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# Ramanujan Cayley graphs of some sporadic and linear groups

**Reza Safakish**
Department of Mathematics, Bu-Ali Sina University, Hamedan, Iran
safakish@basu.ac.ir

**Abstract.** Let $\Gamma$ be a k-regular graph with the second maximum eigenvalue $\lambda$. Then $\Gamma$ is a Ramanujan graph if $\lambda \leq 2\sqrt{k-1}$. Let G be a finite group and $\Gamma = Cay(G, S)$ be a Cayley graph related to G. The aim of this paper is to investigate the Ramanujan Cayley graphs of sporadic groups.

**AMS Subject Classification:** MSC code1; MSC code 2; more
**Keywords and Phrases:** sporadic group, character table, Cayley graph, eigenvalue.

## 1 Introduction

Recently the theory of Ramanujan graphs has received more attention in the literate. It is a wellknown fact that these graphs resolve an extremal problem in communication network theory. On the other hand, they fuse diverse branches of pure mathematics, namely, number theory, representaion theory and algebraic geometry. The aim of the present paper is to determine the Ramanujan Cayley graph $\Gamma = Cay(G, S)$ From the perspective of a normal symmetric generating subset (or NSGS for

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short) where G is a sparse group. It should be noted that computing the spectrum of Cayley graphs was started by a paper of Babai [1] in 1979 and recently, this exciting research topic is received increasing attention by mathematician, see for example [2, 4, 7, 9]. Most of results of this paper are based on Theorem 2.2. In the next section, we give the necessary definitions and some preliminary results and section three contains the main results, namely, computing the Ramanujan Cayley graph of linear and sporadic groups. All graphs and groups considered in this paper are finite. Also all graphs are connected graphs without loops and parallel edges.

## 2 Definitions and Preliminaries

Let $\Gamma$ be a k-regular graph with the second maximum eigenvalue $\lambda$. Therfore if

$$\lambda \leq 2\sqrt{k-1}.$$

then $\Gamma$ it is known as a Ramanujan graph.

In this article, a symmetric subset of a group such as G is a subset such as S of G, where 1 is not a member of S and S is equal to $S^{-1}$. In addition, graph $\Gamma = Cay(G, S)$ with state S is a graph whose vertex set $V(\Gamma) = G$ and two vertices x, y belongs to is $V(\Gamma)$. If y is equal to xs and vice versa for element $s \in S$ they are adjacent. It is a known fact that $Cay(G, S)$ is connected if S generates a group G and vice versa, this is appreciable in [3, 14]. A general linear group $GL(V)$ of vector space V is the set of all $A \in End(V)$ where A is invertible. A representation of group G is a homomorphism $\alpha : G \to GL(V)$ and the degree of $\alpha$ is equal to the dimension of V . A trivial representation is a homomorphism $\alpha : G \to \mathbb{C}^*$ where $\alpha(g) = 1$ for all $g \in G$. Let $\varphi : G \to GL(V)$ be a representation with $\varphi(g) = \varphi_g$, the character $\chi_\varphi : G \to \mathbb{C}$ of $\varphi$ is defined as $\chi_\varphi(g) = tr(\varphi_g)$: An irreducible character is the character of an irreducible representation and the character $\chi$ is linear, if $\chi(1) = 1$. We denote the set of all irreducible characters of G by $Irr(G)$. The number of irreducible characters of G is equal to the number of conjugacy classes of G and the number of linear characters of finite group G is $|G/G'|$ which is $G'$ here, the derivative subgroup of G.

A character table is a matrix whose rows and columns are correspond to the irreducible characters and the conjugacy classes of G, respectively. Examining the spectrum of Cayley graphs will be closely related to the irreducible characters of G. If G is abelian, then the spectrum of $\Gamma = Cay(G,S)$ can easily be determined as follows.

**Theorem 2.1.** *Let S be a symmetric subset of abelian group G. Then the eigenvalues of the adjacency matrix of $Cay(G,S)$ are given by*

$$\lambda_\varphi = \sum_{s\in S} \varphi(s)$$

*where $\varphi \in Irr(G)$.*

Let G be a finite group with symmetric subset S . We recall that S is a normal subset if and only if $S^g = g^{-1}Sg = S$, for all $g \in G$. The following theorem is implicitly contained in [6, 11].

**Theorem 2.2.** *([6]). Let $\alpha$ is the characteristic function of S, $Cay(G,S)$ be a Cayley graph and $\varphi_k (k = 1,\dots,n)$ be an irreducible inequivalent representation of G. Let $d_k$ be the degree of $\varphi_k$ and $\varepsilon_k$ denote the set of eigenvalues of linear map $\sum_{g\in G} \alpha(g)\varphi(g)$. Then*

*i) the set of eigenvalues of A (adjacency matrix of $Cay(G,S)$) equal $\cup_{k=1}^{n} \varepsilon_k$; and*

*ii) if the eigenvalue $\lambda$ occurs with multiplicity $m_k(\lambda)$ in $\sum_{g\in G} \alpha(g)\varphi(g)$, then the multiplicity of $\lambda$ in A is $\sum_{k=1}^{n} d_k m_k(\lambda)$*
*If $\alpha$ be a class function, then*

$$\lambda_k = \frac{|G|}{d_k} \langle \alpha, \overline{\chi}_k \rangle.$$

**Corollary 2.3.** *Let G be a finite group with a normal symmetric subset S . Let A be the adjacency matrix of graph $\Gamma = Cay(G,S)$. Then the eigenvalues of A are given by*

$$[\chi_\chi]^{\chi(1)^2}, \chi \in Irr(G)$$

*where $\lambda_\chi = \frac{1}{\chi(1)} \sum_{s\in S} \chi(s)$.*
*Thus, in a Ramanujan Cayley graph, we have*

$$\sum_{s\in S} \chi(s) \leq 2\chi(1)\sqrt{|S|-1}.$$

*In what follows assume that*

$$\delta_A(B) = \begin{cases} 1 & A \subseteq B \\ 0 & A \nsubseteq B \end{cases}.$$

*Consider the cyclic group* $\mathbb{Z}_n$ *in two separately cases:*

*Case 1.* $n$ *is odd, thus* $C_i = \{x^i, x^{-i}\}(1 \leq i \leq \frac{n-1}{2})$ *are normal symmetric subsets of* $\mathbb{Z}_n$ *and so*

$$S \subseteq \bigcup_{i=1}^{\frac{n-1}{2}} C_i.$$

*For* $0 \leq j \leq n-1, \chi_j(x^i) = \omega^{ij}$ *are all irreducible characters of* $\mathbb{Z}_n$*, where* $x$ *is the generator for* $\mathbb{Z}_n$ *, where* $\omega = e^{\frac{2\pi}{n}i}$ *. Therefore*

$$\lambda_{\chi_j} = \sum_{i=1}^{\frac{n-1}{2}} \delta_{C_i}(S)(\omega^{ij} + \omega^{-ij}).$$

*Case 2.* $n$ *is even, hence all normal symmetric subsets are*

$$C_i = \{x^i, x^{-i}\}(1 \leq i \leq \frac{n}{2} - 2) \, and \, C_{\frac{n}{2}-1} = \{\chi^{n/2}\}.$$

*Therefore,*

$$S \subseteq \bigcup_{i=1}^{\frac{n}{2}-2} C_i.$$

*Similar to the last case, we have*

$$\lambda_{\chi j} = \sum_{i=1}^{\frac{n}{2}-2} \delta_{C_i}(S)(\omega^{ij} + \omega^{-ij}) + (-1)^j \delta_{C_{\frac{n}{2}-1}}(S).$$

*In this situation, consider the dihedral group as*

$$D_{2n} = \langle a, b, a^n = b^2 = 1, b^{-1}ab = a^{-1} \rangle.$$

*Here, by using theorem 2.2, the spectrum of* $Cay(D_{2n}, S)$ *is determined, where* $S$ *is NSGS. Let us to show the conjugacy class of* $g \in G$ *by* $g^G$*. To find the number of conjugate classes of the dihedral group, two separate situations can be considered.*

*Case 1. If n is odd, then $D_{2n}$ has precisely $\frac{1}{2}(n+3)$ conjugacy classes:*

$$\{1\}, \{a^i, a^{-1}\}(1 \leq i \leq (n-1)/2), \{b, ba, \cdots, ba^{n-1}\}.$$

*If n is odd, then $D_{2n}$ has exactly $\frac{1}{2}(n+3)$ conjugate classes as*

$$C_i = \{a^i, a^{-i}\},\ (1 \leq i \leq \frac{n-1}{2})\, and\, C_{\frac{n+1}{2}} = b^{D_{2n}}.$$

*Will have. Therfore, the normal symmetric subsets of $D_{2n}$ are*

*This shows that S is a subset of $\bigcup_{i=1}^{\frac{n+1}{2}} C_i$ and similarly using the table 1 , is obtained*

$$\lambda_{\chi_1} = n\delta_{C_{\frac{n+1}{2}}}(S) + 2\sum_{i=1}^{\frac{n-1}{2}} \delta_{C_i}(S),$$
$$\lambda_{\chi_2} = -n\delta_{C_{\frac{n+1}{2}}}(S) + 2\sum_{i=1}^{\frac{n-1}{2}} \delta_{C_i}(S),$$
$$\lambda_{\psi_j} = \sum_{i=1}^{\frac{n-1}{2}} \delta_{C_i}(S)(\varepsilon^{ij} + \varepsilon^{-ij})(1 \leq j \leq \frac{n-1}{2}),$$

*where $\varepsilon = e^{\frac{2\pi}{n}i}$.*

*Case 2. If n is even, then $D_{2n}$ has precisely $\frac{n}{2}+3$ conjugacy classes:*

$$\{1\}, \{a^{\frac{n}{2}}\}, \{a^i, a^{-i}\}, \{ba^{2j}, \{ba^{2j+1}\}.$$

*So, the normal symmetric subsets of $D_{2n}$ are equal to:*

$$C_i = \{a^i, a^{-i}\},\ (1 \leq i \leq \frac{n}{2} - 1), C_{\frac{n}{2}} = \{a^{n/2}\}, C_{\frac{n}{2}+1} = b^{D_{2n}}\, and\, C_{\frac{n}{2}+2} = ba^{D_{2n}}.$$

*As a result,* $S \subseteq \bigcup_{i=1}^{\frac{n+1}{2}} C_i$ *and by using Table 2, we will have*

$$\lambda_{\chi_1} = \delta_{C_{\frac{n}{2}}}(S) + \frac{n}{2}(\delta_{C_{\frac{n}{2}+1}}(S) + \delta_{C_{\frac{n}{2}+2}}) + 2\sum_{i=1}^{\frac{n}{2}-1}\delta_{C_i}(S),$$

$$\lambda_{\chi_2} = \delta_{C_{\frac{n}{2}}}(S) - \frac{n}{2}(\delta_{C_{\frac{n}{2}+1}}(S) + \delta_{C_{\frac{n}{2}+2}}(S)) + 2\sum_{i=1}^{\frac{n}{2}-1}\delta_{C_i}(S),$$

$$\lambda_{\chi_3} = (-1)^{\frac{n}{2}}\delta_{C_{\frac{n}{2}}}(S) + \frac{n}{2}(\delta_{C_{\frac{n}{2}+1}}(S) - \delta_{C_{\frac{n}{2}+2}}(S)) + 2\sum_{i=1}^{\frac{n}{2}-1}\delta_{C_i}(S)(-1)^j,$$

$$\lambda_{\chi_4} = (-1)^{\frac{n}{2}}\delta_{C_{\frac{n}{2}}}(S) - \frac{n}{2}(\delta_{C_{\frac{n}{2}+1}}(S) - \delta_{C_{\frac{n}{2}+2}}(S)) + 2\sum_{i=1}^{\frac{n}{2}-1}\delta_{C_i}(S)(-1)^j,$$

$$\lambda_{\psi_j} = (-1)^j\delta_{C_{\frac{n}{2}}}(S) + \sum_{i=1}^{\frac{n}{2}-1}\delta_{C_i}(S)(\varepsilon^{ij} + \varepsilon^{-ij})(1 \leq j \leq \frac{n}{2} - 1).$$

*As a special situation, the minimal SNGS of group* $D_{2n}$ *is*

$$\Delta = \begin{cases} b^{D_{2n}} \cup \{a, a^{-1}\}, & 2 \mid n \\ b^{D_{2n}}, & 2 \nmid n \end{cases}.$$

*Hence, the spectrum of Cayley graph* $\Gamma = Cay(D_{2n}, \Delta)$ *is*

- *n is odd:*

$$\{[-n]^1, [n]^1, [0]^{2n-2}\}.$$

*Since* $0 \leq 2\sqrt{n-1}$*, inthiscase* $Cay(D_{2n}, S)$ *is Ramanujan.*

- *n is even:*

$$\{[\pm n/2 \pm 2]^1, [0]^{2n-4}\}.$$

*Since for* $n \geq 6, \frac{n}{2} - 2 \geq 2\sqrt{\frac{n}{2}+1}$, $Cay(D_{2n}, S)$ *is not Ramanujan.*

| g | 1 | $a^r$ | b |
|---|---|---|---|
| $\chi_1$ | 1 | 1 | 1 |
| $\chi_2$ | 1 | 1 | -1 |
| $\psi_j$ | 2 | $\varepsilon^{jr}+\varepsilon^{-jr}$ | 0 |

**Table 1:** The character table of group $D_{2n}$ where n is odd and $1 \leq r, j \leq \frac{n-1}{2}$.

| g | 1 | $a^{\frac{n}{2}}$ | $a^r$ | b | ba |
|---|---|---|---|---|---|
| $\chi_1$ | 1 | 1 | 1 | 1 | 1 |
| $\chi_2$ | 1 | 1 | 1 | -1 | -1 |
| $\chi_3$ | 1 | $(-1)^{\frac{n}{2}}$ | $(-1)^r$ | 1 | -1 |
| $\chi_4$ | 1 | $(-1)^{\frac{n}{2}}$ | $(-1)^r$ | -1 | 1 |
| $\psi_j$ | 2 | $2(-1)^j$ | $\varepsilon^{jr}+\varepsilon^{-jr}$ | 0 | 0 |

**Table 2:** For the group $D_{2n}$ the character table where n is odd and $1 \leq r, j \leq \frac{n}{2} - 1$.

*Considering that all eigenvalues $\Gamma = Cay(D_{2n}, S)$ are symmetric with respect to the origin, based on [6] Theorem 3.2.3 $\Gamma$ must be bipartite*

# 3 Main Results

*By assessment Cayley graphs, even more detailed information about can be obtained. Foe example, the automorphism graph of a Cayley graph whose all eigenvalues are simple is an elementary 2-group. The aim of this section is to investigate Ramanujan Cayley graph $Cay(G, S)$ via character table of $G$ where $S$ is a NSGS of $G$ and $G$ is a sporadic group.*

**Example 3.1.** Consider group $T_{4n}$ with the following presentation:

$$T_{4n} = \langle a, b | a^{2n} = 1, a^n = b^2, b^{-1}ab = a^{-1} \rangle.$$

The conjugacy classes of $T_{4n}$ are

$$\{1\}, \{a^n\}, \{a^m, a^{-m}, 1 \leqslant m \leqslant n-1\},$$
$$\{ba^{2j}, 0 \leqslant j \leqslant n-1\}, \{ba^{2j+1}, 0 \leqslant j \leqslant n-1\}.$$

Let $S = \{a, a^{-1}, b, b^{-1}\}$.

Case 1. n is even, then all irreducible representations of $T_{4n}$ are as follows:

$$id : (a,b) \to (1,1), \qquad \varphi_1 : (a,b) \to (1,-1),$$
$$\varphi_2 : (a,b) \to (-1,1), \qquad \varphi_3 : (a,b) \to (-1,-1)$$

and

$$\psi_k : (a,b) \to (\begin{pmatrix} \varepsilon^k & 0 \\ 0 & \varepsilon^{-k} \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ \varepsilon^{kn} & 0 \end{pmatrix})$$

where $\varepsilon = e^{\frac{2\pi i}{2n}} (0 \leqslant k \leqslant n-1)$. If

$$\varphi_1(a, a^{-1}, b, b^{-1}) = (1, 1, -1, -1),$$

then we conclude thate $\lambda_1 = 0$ and if

$$\varphi_2(a, a^{-1}, b, b^{-1}) = (-1, -1, 1, 1),$$

then $\lambda_2 = 0$. By regarding $\varphi_3$ we achieve $\lambda_3 = -4$. Therefor the second maximum eigenvalue $\lambda$ can be obtained from a non-linear irreducible representation. In other words

$$\lambda_k = 2\cos\frac{2k\pi}{2n} \pm (1 + \cos k\pi).$$

Case 2. n is odd, then all irreducible characters are

$$id : (a,b) \to (1,1), \qquad \varphi_1 : (a,b) \to (-1,i),$$
$$\varphi_2 : (a,b) \to (1,-1), \qquad \varphi_3 : (a,b) \to (-1,i)$$

and

$$\psi_k : (a,b) \to (\begin{pmatrix} \varepsilon^k & 0 \\ 0 & \varepsilon^{-k} \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ \varepsilon^{kn} & 0 \end{pmatrix})$$

where $\varepsilon = e^{\frac{2\pi i}{2n}} (0 \leqslant k \leqslant n-1)$. For $n \geqslant 6$,

$$\frac{\pi}{n} \leq \frac{\pi}{6} \Rightarrow 2\cos\frac{\pi}{n} + 2 \geq 2\cos\frac{\pi}{6} + 2 > 2\sqrt{3}.$$

This means that $Cay(G,S)$ is not Ramanujan. Hence, in this case $Cay(T_{4n}, S)$ is Ramanujan if and only if $n = 1, 3, 5$. Similar to Case 1, the $Cay(G,S)$ is Ramanujan if and only if $n = 1, 3$. Hence we can verify that $Cay(T_{4n}, S)$ is Ramanujan if and only if $n = 1, 2, 3, 4$.

**Example 3.2.** Consider now $U_{6n}$ with the following presentation:

$$U_{6n} = \langle a, b | a^{2n} = b^3 = 1, a^{-1}ba = b^{-1} \rangle$$

and set $S = \{a, a^{-1}, b, b^{-1}\}$, Clearly, S is not normal. For $0 \leq j \leq n-1$ the conjugacy classes of $U_{6n}$ are as follows:

$$\{a^{2j}\}, \{a^{2j}b, a^{2j}b^2\}, \{a^{2j+1}, a^{2j+1}b, a^{2j+1}b^2\}.$$

All irreducible representations are as follows:

$$\psi : (a, b) \to (0, -1),$$
$$\varphi_k : (a, b) \to (\varepsilon^{2k}, 1), 0 \leq k \leq 2n-1,$$

and

$$\psi_k : (a, b) \to (\begin{pmatrix} 0 & \varepsilon^k \\ \varepsilon^{-k} & 0 \end{pmatrix}, \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix})$$

where $\varepsilon = e^{\frac{2\pi i}{2n}}$, $\omega = e^{\frac{2\pi i}{3}}$ For linear representation we have

$$\lambda_k = \psi_k(a) + \psi_k(a^{-1}) + \psi_k(b) + \psi_k(b^{-1}) = \varepsilon^{2k} + \varepsilon^{-2k} + 2 = 2 + 2\cos\frac{2k\pi}{2n}$$

and for non-linear representation we have:

$$\sum_{g \in S} \psi_k = \begin{pmatrix} \omega + \omega^2 & \varepsilon^k + \varepsilon^{-k} \\ \varepsilon^k + \varepsilon^{-k} & \omega + \omega^2 \end{pmatrix} = \begin{pmatrix} -1 & 2\cos\frac{k\pi}{n} \\ 2\cos\frac{k\pi}{n} & -1 \end{pmatrix}.$$

Thus

$$\mu_k = -1 \pm 2\cos\frac{k\pi}{n}.$$

One can see that $|\mu| < 2\sqrt{3}$ and for $n \geq 9, k = 1$ we have

$$2 + 2\cos\frac{k\pi}{n} \geq 2 + 2\cos\frac{k\pi}{9} \geq 2\sqrt{3}.$$

On the other hand, for $n \leq 8, \lambda < 2\sqrt{3}$ and thus $Cay(U_{6n}, S)$ is Ramanujan if and only if $n \leq 8$.

**Example 3.3.** Suppose the group $V_{8n}$ has the following presentation:

$$V_{8n} = \langle a, b | a^{2n} = b^4 = 1, aba = b^{-1}, ab^{-1}a = b \rangle.$$

For $1 \leq r \leq \frac{n-1}{2}$ and $0 \leq s \leq n-1$, the conjugacy classes of $V_{8n}$ are as follows:

$$\{1\}, \{b^2\}, \{a^{2r}, a^{-2r}\}, \{a^{2r}b^2, a^{-2r}b^2\}, \{a^{2s+1}, a^{-2s-1}b^2\},$$
$$\{a^{2l}b, a^{2l}b^3 | 0 \leq l \leq n-1\}, \{a^{2l+1}b, a^{2l+1}b^3, 0 \leq l \leq n-1\}.$$

It is clear that $S = \{a, a^{-1}, b, b^{-1}\}$ is not normal and all irreducible representation of $V_{8n}$ are as follows:

$$\varphi_k : (a, b) \to (\begin{pmatrix} \varepsilon^k & 0 \\ 0 & -\varepsilon^{-k} \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}), 1 \leq k \leq \tfrac{n-1}{2}$$

Hence,

$$\sum_{g \in S} \psi_k = \begin{pmatrix} \varepsilon^{2k} + \varepsilon^{-2k} & 0 \\ 0 & -(\varepsilon^{2k} + \varepsilon^{-2k}) \end{pmatrix}.$$

This yields that $\lambda_k = \pm 2\cos\frac{2k\pi}{n}$ and so $|\lambda_k| =< 2\sqrt{3}$. On the other hand,

$$\sum_{g \in S} \psi_k = \begin{pmatrix} \varepsilon^{k} + \varepsilon^{-k} & 0 \\ 0 & \varepsilon^{k} + \varepsilon^{-k} \end{pmatrix}.$$

implies that $\lambda_k = 2\cos\frac{k\pi}{n}$ and thus $|\lambda_k| =< 2\sqrt{3}$. Therefor, $Cay(V_{8n}, S)$ is Ramanujan.

## 3.1 Liner Groups

*Let $v(n, \mho)$ be an n-dimensional vector space over the field F.*

*Every transvection is a linear transformation $T$ on $v(n, \mho)$ with $\lambda$ equals one as an eigenvalue and such that rank$(T - I_n)$, is equal to 1, where $I_n$ is the edentity transformation on $V(n, \mathbb{F})$. Based on matrixe a transvection $A_{ij}(\alpha)$ where i and j are unequal and alpha is a member of $\mathbb{F}$ , The matrix is different from the identity that has $\alpha$ in row i and column j. It seems that all the transvections elements of $SL(n, \mathbb{F})$.*

**Proposition 3.4.** *[1] If $i, j$ are assumed fixed, Expressions* $\mathcal{A}_{ij} = \{A_{ij}(\alpha) | \alpha \in \mathbb{F}\}$ *create a subgroup of* $SL(n, \mathbb{F})$.

*Subgroups defined by this method are known as root subgroups of* $GL(n, \mathbb{F})$. *By proposition 3.4,Subgroups of the root* $A_{ij} of the group create the group S$ *L(n, F).* $To put it similarly, SL(n, \mathbb{F}) = \langle \mathcal{A}_{ij} : 1 \leqslant i \neq j \leqslant n \rangle$By using proposition 3.4 group $GL(n, \mathbb{F})$ is also generated by the set of all invertible diagonal matrices and all transvections.

**Theorem 3.5.** *In* $GL(n, q)$ *all transvections are conjugate and if* $n$ *is greater than 2, in this case, all transvections in* $SL(n, q)$ *will be conjugate.*

Conjugacy classes of $SL(2, q)$, q is odd. The number of classes of $SL(2, q)$ is $q + 4$ (see [1]) and two following cases hold:

Case 1. q is odd, the character table of $SL(2, q)$ is as reported in Table 3.

| Type | Rep g | No. of CC | $\|[g]\|$ |
|---|---|---|---|
| $\tau_0^{(1)}$ | $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ | 1 | 1 |
| $-\tau_0^{(1)}$ | $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ | 1 | 1 |
| $\tau_{01}^{(2)}$ | $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ | 1 | $\frac{q^2-1}{2}$ |
| $-\tau_{01}^{(2)}$ | $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ | 1 | $\frac{q^2-1}{2}$ |
| $\tau_{0\varepsilon}^{(2)}$ | $\begin{pmatrix} 1 & \varepsilon \\ 0 & 1 \end{pmatrix}$ | 1 | $\frac{q^2-1}{2}$ |
| $-\tau_{0\varepsilon}^{(2)}$ | $\begin{pmatrix} -1 & -\varepsilon \\ 0 & 1 \end{pmatrix}$ | 1 | $\frac{q^2-1}{2}$ |
| $\tau_{k,-k}^{(3)}$ | $\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$ | $\frac{q-3}{2}$ | q(q+1) |
| $-\tau_k^{(4)}$ | $\begin{pmatrix} 0 & 1 \\ -1 & -(r + r^q) \end{pmatrix}$ | $\frac{q-1}{2}$ | $q(q-1)$ |

**Table 3:** The character table of $SL(2, q), q$ is odd:

where, in table 3,

- by No.of CC we introduce number of conjugacy classes of prescribed type of classes,
- by Rep g we mean representation of g.

| Class | $\tau_0^{(1)}$ | $-\tau_0^{(1)}$ | $\tau_{01}^{(2)}$ | $-\tau_{01}^{(2)}$ |
|---|---|---|---|---|
| Repg | $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ | $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ | $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ | $\begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}$ |
| $\|[g]\|$ | 1 | 1 | $\frac{q^2-1}{2}$ | $\frac{q^2-1}{2}$ |
| $\lambda$ | 1 | 1 | 1 | 1 |
| $\psi$ | $q$ | $q$ | 0 | 0 |
| $\psi_{k,1}$ | $q+1$ | $(-1)^{k+1}(q+1)$ | 1 | 1 |
| $\pi_k$ | $q-1$ | $(-1)^{k+1}(q-1)$ | -1 | $(-1)^{k+1}$ |
| $\xi_1$ | $\frac{q+1}{2}$ | $\theta\frac{(q+1)}{2}$ | $\frac{1}{2}(1+\sqrt{\theta q})$ | $\frac{\theta}{2}(1+\sqrt{\theta q})$ |
| $\xi_2$ | $\frac{q+1}{2}$ | $\theta\frac{(q+1)}{2}$ | $\frac{1}{2}(1-\sqrt{\theta q})$ | $\frac{\theta}{2}(1-\sqrt{\theta q})$ |
| $v_1$ | $\frac{q-1}{2}$ | $-\theta\frac{(q-1)}{2}$ | $\frac{1}{2}(-1+\sqrt{\theta q})$ | $\frac{-\theta}{2}(1+\sqrt{\theta q})$ |
| $v_2$ | $\frac{q-1}{2}$ | $-\theta\frac{(q-1)}{2}$ | $\frac{1}{2}(-1-\sqrt{\theta q})$ | $\frac{-\theta}{2}(-1-\sqrt{\theta q})$ |

continued:

| Class | $\tau_{0\varepsilon}^{(2)}$ | $-\tau_{0\varepsilon}^{(2)}$ | $\tau_{k,-k}^{(3)}$ | $-\tau_k^{(4)}$ |
|---|---|---|---|---|
| Repg | $\begin{pmatrix} 1 & \varepsilon \\ 0 & 1 \end{pmatrix}$ | $\begin{pmatrix} -1 & -\varepsilon \\ 0 & 1 \end{pmatrix}$ | $\begin{pmatrix} \alpha & 1 \\ 0 & \alpha^{-1} \end{pmatrix}$ | $\begin{pmatrix} 0 & 1 \\ -1 & -(r+r^q) \end{pmatrix}$ |
| $\|[g]\|$ | $\frac{q^2-1}{2}$ | $\frac{q^2-1}{2}$ | $q(q+1)$ | $q(q-1)$ |
| $\lambda$ | 1 | 1 | 1 | 1 |
| $\psi$ | 0 | 0 | 1 | -1 |
| $\psi_{k,1}$ | 1 | $(-1)^{k+1}$ | $\varepsilon^{(k-1)}+\varepsilon^{-(k-1)}$ | 0 |
| $\pi_k$ | $-1$ | $(-1)^{k+1}$ | 0 | $-(r^k+r^{kq})$ |
| $\xi_1$ | $\frac{1}{2}(1-\sqrt{\theta q})$ | $\frac{\theta}{2}(1-\sqrt{\theta q})$ | $(-1)^k$ | 0 |
| $\xi_2$ | $\frac{1}{2}(1+\sqrt{\theta q})$ | $\frac{\theta}{2}(1+\sqrt{\theta q})$ | $(-1)^k$ | 0 |
| $v_1$ | $\frac{1}{2}(-1-\sqrt{\theta q})$ | $\frac{-\theta}{2}(-1-\sqrt{\theta q})$ | 0 | $(-1)^{m+1}$ |
| $v_2$ | $\frac{1}{2}(-1+\sqrt{\theta q})$ | $\frac{-\theta}{2}(-1+\sqrt{\theta q})$ | 0 | $(-1)^{m+1}$ |

Let

$$A = \begin{pmatrix} 1 & \varepsilon^{2t+1} \\ 0 & 1 \end{pmatrix}, \quad t \neq 0$$

Then for

$$B = \begin{pmatrix} \varepsilon^{t} & 0 \\ 0 & \varepsilon^{-t} \end{pmatrix}$$

we have $B^{-1}AB = \begin{pmatrix} 0 & \varepsilon \\ 0 & 1 \end{pmatrix}$ and $A \in \tau_{0\varepsilon}^{(2)}$. Similarly for $\begin{pmatrix} 1 & \varepsilon^{2t} \\ 0 & 1 \end{pmatrix}$,

$$\begin{pmatrix} \varepsilon^{t} & 0 \\ 0 & \varepsilon^{-t} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \varepsilon^{t} & 0 \\ 0 & \varepsilon^{-t} \end{pmatrix}^{-1} = \begin{pmatrix} 1 & \varepsilon^{2t} \\ 0 & 1 \end{pmatrix}$$

Also all matrixes in form $\begin{pmatrix} 1 & 0 \\ \varepsilon^{2t+1} & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ \varepsilon^{2t} & 1 \end{pmatrix}$ belong to $\tau_{01}^{(2)}$ and $\tau_{0\varepsilon}^{(2)}$ since

$$\begin{pmatrix} 0 & -\varepsilon^{k} \\ \varepsilon^{-k} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \varepsilon^{2t+1} & 1 \end{pmatrix} \begin{pmatrix} 0 & \varepsilon^{k} \\ -\varepsilon^{-k} & 0 \end{pmatrix} = \begin{pmatrix} 1 & \varepsilon \\ 0 & 1 \end{pmatrix}; \; k = (\frac{q-1}{k}) - t$$

$$\begin{pmatrix} 0 & -\varepsilon^{k} \\ \varepsilon^{-k} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \varepsilon^{2t} & 1 \end{pmatrix} \begin{pmatrix} 0 & \varepsilon^{k} \\ -\varepsilon^{-k} & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}; \; k = (\frac{q-1}{k}) - t$$

Thus $S = \tau_{01}^{(2)} \cup \tau_{0\varepsilon}^{(2)}$ is a genarator for $G = SL(2,q)$. The character table of G for this class is as follow:

| Class | $\tau_0^{(1)}$ | $\tau_{01}^{(2)}$ | $\tau_{0\varepsilon}^{(2)}$ |
|---|---|---|---|
| Repg | $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ | $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ | $\begin{pmatrix} 1 & \varepsilon \\ 0 & 1 \end{pmatrix}$ |
| $\|[g]\|$ | 1 | $\frac{q^2-1}{2}$ | $\frac{q^2-1}{2}$ |
| $\lambda$ | 1 | 1 | 1 |
| $\psi$ | $q$ | 0 | 0 |
| $\psi_{k,1}$ | $q+1$ | 1 | 1 |
| $\pi_k$ | $q-1$ | -1 | -1 |
| $\xi_1$ | $\frac{q+1}{2}$ | $\frac{1}{2}(1+\sqrt{\theta q})$ | $\frac{1}{2}(1-\sqrt{\theta q})$ |
| $\xi_2$ | $\frac{q+1}{2}$ | $\frac{1}{2}(1-\sqrt{\theta q})$ | $\frac{1}{2}(1+\sqrt{\theta q})$ |
| $\upsilon_1$ | $\frac{q-1}{2}$ | $\frac{1}{2}(-1+\sqrt{\theta q})$ | $\frac{1}{2}(-1-\sqrt{\theta q})$ |
| $\upsilon_2$ | $\frac{q-1}{2}$ | $\frac{1}{2}(-1-\sqrt{\theta q})$ | $\frac{1}{2}(-1+\sqrt{\theta q})$ |

where, for $q = 4n+1, \theta = 1$ and for $q = 4n+3, \theta = -1$. Therefor all eigenvalues of $Cay(G, S)$ are

$$\begin{aligned}
\mu_1 &= q^2 - 1 = |S|,\\
\mu_2 &= 0,\\
\mu_3 &= \frac{1}{q+1}(q^2-1) = q-1,\\
\mu_4 &= \frac{-1}{q-1}(q^2-1) = -(q+1),\\
\mu_5 &= \frac{2}{q+1}\frac{q^2-1}{2} = q-1 = \mu_6,\\
\mu_7 &= \frac{-2}{q-1}\frac{q^2-1}{2} = -q-1 = \mu_7.
\end{aligned}$$

Hence, the spectrum of $Cay(SL(2,q), S)$ is $\{[0], [-q-1], [q+1], [q^2-1]\}$. Since, $\lambda = q+1$, we can deduce that $Cay(SL(2,q), S)$ is Ramanujan.

Case 2. The number of conjugacy classes of $SL(2,q)$ where $2 \mid q$ is $q+1$. see [1], proposition 4.4.7 On the other hand, the character table of $SL(2,q)$ is as reported in Table 4.

The conjugacy classes and character table of $SL(2,q), q$ is even

| Class | $\tau_0^{(1)}$ | $\tau_0^{(2)}$ | $\tau_{k,-k}^{(2)}$ | $\tau_k^{(4)}$ |
|---|---|---|---|---|
| Repg | $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ | $\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$ | $\begin{pmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{pmatrix}$ | $\begin{pmatrix}0 & 1\\ 1 & r+r^q\end{pmatrix}$ |
| $No.ofCC$ | 1 | 1 | $\frac{q-1}{2}$ | $\frac{q}{2}$ |
| $\|[g]\|$ | 1 | $q^2-1$ | $q(q+1)$ | $q(q-1)$ |
| $\lambda$ | 1 | 1 | 1 | 1 |
| $\psi$ | $q$ | 0 | 1 | -1 |
| $\psi_{k,0}$ | $q+1$ | 1 | $\alpha^k + \alpha^{-k}$ | 0 |
| $\pi$ | $q-1$ | -1 | 0 | $-(r^k + r^{kq})$ |

**Table 4:** The character table of $SL(2,q), q$ is even.

Let q be even. we have

$$\begin{pmatrix} \varepsilon^{\frac{q}{2}} & 0 \\ 0 & \varepsilon^{-\frac{q}{2}} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \varepsilon^{\frac{q}{2}} & 0 \\ 0 & \varepsilon^{-\frac{q}{2}} \end{pmatrix}^{-1} = \begin{pmatrix} 1 & \varepsilon \\ 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \varepsilon \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -\varepsilon \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \varepsilon \\ 0 & 1 \end{pmatrix}.$$

It is not difficult to see that $S = \tau_0^{(2)}$ is a ganarator of $G = SL(2, q)$ and eigenvalues of $Cay(G, S)$ is as follow:

$$\begin{aligned} \mu_1 &= q^2 - 1 = |S|, \\ \mu_2 &= 0, \\ \mu_3 &= \frac{1}{q+1}(q^2 - 1) = q - 1, \\ \mu_4 &= \frac{-1}{q-1}(q^2 - 1) = -(q + 1). \end{aligned}$$

Therefore $\lambda = q + 1$ and hence $Cay(G, S)$ is Ramanujan.

## 3.2 Mathieu Groups

We find from GAP, The conjugacy classes of mathieu group $G = M(9)$ are as follow:

$$A = \{()^G, (2,3,8,6)(4,7,5,9)^G, (2,4,8,5)(3,9,6,7)^G, (2,7,8,9)(3,4,6,5)^G, (2,8)(3,6)(4,5)(7,9)^G, (1,2,8)(3,9,4)(5,7,6)^G\}.$$

Thus, the eigenvalues of $Cay(G, S)$, when $S = a^G \cup b^G, a^G, b^G \in A$ are

$[36, -36, 0, 0, 0, 0]$, $[36, 0, -36, 0, 0, 0]$,
$[27, -9, -9, 27, -9, 0]$, $[26, -10, -10, 26, 8, 1]$,
$[36, 0, 0, -36, 0, 0]$, $[27, -9, 27, -9, -9, 0]$,
$[26, -10, 26, -10, 8, -1]$, $[27, 27, -9, -9, -9, 0]$,
$[26, 26, -10, -10, 8, -1]$, $[17, 17, 17, 17, -1, -1]$.

It yields that $Cay(G,S)$ is Ramanujan. In special case

$$S=\{(2,8)(3,6)(4,5)(7,9)G,(1,2,8)(3,9,4)(5,7,6)^G\}$$

and S is set with minimum size.
The conjugacy classes of mathieu group $G=M(10)$ are as follow,

$$\{()^G,(3,4,9,7)(5,8,6,10)^G,(3,5,9,6)(4,10,7,8)^G,$$
$$(3,9)(4,7)(5,6)(8,10)^G,(2,3,9)(4,10,5)(6,8,7)^G,$$
$$(1,2)(3,4,5,10,9,7,6,8)^G,(1,2)(3,7,5,8,9,4,6,10)^G,$$
$$(1,2,3,7,6)(4,8,5,9,10)^G\}.$$

The eigenvalues of $Cay(G,S)$ for $S=a^G, a^G\in A$are

$$[1,1,1,1,1,1,1,1],[180,-180,-20,20,0,0,0,0],$$
$$[90,90,10,10,-18,0,0,0],[45,45,5,5,9,-9,-9,0],$$
$$[80,80,0,0,8,8,8,-10],$$
$$[90,-90,10,-10,0,-9*E(8)-9*E(8)^3,9*E(8)+9*E(8)^3,0],$$
$$[90,-90,10,-10,0,9*E(8)+9*E(8)^3,-9*E(8)-9*E(8)^3,0],$$
$$[144,144,-16,-16,0,0,0,9].$$

For $S=(3,9)(4,7)(5,6)(8,10)^G$ eigenvalues of M(10) are, $\{45,45,5,5,9,-9,-9,0\}$ and in this case $Cay(G,S)$ is Ramanujan. The conjugacy classes of mathieu group $G=M(11)$ are also as follows,

$$A=\{()^G,(1,11,2,5,3,8,10,9,7,6,4)^G,(1,4,6,7,9,10,83,5,2,11)^G,$$
$$(2,5)(3,10)(4,9)(7,8)^G,(2,7,5,8)(3,9,10,4)^G,$$
$$(1,5,6,11,7,8,2,10)(4,9)^G,(1,10,2,8,7,11,6,5)(4,9)^G,$$
$$(1,11,6)(2,4,3)(5,910)^G(1611)(2,10,4,5,3,9)(7,8)^G,$$
$$(1,5,8,3,10)(2,11,7,9,6)^G\}.$$

If

$$S=\{(1,4,6,11,8,7,10,2,3,9,5)^G,$$
$$(1,3,4)(2,10)(5,7,11,6,9,8)^G,(2,6,10,5)(7,11,9,8)^G\}$$

then, all eigenvalues of $Cay(G, S)$ are,

$$[3030, -6, 60, 60, -90, 45*E(11)^2 + 45*E(11)^6 + 45*E(11)^7 \\ + 45*E(11)^8 + 45*E(11)^10, \\ 45*E(11) + 45*E(11)^3 + 45*E(11)^4 \\ + 45*E(11)^5 + 45*E(11)^9, 30, 38, -42].$$

Since $2\sqrt{k-1} = 2\sqrt{3030-1} = 110$,

$$|45*E(11)^2 + 45*E(11)^6 \\ + 45*E(11)^7 + 45E(11)^8 + 45*E(11)^{10}| = 77.94228629 \\ |45*E(11) + 45*E(11)^3 + 45*E(11)^4 \\ + 45*E(11)^5 + 45*E(11)^9| = 77.94228635$$

It yields that $\lambda = 90$ and so $Cay(G, S)$ is Ramanujan.

### 3.3 Suzuki Group

In reference [15], Suzuki introduces a group G as a ZT-group if G acts on the set Ω in such a way that it has the following four conditions (1) G acts on symbols $1+N$ as a doubly transitive group,(2) G acts on $1+N$ symbols as a doubly transitive group, (3) G does not contain any regular subgroups of order $1+N$, and (4) N is even. He showed in [15] that for every prime power $q = 2^{2s+1}$, there exists a unique ZT group Sz(q) of order $q^2(q-1)(q^2+1)$, which was later called the Suzuki group. When $q > 2$, this group is simple. assuming that where a is the element where G acts on it and $H = G_a$. By [15], According to conditions (1) and (2), it is concluded which is H a Frobenius group on Ω n fag. If you use a known result Frobenius to prove that H contains a regular normal subgroup Q of order N in such a way that every non-identity element of Q only leaves it. The symbol is fixed. Supoose b is a member of Ω
$a\}$ opposite to a, $K = H_b$ and x is a member of $N_G(K)$ is involution. Then it can be seen that the Suzuki group includes elements such as y and z, in a way that y is an involution and xyx = z 1xz, and the cyclic subgroup $A_0$ has order q-1, $A_1$ has order q+r-1 And finally, the subgroup $A_2$ has order $q - r + 1$.

The conjugacy classes of S z(q) in the following, with a little thought, you can calculate as you can see:

$$\{e\}, y^{Sz(q)}, z^{Sz(q)}, (z^{-1})^{Sz(q)}, b_0^{Sz(q)}, b_1^{Sz(q)}, b_2^{Sz(q)}$$

of lengths $1, (q-1)(q^2+1)\frac{1}{2}(q-1)(q^2+1), \frac{1}{2}(q-1)(q^2+1), q^2(q-1)(q+r+1), q^2(q+r+1)(q-r+1)$, and $q^2(q-1)(q-r+1)$,respectively. Here, $b_0; b_1$ and $b_2$ are non-identity alements of $A_i$ ,Here i is one of the numbers 0, 1 or 2 , respectively. Note that there are $\frac{q-r}{2}\frac{q}{2}-1$ and $\frac{q+r}{4}$ conjugacy classes of types $b_0^{Sz(q)}, b_1^{Sz(q)}$ and $b_2^{Sz(q)}$, respectively. remember the Suzuki group $Sz(q)$ whit $q = 2^{2s+1}, r = 2^{s+1}$ and $s$ is positive number. The conjugacy class $S = ySz(q)$ and the normal subset $T = z^{Sz(q)} \cup (z^{-1})^{Sz(q)}$ are a minimal NSGS and second minimal NSGS of S z(q), respectively. Moreover, $|S| = (q-1)(q^2+1), |T| = q(q-1)(q^2+1)$ and the simple eigenvalues of $Cay(Sz(q), S)$ are $|S|$ and $Cay(Sz(q), T)$ are $|T|$ . The eigenvalues of $Cay(Sz(q), S)$ are :

$$0, -(q^2+1), (q-1), \frac{(1+q^2)(r-1)}{q-r+1}, \frac{-(1+q^2)(r+1)}{q+r-1}.$$

therefore $|1+q^2| \not< 2\sqrt{|S|-1}$ and $Cay(Sz(q), S)$ is not Ramanujan graph. The $Cay(Sz(q), T)$ has eigenvalues:

$$0, q(q-1), \frac{-q(q^2+1)}{q-r+1}, \frac{-q(1+q^2)}{q+r-1}.$$

in this case $Cay(Sz(q), S)$ is a Ramanujan.

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**Reza Safakish**
Department of Mathematics
Assistant Professor of Mathematics
Author's Affiliation
Hamedan, Iran
E-mail: safakish@basu.ac.ir