^2 - 3x - \frac{9}{2} & 2 \leqslant x < 3 \\ 0 & 3 \leqslant x \end{cases}$$

In general, spline of degree n with nods in $a\mathbb{Z}$ is denoted by $S^n(a\mathbb{Z})$. The following important theorem for these wavelets holds (see [10]).

Theorem 2.1. *If* $v_j = S^n(2^{-j}\mathbb{Z}) \bigcap H^{n-1}(\mathbb{R})$ *then* $\{v_j\}_{j \in \mathbb{Z}}$ *is a MRA for* $H^{n-1}(\mathbb{R})$ *with* $\{N_n(x-k) : k \in \mathbb{Z}\}$ *as a basis (Riesz basis) for* v_0.

By use of the square B-spline scaling function we define

$$v_j = \overline{span\{N_2(2^j x - k), k \in \mathbb{Z}\}}$$

and

$$v_j^{'} = \overline{span\{N_2(2^j t - k), k \in \mathbb{Z}\}}.$$

Since $\{v_j\}_{j\in\mathbb{Z}}$ and $\{v_j^{'}\}_{j\in\mathbb{Z}}$ are MRA for $H^1(\mathbb{R})$, if we define

$$\begin{aligned} V_j &= v_j \bigotimes v_j^{'} \\ &= \overline{span}\{N_2(2^j x - k)N_2(2^j t - \ell); N_2(2^j x - k) \in v_j, \\ &\quad N_2(2^j t - \ell) \in v_j^{'}\}, \end{aligned}$$

then we will have the following theorem for approximation schemes.

Theorem 2.2. $\bigcup_{j\in\mathbb{Z}} V_j$ *is dense in* $C^{'}(\mathbb{R}^2)$.

Proof. By theorem 2.1 $\{V_j\}_{j\in\mathbb{Z}}$is a MRA for $H^1(\mathbb{R})$. Hence by MRA2 we have $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = H^1(\mathbb{R}^2)$; moreover ... $\subset V_{j-1} \subset V_j \subset V_{j+1}$ (MRA1) and therefore $\bigcup_{j\in\mathbb{Z}} V_j = \lim_{j\to\infty} V_j$. Since $C^{'}(\mathbb{R}^2)$ is dense in $H^1(\mathbb{R}^2)$ ([10]), and $\lim_{j\to\infty} V_j \subseteq C^{'}(\mathbb{R}^2)$, $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $C^{'}(\mathbb{R}^2)$. $\square$

3. Classical Optimal Control Problem

A classical optimal control problem has the following form

$$\text{Minimize } u \in \mathcal{U}: \qquad \int_J f_0(t, x(t), u(t))dt \equiv I(p)$$

$$\text{Subject to :} \qquad \dot{x} = g(t, x(t), u(t)), \forall t \in J$$

in which

- $J = [t_a, t_b]$ is the time interval and $J^0 = (t_a, t_b)$;
- for a bounded and closed set $U \subset \mathbb{R}^n$ the a.e. Lebesgue measurable function $u(t) : J \to U$ is the control function;
- for a bounded and closed set $U \subset \mathbb{R}^n$ the absolutely continuous function $x(t) : J \to A$ is the trajectory;
- let $\Omega = J \times A \times U$, it is supposed that $f_0 : \Omega \to \mathbb{R}$ is continuous.

The pair $p = (x, u)$ is called *admissible* if it satisfies in the above conditions; the set of all admissible pairs is denoted by W.

Problems may arise in the quest for the finding the optimal pair: It is difficult to determine the solution of the differential equations, to identify an admissible pair, to find a general applicable approximation method to estimate the optimal control and its related trajectory at the same time, and etc. We therefore change the problem into a measure theoretical one. This idea first was propounded by L. C. Young (1969) in his book ([21]), even Rosenbloom (1952) and Ghouila-Houri (1967) had done some related works previously (see [19]). In 1986 Rubio theorized this method in his book ([19]). Afterwards, base on this method (embedding method), a lot of papers was published in the area of optimal control and optimal shape design theory (such as papers by Rubio ([20]), Kamyad ([15]), Farahi ([9]), Fakharzadeh ([8]), etc).

The basis of this method is replacing the admissible pairs with positive Radon measures. First the problem will be defined in a variational form. Then it will change to a measure theoretical one in which its solution can be obtained from a linear programming problem approximately. Finally the nearly optimal pair of control and trajectory will be identify from the solution of the linear programming problem.

3.1 Metamorphosis and Existence

Rubio in ([19]) proved that each optimal pair is satisfied in the following properties:

$$\begin{aligned}
&\int_J \varphi^g[t, x(t), u(t)]dt = \varphi(t_b, x_b) - \varphi(t_a, x_a) \equiv \triangle\varphi && \forall \varphi \in C^{'}(B) \\
&\int_J \psi_j^g[t, x(t), u(t)]dt = 0 && j = 1, ..., n, \forall \psi \in D(J^0) \\
&\int_J f(t, x(t), u(t))dt = a_f && \forall f \in C_1(\Omega)
\end{aligned}$$

where a_f is the Lebesgue integral of f on J, $C^{'}(B)$ is the space of real-valued continuously differentiable functions on B such that they and their first derivatives are bounded on B, $D(J^0)$ is the space of infinitely differentiable real-valued functions with compact support in J^0, $C_1(\Omega)$ is the space of functions in $C(\Omega)$ which depend only on the variable t

and moreover, here we have:

$$\varphi^g(t,x,u) = \varphi_x(t,x)g(t,x,u) + \varphi_t(t,x).$$

For a given admissible pair p, let

$$\Lambda_p : F \in C(\Omega) \longrightarrow \int_J F(t,x(t),u(t))dt,$$

we have the following proposition.

Proposition 3.1. *Transformation $p \longrightarrow \Lambda_p$ from W to the set of linear mappings Λ_p is one to one.*

In the other hand, from Riesz Representation Theorem, one can present each Λ_p by a measure $\mu_p \in M^+(\Omega)$ (the set of all positive Radon measure on Ω) which satisfies in the above properties. Now to overcome the difficulties, we will extend the problem over the set of all pairs of measures in $\mathcal{M}^+(\Omega)$ satisfying the conditions mentioned in (1). Hence we have the following new problem.

$$\begin{aligned}
\text{Minimize}: \quad & \mu(f_0) \qquad \mu \in M^+(\Omega) \\
\text{Subject to}: \quad & \mu(\varphi^g) = \triangle\varphi \qquad \forall\varphi \in C'(B) \\
& \mu(\psi_j^g) = 0 \qquad j = 1,2,...,n, \forall\psi \in D(J^0) \\
& \mu(f) = a_f \qquad \forall f \in C_1(\Omega)
\end{aligned}$$

The above representation of the problem has many advantages. Let Q be the solution space of (1), then

- existence of an optimal pair of measures in Q is guaranteed;
- functions in (1) are linear in their arguments so the problem is Linear;
- Since extending the underlying space $\inf_Q \mu(f_0) \leqslant \inf_W I(p)$; thus the minimization is *global*.

Although the problem (1) is linear the underling space still has infinite dimension. It would be more appropriate if some how one could obtain its solution from the solution of a finite linear programming problem.

4. Approximation

Now we explain how to approximate the optimal pair for the classical problem. This fact is done in three steps. First the number of constraints be fixed. Then by discretizing, and using the properties of the optimal measure, the problem changes into a finite linear programming one. Finally the last step is related to the applications of wavelets.

Step one: We choose a countable total sets in $C'(B)$, $D(J^0)$ and $C_1(\omega)$. For the first set of equations in (1), we choose the function $\{\varphi_j\}_{j\in\mathbb{Z}}$ as

$$\varphi_j = \sum_{k\in\mathbb{Z}^2} N_2(2^j x - k_1)N_2(2^j t - k_2) \quad , \quad k = (k_1, k_2). \tag{1}$$

Since V_j's are dense in $C'(B)$ (theorem 2.2), the set of finite linear combinations of φ_j's are dense in $C'(B)$. We remind that up to now researchers usually have used the set of polynomials $\{t, x, tx, t^2x, ...\}$ which may cause some problems in feasibility.

For the second and third sets of equations in (1), the same as in [19], we respectively select

$$\psi_j = \sin(2\pi j \frac{t - t_a}{\triangle t}) \ , \psi_j = 1 - \cos(2\pi j \frac{t - t_a}{\triangle t}), \ j = 1, 2, ...$$

$$f_s(t) = \begin{cases} 1 & t \in J_s \\ 0 & t \notin J_s \end{cases}$$

where $J_s = [t_a + (s-1)d, \ \ t_a + sd]$ and $d = \frac{\triangle t}{M_3}$ for a given M_3. Although the last function is not continuous, their linear combinations can approximate a function in $C_1(\Omega)$ arbitrarily well; also they are very important for determining the optimal control. Moreover the following proposition (see [19]) shows that the solution can be approximated just by finite number of these functions.

Proposition 4.1. *Let M_1, M_2 and M_3 be the positive integers. Consider the problem of minimizing the function $\mu \longrightarrow \mu(f_0)$ over the set $Q(M_1, M_2, M_3)$ of measures in $\mathcal{M}^+(\Omega)$ satisfying*

$$\mu(\varphi_j^g) = \triangle\varphi_j \qquad j = 1, 2, \ldots, M_1$$

$$\mu(\psi_i^g) = 0 \qquad i = 1, 2, \ldots, M_2$$

$$\mu(f_s) = a_s \qquad s = 1, 2, \ldots, M_3$$

then $\inf_{Q(M_1,M_2,M_3)} \mu(f_0)$ *tends to* $\inf_{Q} \mu(f_0)$ *when* $M_1, M_2, M_3 \longrightarrow \infty$.

Step two: By the results of Rosenbloom works in [18], the optimal measure of (2), μ^*, has the form

$$\mu^* = \sum_{k=1}^{M_1+M_2+M_3} \alpha_k^* \delta(z_k^*) \text{ where } z_k^* \in \Omega, \alpha_k^* \geqslant 0$$

and $\delta(z_k^*)$ is the unitary atomic measure with support the singleton set $\{z_k^*\}$. But applying this fact convert (2) to a nonlinear one in which its unknowns are the coefficients $\{\alpha_k^*\}_{i=1}^{M_1+M_2+M_3}$ and the supports $\{z_k^*\}_{i=1}^{M_1+M_2+M_3}$. However as shown in [19], it is sufficient to select the points z_k^*'s from a dense subset of Ω. Now by applying a discretization on Ω with nods in a dense subset of it, one is able to approximate the solution of (1) by the solution of a finite linear programming problem.

Step three: Although the number of φ_j in (2) is finite, each one is defined by an infinite series in which its calculations, indeed, is impossible (especially in numerical works). Considering the ability of wavelets in approximation (section 2), by selecting the finite number of wavelets in (1) each φ_j can be calculated approximately. Therefor if the number of wavelets in this calculation increases, the calculation will be more accurate. Hence, by choosing M_1, we select $K_{1M_1} \times K_{2M_1}$ of this terms and then obtain the nearly optimal solution of (1) from the solution of the following FLP problem. (We mention, to compare with Rubio's method, this one has an extra approximation step, one must remember the abilities of wavelets, specially in sobolove spaces and when the control system is governed by a partial differential equation.)

$$\begin{aligned}
\text{Minimize :} \quad & \sum_{j=1}^{N} \alpha_j f_0(z_j) \\
\text{Subject to :} \quad & \sum_{j=1}^{N} \alpha_j \Big(\sum_{(k_1,k_2)} N_2(2^m x_j - k_1) N_2(2^m t_j - k_2) \Big)^g \\
& = \triangle \sum_{(k_1,k_2)} N_2(2^m x_j - k_1) N_2(2^m t_j - k_2)
\end{aligned}$$

$$k_1 = 0,1,2,...,K_{1m}, \quad k_2 = 0,1,2,...,K_{2m}, \quad m = 0,1,...,M_1$$

$$\begin{aligned}
\sum_{j=1}^{N} \alpha_j \psi_h^g(z_j) = 0 \qquad & h = 1,2,...,M_2 \\
\sum_{j=1}^{N} \alpha_j f_s(z_j) = a_s \qquad & s = 1,2,...,M_3 \\
\alpha_j \geqslant 0 \qquad & j = 1,2,...,N
\end{aligned}$$

5. Numerical Example

The same as the example 3 in [19] chapter 5, we assume $f_0 = u^2$, $\dot{x}(t) = \frac{1}{2}x + u$ and $x(0) = 0$. Each interval $J = A = U = [0,1]$ was divided into 10 equal subintervals and a point in each was selected; hence $N = 1000$. Also for $M_1 = 3$, we had

$$\varphi_1(t,x) = N_2(x)N_2(t),$$

$$\varphi_2(t,x) = \sum_{k=0}^{3}\sum_{\ell=0}^{3} N_2(2^2x-k)N_2(2^2t-\ell),$$
$$\varphi_3(t,x) = \sum_{k=0}^{31}\sum_{\ell=0}^{31} N_2(2^5x-k)N_2(2^5t-\ell);$$

hence $\triangle\varphi_1 = 0.25$, $\triangle\varphi_2 = 0$ and $\triangle\varphi_3 = 0$. Finally $M_2 = 7$ and $M_3 = 10$ and therefor each $a_s = 0.1$ were chosen. In this manner a linear programming problem like (2) with 1000 unknowns and 20 equation was set up. Then this problem was solved by revised simplex method with subroutine DLPRS in IMSL library of Compaq Visual Fortran6. The optimal value of objective function was 0.145218. Afterwards, as explained in [19], the nearly optimal pair of control and trajectory functions are obtained by help of Maple8, and drawn in figures (3) and (4). (We remind that the optimal value from Rubio's method in [19] is 0.1451 and the main result is 0.14549.)

Conclusion. *The explained method was the first attempt to solve optimal control problems by applying measures and wavelets. Although in this Manner a step in approximation scheme and a state in discontinuity are increased, but regarding the wide and growing up abilities of wavelets, it could be much more applicable especially for more complicated cases.*

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