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The Maximal Ideal Space of $C(K, \mathcal{A})$

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Abstract. Let $C(K, \mathcal{A})$ denote the space of all continuous $\mathcal{A}$-valued functions on the compact Hausdorff space K, where $\mathcal{A}$ is a commutative Banach algebra. In this paper we show that the maximal ideal space of $C(K, \mathcal{A})$ can be identified with $K \times \mathcal{M}$, where $\mathcal{M}$ denotes the maximal ideal space of $\mathcal{A}$.

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1. Introduction

The most important problem concerning commutative Banach algebras is characterizing its maximal ideal space. Though many commutative Banach algebras, including $C(X)$ for a compact Hausdorff space X and many function algebras, have a known maximal ideal space, there are many important commutative Banach algebras including H^∞ for which the topological properties of their maximal ideal spaces are not fully understood [1].

Here we show that the maximal space of the algebra of all continuous functions from a compact Hausdorff space into a Banach algebra has a simple characterization.

Let K be an arbitrary compact Hausdorff space and $\mathcal{A}$ be a Banach space. Denote by $C(K,\mathcal{A})$ the space of all continuous $\mathcal{A}$-valued functions defined on K equipped with the norm

$$||f|| = \sup_{k\in K} ||f(k)||_{\mathcal{A}}.$$

Then $C(K,\mathcal{A})$ will be a Banach space and if $\mathcal{A}$ is a commutative Banach algebra, then $C(K,\mathcal{A})$ is a commutative Banach algebra. In this case, we shall show that the maximal ideal space of $C(K,\mathcal{A})$ can be identified with $K\times\mathcal{M}$, where $\mathcal{M}$ denotes the maximal ideal space of $\mathcal{A}$ equipped with the weak* topology. We remind that $\mathcal{A}$ need not be unital.

2. Main Results

The following representation theorem is due to Singer [4]. For a nice proof see Hensgen [3].

Theorem 1. *Let $\mathcal{A}$ be a Banach space. The dual $C(K,\mathcal{A})^*$ of $C(K,\mathcal{A})$ can be identified with $M(K,\mathcal{A}^*)$, the space of all regular Borel $\mathcal{A}^*$-valued measures on K having finite variation. The action of an element $\Phi\in C(K,\mathcal{A})^*$ corresponding to $F\in M(K,\mathcal{A}^*)$ on an element $g\in C(K,\mathcal{A})$*

is then given by

$$\Phi(g) = \int_K < g(k),\ dF(k) > .$$

Note that for an element $F \in M(K, \mathcal{A}^*)$ and $g \in C(K, \mathcal{A})$, $d\mu_g =< g, dF >$ defines a regular Borel measure on K.

Let $M(K)$ denote the space of all regular Borel measures on K. Obviously, each element $\mu \times \varphi \in M(K) \times \mathcal{A}^*$ is an element of $C(K, \mathcal{A})^*$ acting on an element $g \in C(K, \mathcal{A})$ as

$$\mu \times \varphi(g) = \int_K \varphi(g(k)) d\mu.$$

Now if $\mu = \delta_{k_0}$ is a point mass measure at some point $k_0 \in K$, then the action of $\Phi = \mu \times \varphi$ simply becomes

$$\mu \times \varphi(g) = \varphi(g(k_0)). \tag{1}$$

In this case we say that Φ is supported at the single point k_0. Looking at (1) reveals that if $f \in C(K)$, then

$$\Phi(fg) \in \mathrm{Im}(f), \tag{2}$$

for all $g \in C(K, \mathcal{A})$ with $\Phi(g) = 1$. In other words if $\Phi(g) = 1$, then the measure ν defined on $C(K)$ by $\nu(f) = \int_K f(k) < g(k),\ dF(k) >$ has the property

$$\int_K f(k) d\nu \in \mathrm{Im}(f), \quad \text{for all} \ \ f \in C(K),$$

and such measures are supported at a single point by Lemma 2.5 of [2].

In fact, as the following lemma shows, the converse is also true, i. e. if an element $\Phi \in C(K, \mathcal{A})^*$ satisfies (2) for all $g \in C(K, \mathcal{A})$ with $\Phi(g) = 1$, then Φ is supported at a single point.

Lemma 2. *Let* $\Phi \in C(K, \mathcal{A})^*$ *satisfy*

$$\Phi(fg) \in Im\,(f)$$

for every $f \in C(K)$ *and* $g \in C(K, \mathcal{A})$ *with* $\Phi(g) = 1$. *Then* Φ *is supported by a single point.*

Proof. There exists $F \in M(K, \mathcal{A}^*)$ such that for all $g \in C(K, \mathcal{A})$,

$$\Phi(g) = \int_K < g(k),\ dF(k) > .$$

Choose $g_0 \in C(K, \mathcal{A})$ with $\Phi(g_0) = 1$. Then for every $f \in C(K)$ we have

$$\int_K f(k) < g_0(k), dF(k) > \in \mathrm{Im}\,(f).$$

By Lemma 2.5 of [2], the measure $< g_0, dF >$ is supported by a single point say $k_{g_0} = k_0$ in K. Thus the relation

$$\int_K f(k) < g_0(k), dF(k) > = f(k_0) \tag{3}$$

holds for all $f \in C(K)$. To show that k_0 is independent of g_0 let $g_1 \in \mathcal{A}$ with $\Phi(g_1) = 1$. Hence $\Phi(g_2) = 1$, where $g_2 = (g_0 + g_1)/2$. Suppose the measures $< g_1, dF >$ and $< g_2, dF >$ are supported by k_1 and k_2, respectively. Therefore (3) implies that $f(k_2) = \frac{f(k_0)+f(k_1)}{2}$ for all $f \in C(K)$. Consequently $k_0 = k_1 = k_2$. In general we have

$$\Phi(g) =< g(k_0), F(k_0) >= F(k_0)(g(k_0)) \tag{4}$$

for every $g \in C(K,\mathcal{A})$ and for some $k_0 \in K$. □

Theorem 3. *Let K be a compact Hausdorff space and let $\mathcal{A}$ be a commutative Banach algebra with maximal ideal space $\mathcal{M}$. Then the maximal ideal space $\mathcal{M}_{C(K,\mathcal{A})}$ of $C(K,\mathcal{A})$ can be identified with the space $K \times \mathcal{M}$. The action of an element $(k,\varphi) \in K \times \mathcal{M}$ on an element $g \in C(K,\mathcal{A})$ is given by $g \mapsto \varphi(g(k))$.*

Proof. Let Φ be a nonzero multiplicative linear functional on $C(K,\mathcal{A})$. Fix $g \in C(K,\mathcal{A})$ with $\Phi(g) = 1$. Then $\Phi(f_1 f_2 g) = \Phi(f_1 g)\Phi(f_2 g)$ for every $f_1, f_2 \in C(K)$. In this way Φ defines a multiplicative linear functional on $C(K)$, and because the maximal ideal space of $C(K)$ is K we have $\Phi(fg) \in \mathrm{Im}(f)$, $f \in C(K)$. By Lemma 2 we see that Φ is supported by a single point k_0. Now if Φ is represented by $F \in M(K,\mathcal{A}^*)$, then by relation (4),

$$\Phi(g) = F(k_0)(g(k_0)), \text{ for all } g \in C(K,\mathcal{A}).$$

Since Φ is not identically zero, $F(k_0)$ would also be nonzero and by letting g vary in constant functions, it follows that $F(k_0) \in \mathcal{M}$. Therefore, we have the identification $\Lambda: \Phi \mapsto (k_0, F(k_0))$ from $\mathcal{M}_{C(K,\mathcal{A})} \to K \times \mathcal{M}$.

We now prove that this identification is unique. If $\Phi \in \mathcal{M}_{C(K,\mathcal{A})}$ corresponds to two elements (k_1, φ_1) and (k_2, φ_2) in $K \times \mathcal{M}$, then for all $f \in C(K, \mathcal{A})$ we have $\Phi(f) = \varphi_1(f(k_1)) = \varphi_2(f(k_2))$. Letting f be a constant function we have $\varphi_1 = \varphi_2$. Choose $x \in \mathcal{A}$ such that $\varphi_1(x) \neq 0$ and if $k_1 \neq k_2$ choose $f \in C(K)$ such that $f(k_1) = 0$ and $f(k_2) = 1$. Then $\Phi(fx) = \varphi_1(f(k_1)x) = 0$ and $\Phi(fx) = \varphi_2(f(k_2)x) \neq 0$. This contradiction shows that $k_1 = k_2$. Hence the identification Λ is well-defined.

It is clear that Λ is one to one. On the other hand each $(k, \varphi) \in K \times \mathcal{M}$ induces an element $\Phi \in \mathcal{M}_{C(K,\mathcal{A})}$ acting as $\Phi(f) = \varphi(f(k))$ and as above Φ is identified with (k, φ). Hence the identification Λ is onto.

Now we prove that Λ and Λ^{-1} are continuous. If $\mathcal{A}$ is assumed to be unital, the continuity of Λ implies that of Λ^{-1}, since $\mathcal{M}_{C(K,\mathcal{A})}$ is compact in this case.

Let $\Phi_\alpha \to \Phi$ weak * in the space $\mathcal{M}_{C(K,\mathcal{A})}$ and let Φ_α correspond to $F_\alpha \in M(K, \mathcal{A}^*)$ and Φ to $F \in M(K, \mathcal{A}^*)$. Then there are $k_0, k_\alpha \in K$ such that $\Lambda\Phi_\alpha = (k_\alpha, F_\alpha(k_\alpha))$ and $\Lambda\Phi = (k_0, F(k_0))$. Thus, for all $g \in C(K, \mathcal{A})$, $F(k_\alpha)(g(k_\alpha)) \to F(k_0)(g(k_0))$. Again letting g vary in constant functions implies that $F_\alpha(k_\alpha) \to F(k)$ weak * in $\mathcal{M}$. Now for an element $g_0 \in C(K,\mathcal{A})$ with $\Phi(g_0) = 1$ and for all $f \in C(K)$,

$$\int_K f(k) < g_0(k), dF_\alpha(k) > \longrightarrow \int_K f(k) < g_0(k_0), dF(k) > .$$

This shows that the measures $< g_0, dF_\alpha >$ converge weak * in $M(K)$ to the measure $< g_0, dF >$ which is just the point mass at k_0. Also for each α the measure $< g_0, dF_\alpha >$ is zero or is supported at the point k_α. In each case there exists a complex number a_α such that $< g_0, dF_\alpha >= a_\alpha d\delta_{k_\alpha}$. Thus for all $f \in C(K)$ we have $a_\alpha f(k_\alpha) \to f(k_0)$ from which we easily conclude that $k_\alpha \to k_0$ in K. Therefore $(k_\alpha, F(k_\alpha)) \to (k_0, F(k_0))$ in $K \times \mathcal{M}$ and this implies the continuity of Λ.

Conversely, let $(k_\alpha, \varphi_\alpha) \to (k, \varphi)$ in $K \times \mathcal{M}$ and $\Phi_\alpha = \Lambda^{-1}(k_\alpha, \varphi_\alpha)$, $\Phi = \Lambda^{-1}(k, \varphi)$. Then for a fixed $f \in C(K,\mathcal{A})$,

$$\begin{aligned} |\Phi_\alpha(f) - \Phi(f)| &= |\varphi_\alpha(f(k_\alpha) - \varphi(f(k))| \\ &\leqslant |\varphi_\alpha(f(k_\alpha)) - \varphi_\alpha(f(k))| + |\varphi_\alpha(f(k)) - \varphi(f(k))| \\ &\leqslant \|f(k_\alpha) - f(k)\| + |\varphi_\alpha(f(k)) - \varphi(f(k))| \end{aligned}$$

The right hand side of the above inequality converges to zero by noting that f is continuous, $k_\alpha \to k$ and $\varphi_\alpha \to \varphi$ weak * in $\mathcal{M}$. This shows that $\Phi_\alpha \to \Phi$ weak * in $\mathcal{M}_{C(K,\mathcal{A})}$ and this implies the continuity of Λ^{-1}. □

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