The Maximal Ideal Space of $C(K, \mathcal{A})$

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Abstract. Let $C(K, \mathcal{A})$ denote the space of all continuous $\mathcal{A}$-valued functions on the compact Hausdorff space $K$, where $\mathcal{A}$ is a commutative Banach algebra. In this paper we show that the maximal ideal space of $C(K, \mathcal{A})$ can be identified with $K \times \mathcal{M}$, where $\mathcal{M}$ denotes the maximal ideal space of $\mathcal{A}$.

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1. Introduction

The most important problem concerning commutative Banach algebras is characterizing its maximal ideal space. Though many commutative Banach algebras, including $C(X)$ for a compact Hausdorff space $X$ and many function algebras, have a known maximal ideal space, there are many important commutative Banach algebras including $H^\infty$ for which the topological properties of their maximal ideal spaces are not fully understood [1].
Here we show that the maximal space of the algebra of all continuous functions from a compact Hausdorff space into a Banach algebra has a simple characterization.

Let $K$ be an arbitrary compact Hausdorff space and $A$ be a Banach space. Denote by $C(K, A)$ the space of all continuous $A$-valued functions defined on $K$ equipped with the norm

$$||f|| = \sup_{k \in K} ||f(k)||_A.$$ 

Then $C(K, A)$ will be a Banach space and if $A$ is a commutative Banach algebra, then $C(K, A)$ is a commutative Banach algebra. In this case, we shall show that the maximal ideal space of $C(K, A)$ can be identified with $K \times M$, where $M$ denotes the maximal ideal space of $A$ equipped with the weak* topology. We remind that $A$ need not be unital.

2. Main Results

The following representation theorem is due to Singer [4]. For a nice proof see Hensgen [3].

**Theorem 1.** Let $A$ be a Banach space. The dual $C(K, A)^*$ of $C(K, A)$ can be identified with $M(K, A^*)$, the space of all regular Borel $A^*$-valued measures on $K$ having finite variation. The action of an element $\Phi \in C(K, A)^*$ corresponding to $F \in M(K, A^*)$ on an element $g \in C(K, A)$
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is then given by

$$\Phi(g) = \int_K < g(k), dF(k) > .$$

Note that for an element $F \in M(K, \mathcal{A}^*)$ and $g \in C(K, \mathcal{A})$, $d\mu_g = < g, dF >$ defines a regular Borel measure on $K$.

Let $M(K)$ denote the space of all regular Borel measures on $K$. Obviously, each element $\mu \times \varphi \in M(K) \times \mathcal{A}^*$ is an element of $C(K, \mathcal{A})^*$ acting on an element $g \in C(K, \mathcal{A})$ as

$$\mu \times \varphi(g) = \int_K \varphi(g(k))d\mu.$$ 

Now if $\mu = \delta_{k_0}$ is a point mass measure at some point $k_0 \in K$, then the action of $\Phi = \mu \times \varphi$ simply becomes

$$\mu \times \varphi(g) = \varphi(g(k_0)). \quad (1)$$

In this case we say that $\Phi$ is supported at the single point $k_0$. Looking at (1) reveals that if $f \in C(K)$, then

$$\Phi(fg) \in \text{Im}(f), \quad (2)$$

for all $g \in C(K, \mathcal{A})$ with $\Phi(g) = 1$. In other words if $\Phi(g) = 1$, then the measure $\nu$ defined on $C(K)$ by $\nu(f) = \int_K f(k) < g(k), dF(k) >$ has the property

$$\int_K f(k)d\nu \in \text{Im}(f), \text{ for all } f \in C(K),$$

for all $g \in C(K, \mathcal{A})$ with $\Phi(g) = 1$. In other words if $\Phi(g) = 1$, then the
and such measures are supported at a single point by Lemma 2.5 of [2].

In fact, as the following lemma shows, the converse is also true, i.e. if an element \( \Phi \in C(K, \mathcal{A})^* \) satisfies (2) for all \( g \in C(K, \mathcal{A}) \) with \( \Phi(g) = 1 \), then \( \Phi \) is supported at a single point.

**Lemma 2.** Let \( \Phi \in C(K, \mathcal{A})^* \) satisfy

\[
\Phi(fg) \in \text{Im } f
\]

for every \( f \in C(K) \) and \( g \in C(K, \mathcal{A}) \) with \( \Phi(g) = 1 \). Then \( \Phi \) is supported by a single point.

**Proof.** There exists \( F \in M(K, \mathcal{A}^*) \) such that for all \( g \in C(K, \mathcal{A}) \),

\[
\Phi(g) = \int_K < g(k), dF(k) >.
\]

Choose \( g_0 \in C(K, \mathcal{A}) \) with \( \Phi(g_0) = 1 \). Then for every \( f \in C(K) \) we have

\[
\int_K f(k) < g_0(k), dF(k) > \in \text{Im } f.
\]

By Lemma 2.5 of [2], the measure \( < g_0, dF > \) is supported by a single point say \( k_{g_0} = k_0 \) in \( K \). Thus the relation

\[
\int_K f(k) < g_0(k), dF(k) > = f(k_0)
\]

holds for all \( f \in C(K) \). To show that \( k_0 \) is independent of \( g_0 \) let \( g_1 \in \mathcal{A} \) with \( \Phi(g_1) = 1 \). Hence \( \Phi(g_2) = 1 \), where \( g_2 = (g_0 + g_1)/2 \). Suppose
the measures $< g_1, dF >$ and $< g_2, dF >$ are supported by $k_1$ and $k_2$, respectively. Therefore (3) implies that $f(k_2) = \frac{f(k_0) + f(k_1)}{2}$ for all $f \in C(K)$. Consequently $k_0 = k_1 = k_2$. In general we have

$$\Phi(g) = < g(k_0), F(k_0) > = F(k_0)(g(k_0))$$

(4) for every $g \in C(K, \mathcal{A})$ and for some $k_0 \in K$. □

**Theorem 3.** Let $K$ be a compact Hausdorff space and let $\mathcal{A}$ be a commutative Banach algebra with maximal ideal space $\mathcal{M}$. Then the maximal ideal space $\mathcal{M}_{C(K, \mathcal{A})}$ of $C(K, \mathcal{A})$ can be identified with the space $K \times \mathcal{M}$. The action of an element $(k, \varphi) \in K \times \mathcal{M}$ on an element $g \in C(K, \mathcal{A})$ is given by $g \mapsto \varphi(g(k))$.

**Proof.** Let $\Phi$ be a nonzero multiplicative linear functional on $C(K, \mathcal{A})$. Fix $g \in C(K, \mathcal{A})$ with $\Phi(g) = 1$. Then $\Phi(f_1 f_2 g) = \Phi(f_1 g) \Phi(f_2 g)$ for every $f_1, f_2 \in C(K)$. In this way $\Phi$ defines a multiplicative linear functional on $C(K)$, and because the maximal ideal space of $C(K)$ is $K$ we have $\Phi(f g) \in \text{Im}(f)$, $f \in C(K)$. By Lemma 2 we see that $\Phi$ is supported by a single point $k_0$. Now if $\Phi$ is represented by $F \in M(K, \mathcal{A}^*)$, then by relation (4),

$$\Phi(g) = F(k_0)(g(k_0)), \text{ for all } g \in C(K, \mathcal{A}).$$

Since $\Phi$ is not identically zero, $F(k_0)$ would also be nonzero and by letting $g$ vary in constant functions, it follows that $F(k_0) \in \mathcal{M}$. Therefore,
we have the identification $\Lambda : \Phi \mapsto (k_0, F(k_0))$ from $\mathcal{M}_{C(K,A)} \to K \times \mathcal{M}$.

We now prove that this identification is unique. If $\Phi \in \mathcal{M}_{C(K,A)}$ corresponds to two elements $(k_1, \varphi_1)$ and $(k_2, \varphi_2)$ in $K \times \mathcal{M}$, then for all $f \in C(K,A)$ we have $\Phi(f) = \varphi_1(f(k_1)) = \varphi_2(f(k_2))$. Letting $f$ be a constant function we have $\varphi_1 = \varphi_2$. Choose $x \in A$ such that $\varphi_1(x) \neq 0$ and if $k_1 \neq k_2$ choose $f \in C(K)$ such that $f(k_1) = 0$ and $f(k_2) = 1$. Then $\Phi(f x) = \varphi_1(f(k_1)x) = 0$ and $\Phi(f x) = \varphi_2(f(k_2)x) \neq 0$. This contradiction shows that $k_1 = k_2$. Hence the identification $\Lambda$ is well-defined.

It is clear that $\Lambda$ is one to one. On the other hand each $(k, \varphi) \in K \times \mathcal{M}$ induces an element $\Phi \in \mathcal{M}_{C(K,A)}$ acting as $\Phi(f) = \varphi(f(k))$ and as above $\Phi$ is identified with $(k, \varphi)$. Hence the identification $\Lambda$ is onto.

Now we prove that $\Lambda$ and $\Lambda^{-1}$ are continuous. If $\mathcal{A}$ is assumed to be unital, the continuity of $\Lambda$ implies that of $\Lambda^{-1}$, since $\mathcal{M}_{C(K,A)}$ is compact in this case.

Let $\Phi_\alpha \to \Phi$ weak * in the space $\mathcal{M}_{C(K,A)}$ and let $\Phi_\alpha$ correspond to $F_\alpha \in M(K,A^*)$ and $\Phi$ to $F \in M(K,A^*)$. Then there are $k_0, k_\alpha \in K$ such that $\Lambda \Phi_\alpha = (k_\alpha, F_\alpha(k_\alpha))$ and $\Lambda \Phi = (k_0, F(k_0))$. Thus, for all $g \in C(K,A)$, $F(k_\alpha)(g(k_\alpha)) \to F(k_0)(g(k_0))$. Again letting $g$ vary in constant functions implies that $F_\alpha(k_\alpha) \to F(k)$ weak * in $\mathcal{M}$. Now for
an element $g_0 \in C(K, A)$ with $\Phi(g_0) = 1$ and for all $f \in C(K)$,
\[
\int_K f(k) < g_0(k), dF_\alpha(k) \longrightarrow \int_K f(k) < g_0(k_0), dF(k) > .
\]
This shows that the measures $< g_0, dF_\alpha >$ converge weak * in $M(K)$
to the measure $< g_0, dF >$ which is just the point mass at $k_0$. Also for
each $\alpha$ the measure $< g_0, dF_\alpha >$ is zero or is supported at the point $k_\alpha$.
In each case there exists a complex number $a_\alpha$ such that $< g_0, dF_\alpha > =
a_\alpha d\delta_{k_\alpha}$. Thus for all $f \in C(K)$ we have $a_\alpha f(k_\alpha) \to f(k_0)$ from which we
easily conclude that $k_\alpha \to k_0$ in $K$. Therefore $(k_\alpha, F(k_\alpha)) \to (k_0, F(k_0))$
in $K \times M$ and this implies the continuity of $\Lambda$.

Conversely, let $(k_\alpha, \varphi_\alpha) \to (k, \varphi)$ in $K \times M$ and $\Phi_\alpha = \Lambda^{-1}(k_\alpha, \varphi_\alpha)$,
$\Phi = \Lambda^{-1}(k, \varphi)$. Then for a fixed $f \in C(K, A)$,
\[
|\Phi_\alpha(f) - \Phi(f)| = |\varphi_\alpha(f(k_\alpha)) - \varphi(f(k))| \\
\leq |\varphi_\alpha(f(k_\alpha)) - \varphi_\alpha(f(k))| + |\varphi_\alpha(f(k)) - \varphi(f(k))| \\
\leq ||f(k_\alpha) - f(k)|| + |\varphi_\alpha(f(k)) - \varphi(f(k))|
\]
The right hand side of the above inequality converges to zero by noting
that $f$ is continuous, $k_\alpha \to k$ and $\varphi_\alpha \to \varphi$ weak * in $M$. This shows that
$\Phi_\alpha \to \Phi$ weak * in $M_{C(K, A)}$ and this implies the continuity of $\Lambda^{-1}$. □

References


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