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Original Research Paper

Iterative Cubic Trapezoidal Rule for The Solution of Mixed Integral Equations

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Abstract. In this study, an efficient iterative numerical technique based on a trapezoidal formula will be proposed to solve nonlinear (mixed) Volterra and Fredholm integral equations of the second kind in any dimension. Also, we will prove the rate of convergence and the convergence analysis of the iterative method. Furthermore, some numerical examples are considered to confirm the applicability of the method.

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1 Introduction

Integral equations are an important subject within pure and applied mathematics and are used as mathematical models to describe many

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and varied concrete physical phenomena. They also occur as reformulations of other mathematical problems as well as in chemistry, physics, engineering, and biological science [2, 5, 20, 24, 28].

Numerous numerical methods for solving integral equations have been studied and developed by many authors. We refer the readers to the books [5, 28] for the results of numerical methods in one-dimensional integral equations. In view of applications, however, functions that are defined on the Cartesian product of intervals in higher-dimensional Euclidean space are also important. In this direction, the solution of multi-dimensional (mixed) Volterra and Fredholm integral equations of the second kind has been extensively studied over the years. Several numerical solution methods for solving multi-dimensional (mixed) Volterra and Fredholm integral equations of the second kind exist in the literature, including iterative method, Prentice's Euler-type and Micula's cubature approach, e.g. see [6, 12–14, 16, 21, 21–23, 25].

In this paper, we introduce and establish an iterative method based on the m-dimension (briefly, m-D) trapezoidal rule with a novel proof for solving fixed points of (mixed) Volterra and Fredholm integral equations and show the rate of convergence, which is defined as

$$B(x)(s) := x(s) = f(s) + \int_I K(s,t)h(t,x(t))dt, \qquad s,t \in I, \tag{1}$$

where "$\int$" stands for Lebesgue's integral, x is an unknown real-valued function, f (source function) and K (kernel function) are the given suitable functions, and I is one of the following m-D cubes:

(a) Fredholm integral equations: where $I := [a_1, b_1] \times \cdots \times [a_m, b_m] \subseteq \mathbb{R}^m$, and $a_1 < b_1, \ldots, a_m < b_m$ are fixed numbers.

(b) Volterra integral equations: where $I := [a_1, s_1] \times \cdots \times [a_m, s_m] \subseteq \mathbb{R}^m$ and $a_1 \leq s_1 \leq b_1, \ldots, a_m \leq s_m \leq b_m$.

(c) Mixed integral equations: where $I := [a_1, s_1] \times \cdots \times [a_m, s_m] \subseteq \mathbb{R}^m$, $a_1 \leq s_1 \leq b_1, \ldots, a_m \leq s_m \leq b_m$ and for some $i = 1, \ldots, m$, not necessarily all cases, s_i is equal to b_i.

The advantages of the proposed method are simplicity, accuracy, precise convergence control, and its application in various dimensions. We also show that the rate of convergence of the method is $O(\frac{1}{n^2})$. Numerical results are reported in Section 4, which confirm that the implementation of the method is considerably fast and highly accurate. This method can be applied to solve linear and nonlinear (mixed) Volterra integral equations or any combination of them (see Examples below).

2 Preliminaries

In the sequel, let d be a metric on $\mathbb{R}^m$, and $C(I)$ be the Banach space of all continuous mappings x from I into $\mathbb{R}$ with the uniform norm. If $x : I \to \mathbb{R}$ be a bounded function, the oscillation of x on I is the quantity

$$\omega_\delta(x) := \omega_\delta(x, I) = \sup\{|x(t_1) - x(t_2)| : t_1, t_2 \in I, d(t_1, t_2) \leq \delta\}.$$

If $x \in C(I)$, then $\omega_I(x)$ is also called the uniform modulus of continuity of x.

Similarly, for a bounded function $x : I \times I \to \mathbb{R}$ we denote

$$\omega^1_\delta(s)(x) := \sup\{|x(s, t_1) - x(s, t_2)| : t_1, t_2 \in I, d(t_1, t_2) \leq \delta\}, \qquad s \in I.$$

The following properties will be very useful in what follows.

Theorem 2.1. *Let $x \in C(I)$. The following properties hold:*

(i) $|x(t) - x(s)| \leq \omega_{d(t,s)}(x)$ *for all* $t, s \in I$,

(ii) $\omega_\delta(x)$ *is an non-decreasing mapping in* δ,

(iii) $\omega_{\delta_1+\delta_2}(x) \leq \omega_{\delta_1}(x) + \omega_{\delta_2}(x)$ *for any* $\delta_1, \delta_2 \geq 0$,

(iv) $\omega_{n\delta}(x) \leq n\omega_\delta(x)$ *for any* $\delta \geq 0$ *and* $n \in \mathbb{N}$,

(v) $\omega_{\lambda\delta}(x) \leq (\lambda + 1)\omega_\delta(x)$ *for any* $\delta, \lambda \geq 0$,

(vi) If $I \subseteq J$, *then* $\omega_\delta(x, I) \leq \omega_\delta(x, J)$, *for all* $\delta > 0$,

(vii) $\omega_{(\cdot)}(x)$ *is continuous at 0 iff* $x \in C(I)$ *(see [15]),*

(viii) $\omega^1_\delta(\cdot)(x)$ *and* $\omega^2_\delta(\cdot)(x)$ *belong to* $C(I)$, *for all* $\delta > 0$ *and* $x \in C(I \times I)$ *(see [10, Page 187]).*

Two sets in $\mathbb{R}^m$ are said to overlap if they have a common interior point (their interiors have non-empty intersection). Suppose that $[a,b] \subseteq \mathbb{R}$ be a closed interval. A partition of $[a,b]$, i.e., a sequence of non-overlap interval $I_1 = [t_0 := a, t_1], I_2 = [t_1, t_2], \ldots, I_n = [t_{n-1}, t_n := b]$ such that $|I_i| > 0$. Note that we have $\sum_{i=1}^{n} |I_i| = |[a,b]| = b - a$. Let $\{I_i\}_1^n \prec [a,b]$ denotes a partition for $[a,b]$. If $I = \prod_{i=1}^{m} [a_i, b_i] = [a_1, b_1] \times \cdots \times [a_m, b_m] \subseteq \mathbb{R}^m$ be a m-D cube then a partition for I is produced by Cartesian product of partitions on each sets $[a_i, b_i], i = 1, \cdots, m$, i.e., if

$$\{J_{i,j_i} := [t_{i,j_i}, t_{i,j_i+1}]\}_{j_i=0}^{n_i-1} \prec [a_i, b_i] \tag{2}$$

be a given partition for $[a_i, b_i]$, where $n_i \in \mathbb{N}$ are fixed integers, then each element I_k in the partition $\{I_k\}_{k=1}^{r} \prec I, r = n_1 n_2 \cdots n_m$ has the form $I_k = \prod_{i=1}^{m} J_{i,j_i}$. Also, note that we get $|I| = \sum_{k=1}^{r} |I_k| = \prod_{i=1}^{m} (b_i - a_i)$.

Theorem 2.2. *(1) Let $x : I \to \mathbb{R}$ a bounded mapping, and $x \in L^1(I)$, where $L^1(I)$ is the space of integrable functions on I inv Lebesgue's sense. Then, for any partitions $\{I_k\}_{k=1}^{r} \prec I$ and any points $\xi_{i,j_i} \in [t_{i,j_i}, t_{i,j_i+1}], 1 \le i \le m, 0 \le j_i \le n_i - 1$, we have*

$$\begin{aligned}
&\Big| \int_I x(t)dt - \sum_{j_1=0}^{n_1-1} \cdots \sum_{j_m=0}^{n_m-1} (t_{1,j_1+1} - t_{1,j_1}) \cdots (t_{m,j_m+1} - t_{m,j_m}) \\
&\qquad x(\xi_{1,j_1}, \cdots, \xi_{m,j_m}) \Big| \\
&\le \sum_{j_1=0}^{n_1-1} \cdots \sum_{j_m=0}^{n_m-1} (t_{1,j_1+1} - t_{1,j_1}) \cdots (t_{m,j_m+1} - t_{m,j_m}) \\
&\qquad \omega_\delta(x, [t_{1,j_1}, t_{1,j_1+1}] \times \cdots \times [t_{1,j_m}, t_{1,j_m+1}]), \\
&\le \sum_{j_1=0}^{n_1-1} \cdots \sum_{j_m=0}^{n_m-1} (t_{1,j_1+1} - t_{1,j_1}) \cdots (t_{m,j_m+1} - t_{m,j_m}) \omega_\delta(x) \\
&= |I| \omega_\delta(x),
\end{aligned}$$

(2) Also, if $x : I \times I \to \mathbb{R}$ be a bounded mapping and if $x(s, \cdot) \in L^1(I)$,

for all $s \in I$ then

$$\Big|\int_I x(s,t)dt - \sum_{j_1=0}^{n_1-1}\cdots\sum_{j_m=0}^{n_m-1}(t_{1,j_1+1}-t_{1,j_1})\cdots(t_{m,j_m+1}-t_{m,j_m})$$
$$x(s,\xi_{1,j_1},\cdots,\xi_{m,j_m})\Big|$$
$$\leq \sum_{j_1=0}^{n_1-1}\cdots\sum_{j_m=0}^{n_m-1}(t_{1,j_1+1}-t_{1,j_1})\cdots(t_{m,j_m+1}-t_{m,j_m})$$
$$\omega^1_\delta(s)(x,[t_{1,j_1},t_{1,j_1+1}]\times\cdots\times[t_{1,j_m},t_{1,j_m+1}]),$$
$$\leq \sum_{j_1=0}^{n_1-1}\cdots\sum_{j_m=0}^{n_m-1}(t_{1,j_1+1}-t_{1,j_1})\cdots(t_{m,j_m+1}-t_{m,j_m})\omega^1_\delta(s)(x)$$
$$= |I|\omega^1_\delta(s)(x),$$

where

$$\delta = \max\{d((t_{1,j_1},\ldots,t_{m,j_m}),(t_{1,j_1+1},\ldots,t_{m,j_m+1})),$$
$$0\leq j_1\leq n_1-1,\ldots,0\leq j_m\leq n_m-1\}.$$

Proof.

(1) It is known that integrals are additively related to intervals. This brings us to

$$\Big|\int_I x(t)dt - \sum_{j_1=0}^{n_1-1}\cdots\sum_{j_m=0}^{n_m-1}(t_{1,j_1+1}-t_{1,j_1})\cdots(t_{m,j_m+1}-t_{m,j_m})$$
$$x(\xi_{1,j_1},\cdots,\xi_{m,j_m})\Big|$$
$$\leq \sum_{j_1=0}^{n_1-1}\cdots\sum_{j_m=0}^{n_m-1}\Big|\int_{t_{m,j_1}}^{t_{m,j_1+1}}\cdots\int_{t_{1,j_1}}^{t_{1,j_1+1}}$$
$$x(t_1,\cdots,t_m)-x(\xi_{1,j_1},\cdots,\xi_{m,j_m})dt_1\cdots dt_m\Big|$$
$$\leq \sum_{j_1=0}^{n_1-1}\cdots\sum_{j_m=0}^{n_m-1}\int_{t_{m,j_1}}^{t_{m,j_1+1}}\cdots\int_{t_{1,j_1}}^{t_{1,j_1+1}}|x(t_1,\cdots,t_m)$$

$$- x(\xi_{1,j_1}, \cdots, \xi_{m,j_m})\Big| dt_1 \cdots dt_m.$$

From parts (i), (ii) and (vi) of Theorem 2.1 we conclude that

$$\sum_{j_1=0}^{n_1-1} \cdots \sum_{j_m=0}^{n_m-1} \int_{t_{m,j_1}}^{t_{m,j_1+1}} \cdots \int_{t_{1,j_1}}^{t_{1,j_1+1}} \Big| x(t_1, \cdots, t_m)$$
$$- x(\xi_{1,j_1}, \cdots, \xi_{m,j_m})\Big| dt_1 \cdots dt_m.$$
$$\leq \sum_{j_1=0}^{n_1-1} \cdots \sum_{j_m=0}^{n_m-1} (t_{1,j_1+1} - t_{1,j_1}) \cdots (t_{m,j_m+1} - t_{m,j_m})$$
$$\omega_\delta(x, [t_{1,j_1}, t_{1,j_1+1}] \times \cdots \times [t_{1,j_m}, t_{1,j_m+1}]),$$
$$\leq \sum_{j_1=0}^{n_1-1} \cdots \sum_{j_m=0}^{n_m-1} (t_{1,j_1+1} - t_{1,j_1}) \cdots (t_{m,j_m+1} - t_{m,j_m}) \omega_\delta(x)$$
$$= |I| \omega_\delta(x),$$

which completes the proof.

(2) is similar to (1).

□

3 Method of Numerical Solution

Here, we introduce an iterative method to solve Eq. (1) for case (a), the cases (b) and (c) are similar, e.g. see equations (13), (14) and examples 4.1-4.3 below. Let $m, n_1, \ldots, n_m \in \mathbb{N}$ be fixed and consider Eq. (1) with kernel K on $I \times I$ and choose partition (2) on I with

$$t_{i,j} = a_i + j_i h_i, \tag{3}$$

where $h_i = \frac{b_i - a_i}{n_i}$, $i = 1, \ldots, m$, $j_i = 0, \ldots, n_i$. Take $x \in C(I)$ and denote

$$\hat{x}(s) = f(s) + \frac{\prod_{i=1}^m h_i}{2^m} \sum_{j_1=0}^{n_1-1} \cdots \sum_{j_m=0}^{n_m-1} \sum_{i_1,\ldots,i_m=0}^{1} [K(s, t_{1,j_1+i_1}, \ldots, t_{m,j_m+i_m})$$
$$h((t_{1,j_1+i_1}, \ldots, t_{m,j_m+i_m}), x(t_{1,j_1+i_1}, \ldots, t_{m,j_m+i_m}))], \tag{4}$$

for all $s \in I$. Notice that $x \to \widehat{x}$ defines a nonlinear operator from $C(I)$ into itself, where $K(\cdot, t) \in C(I)$, for all $t \in I$. Take an initial value $u_0 \in C(I)$, then we shall show that the Picard iterative procedure

$$u_r(s) = \widehat{u_{r-1}}(s) \qquad s \in I, r \in \mathbb{N}, \tag{5}$$

obtained by computing the corresponding m-D integral equation with the trapezoidal formula, gives the approximate solution of Eq. (1) in I.

3.1 Convergence Analysis and the Rate of Convergence

Here, we obtain an error estimate between the exact solution and the approximate solution for the given integral Eq. (1). Before we state the main result of this section the following notions are needed. Let K on $I \times I$ and h on $I \times \mathbb{R}$ be bounded function, for $s \in I$ and $u \in C(I)$ define

$$\begin{aligned} \omega_\delta^1(s)\big(Khu\big) &:= \sup_{t_1,t_2 \in I} \{\big|K(s,t_1)h(t_1,u(t_1)) \\ &\qquad - K(s,t_2)h(t_2,u(t_2))\big| \ ; \ d(t_1,t_2) \le \delta\}, \\ \omega_\delta\big(hu\big) &:= \sup_{t_1,t_2 \in I} \{\big|h(t_1,u(t_1)) - h(t_2,u(t_2))\big| \ ; \ d(t_1,t_2) \le \delta\}, \\ \omega_\delta^2(t)\big(h\big) &:= \sup_{s_1,s_2 \in I} \{\big|h(s_1,t) - h(s_2,t)\big| \ ; \ d(s_1,s_2) \le \delta\}, \qquad t \in \mathbb{R}, \end{aligned}$$

and $M_K := \sup_{s,t \in I} |K(s,t)|$ and $M_h := \sup_{s \in I, t \in \mathbb{R}} |h(s,t)|$.

Lemma 3.1. *Let $u \in C(I)$, $K \in C(I \times I)$ and $h \in C(I \times \mathbb{R})$ be continuous functions,*

(a) $\omega_\delta^1(s)\big(Khu\big) \le M_K \omega_\delta(hu) + M_h \omega_\delta^1(s)(K), \qquad \forall s \in I.$

(b) $\lim_{\delta \to 0} \omega_\delta^1(s)(K) = 0, \qquad \forall s \in I.$

(c) $\lim_{\delta \to 0} \sup_{s \in I} \omega_\delta^1(s)(K) = 0.$

(d) Let $x_r \in C(I)$ be a sequence defined as Theorem 3.2 and $f \in C(I)$, then

$$\omega_\delta(hx_k) \le L\omega_\delta(f) + LM_h \int_I \omega_\delta^2(t)(K)dt + \sup_{|t| \le M_0} \omega_\delta^2(t)(h), \tag{6}$$

where $M_0 = \sup_{k \in \mathbb{N}} \parallel x_k \parallel_u$. Moreover, $\lim_{\delta \to 0} \omega_\delta(hx_k) = 0$.

(e) $\lim_{\delta\to 0}\sup_{s\in I}\omega_\delta^1(s)\big(Khu\big)=0.$

Proof. For all $s,t,t'\in I$, we get

$$\begin{aligned}|K(s,t)h(t,u(t))-K(s,t')h(t',u(t'))|\\ &\le |K(s,t)h(t,u(t))-K(s,t)h(t',u(t'))|\\ &+|K(s,t)h(t',u(t'))-K(s,t')h(t',u(t'))|\\ &\le |K(s,t)||h(t,u(t))-h(t',u(t'))|\\ &+|h(t',u(t'))||K(s,t)-K(s,t')|,\end{aligned}$$

therefore, (a) is obtained.

(b) The function $K(s,\cdot)$ is in $C(I)$, for all $s\in I$, so by Theorem 2.1-(vii), the function $\omega_\delta(\cdot)(K)$ pointwise converges to 0, as $\delta\to 0$.

(c) Using part (b), the compactness of the space, and Theorem 2.1 parts (ii) and (viii), from Dini's theorem (see, for instance, [3, Theorem 24.2]) it is concluded that the convergent is uniform, i.e.

$$\lim_{\delta\to 0}\|\omega_\delta^1(\cdot)(K)\|_u=\lim_{\delta\to 0}\sup_{s\in I}\omega_\delta^1(s)(K)=0.$$

(d) For all $s_1,s_2\in I, d(s_1,s_2)\le\delta, k\in\mathbb{N}$, we get

$$\begin{aligned}|x_{k+1}(s_1)-x_{k+1}(s_2)|&=|B(x_k)(s_1)-B(x_k)(s_2)|\\ &\le |f(s_1)-f(s_2)|+\int_I|K(s_1,t)-K(s_2,t)||h(t,x_k(t))|dt\\ &\le \omega_\delta(f)+M_h\int_I\omega_\delta^2(t)(K)dt,\end{aligned}\tag{7}$$

and

$$\begin{aligned}|h(t_1,x_k(t_1))-h(t_2,x_k(t_2))|&\le|h(t_1,x_k(t_1))-h(t_2,x_k(t_2))|\\ &+|h(t_1,x_k(t_2))-h(t_2,x_k(t_2))|\\ &\le L|x_k(t_1)-,x_k(t_2)|+|h(t_1,x_k(t_2))-h(t_2,x_k(t_2))|.\end{aligned}\tag{8}$$

Notice that x_k is a convergent sequence, so it is bounded, and $M_0=\sup_{k\in\mathbb{N}}\parallel x_k\parallel_u$ exits. Combine (7) and (8), inequality (6) is obtained. Moreover, from Theorem 2.1-(vii) we have $\lim_{\delta\to 0}\omega_\delta(f)=0$ and similar to (c) we have $\lim_{\delta\to 0}\int_I\omega_\delta^2(t)(K)\le\lim_{\delta\to 0}|I|\sup_{t\in I}\omega_\delta^2(t)(K)\to 0$ and $\lim_{\delta\to 0}\sup_{|t|\le M_0}\omega_\delta^2(t)(h)=0$. These prove part (d).

(e) is obtained from previous parts. □

Now, we shall prove the existence and uniqueness of the solution of Eq. (1) that will be used in the next section.

Theorem 3.2. *Assume that*

(I)° $f : I \to \mathbb{R}$ *belong to* $C(I)$,

(II)° $K : I \times I \to \mathbb{R}$ *is a continuous function such that* $K(s, \cdot)^2$ *belong to* $L^1(I)$*, for any* $s \in I$.

(III)° $h : I \times \mathbb{R} \to \mathbb{R}$ *is a bounded function such that the function* $h(s, \cdot) : \mathbb{R} \to \mathbb{R}$*, for all* $s \in I$*, is Lipschitz on* $\mathbb{R}$ *with Lipschitz constant* $L > 0$ *i.e.,*

$$|h(s, u) - h(s, v)| \leq L|u - v|, \qquad s \in I, u, v \in \mathbb{R},$$

where $h^2 \in L^1(I \times \mathbb{R})$ *and* $L > 0$.

(IV)° $\kappa := L \sup_{s \in I} \parallel K(s, \cdot) \parallel_{L_1} < 1$,

then (1) *has a unique solution* $u^* \in C(I)$*. Moreover, for any* $u_0 \in C(I)$*, the Picard sequence defined by* $x_{r+1} = B(x_r)$ *with initial value* $x_0 := u_0$ *converges to* u^* *and the following error*

$$\parallel x_r - u^* \parallel_u \leq \frac{\kappa^r}{1 - \kappa} \parallel x_0 - B(x_0) \parallel_u . \tag{9}$$

fulfills the estimate.

Proof. Note that conditions on K and h and Cauchy-Schwarz inequality imply that $K(s, \cdot)h(\cdot, x(\cdot)) \in L^1(I)$, for all $s \in I$, and so the integral in (1) well defined. Also, for any $x \in C(I)$ we have

$$\begin{aligned}
\omega_\delta(B(x)) \leq{} & \sup\{|f(s_1) - f(s_2)| : s_1, s_2 \in I, d(s_1, s_2) \leq \delta\} \\
& + \sup \left\{ \left| \int_I K(s_1, t)h(t, x(t))dt - \int_I K(s_2, t)h(t, x(t))dt \right| \right. \\
& \qquad \left. s_1, s_2 \in I, d(s_1, s_2) \leq \delta \right\} \\
& \leq \omega_\delta(f) + \int_I \omega_\delta^2(t)(K)|h(t, x(t))|dt.
\end{aligned}$$

Take the limit as $\delta \to 0$, then by Lebesgue's monotone convergence theorem and Theorem 2.1-(vii) we have $\omega_\delta(B(x)) \to 0$, so B maps $C(I)$ into itself. We show that the operator B is a contraction. We have

$$\begin{aligned}
|B(x)(s) - B(y)(s)| &= \left| \int_I K(s,t)[h(t,x(t)) - h(t,y(t))]dt \right| \\
&\leq L \int_I |K(s,t)||x(t) - y(t)|dt \\
&\leq L \parallel K(s,\cdot) \parallel_{L_1} \parallel x - y \parallel_u,
\end{aligned}$$

for all $s \in I, x, y \in C(I)$, thus,

$$\parallel B(x) - B(y) \parallel_u \leq \kappa \parallel x - y \parallel_u .$$

Let $x_0 \in C(I)$, and define the Picard iterative sequence $x_r = B(x_{r-1})$, $r \in \mathbb{N}$. In virtue of the Banach contraction principle and crucial condition (IV)°, we infer that integral equation (1) has a unique solution and the error estimation (9) holds. □

Remark 3.3. In cases (b) and (c) the condition (IV)° is superfluous and it can be removed. Since there exists $1 \leq i \leq m$ such that $I \subseteq [a_1, b_1] \times \cdots \times [a_i, s_i] \times \cdots \times [a_m, b_m]$ and we have

$$\begin{aligned}
|B(x)(s) - B(y)(s)| &= \left| \int_I K(s,t)[h(t,x(t)) - h(t,y(t))]dt \right| \\
&\leq LM_K \frac{|I|}{|b_i - a_i|}(s_i - a_i) \parallel x - y \parallel_u,
\end{aligned}$$

for all $s = (s_1, \ldots, s_m) \in I$, where $M_K = \sup_{s,t\in I} |K(s,t)|$. We compute

$$\begin{aligned}
|B^2(x)(s) - B^2(y)(s)| &= \left| \int_I K(s,t)[h(t,Bx(t)) - h(t,By(t))]dt \right| \\
&\leq LM_K \int_I |Bx(t) - By(t)|dt \\
&\leq (LM_K)^2 \frac{|I|}{|b_i - a_i|} \int_I (t_i - a_i) \parallel x - y \parallel_u dt \\
&\leq \left(\frac{LM_K|I|}{|b_i - a_i|} \right)^2 \frac{(s_i - a_i)^2}{2} \parallel x - y \parallel_u .
\end{aligned}$$

Thus, inductively we get

$$\begin{aligned}\| B^n(x) - B^n(y) \|_u &\leq \frac{(LM_K|I|)^n}{n!|b_i - a_i|^n}(s_i - a_i)^n \| x - y \|_u \\ &\leq \frac{(LM_K|I|)^n}{n!} \| x - y \|_u .\end{aligned}$$

Since $\lim_{n\to+\infty} \frac{(LM_K|I|)^n}{n!} = 0$, there exists some $n_0 \in \mathbb{N}$ such that B^n is a contraction for all $n \geq n_0$. So, the above proof can be applied to $B^n, n \geq n_0$. Suppose that $u^* \in C(I)$ be a unique fixed point of B^{n_0} then u^* is a unique fixed point for B. We note that $B(u^*) = B(B^{n_0}(u^*)) = B^{n_0}(B(u^*))$, ie. $B(u^*)$ is a fixed point of B^{n_0}, thus, we have $B(u^*) = u^*$. Every fixed point of B is also a fixed point of B^n, for all $n \in \mathbb{N}$, so u^* is the unique possible fixed point for all $B^n, n \in \mathbb{N}$ and Banach contraction principle implies that Picard sequence $x_k = B(x_{k-1})$ converges to u^*.

Theorem 3.4. *Let f, K and h satisfy the conditions of Theorem 3.2. If $u_0 \in C(I)$ be an arbitrary initial value, then the iterative procedure (5) converges to the unique solution of Eq. (1), u^*.*

Proof. We prove the theorem for case (a), the others are similar. Consider iterative procedure (5) and let x_k defined by Picard sequence $x_k = B(x_{k-1})$ with the initial value $x_0 := u_0$. For each $\kappa < \kappa' < 1$ and $s \in I$ from Theorems 2.1-(c) and 2.2-(2) for $0 < \varepsilon(s) \leq (\kappa' - \kappa)/L$ there exists $\delta > 0$ such that

$$\begin{aligned}&\left| \int_I K(s,t)dt - \sum_{j_1=0}^{n_1-1} \cdots \sum_{j_m=0}^{n_m-1} \prod_{i=1}^{m} h_i K(s, t_{1,j_1+i_1}, \dots, t_{m,j_m+i_m}) \right| \\ &\qquad\qquad \leq |I|\omega_{\delta(s)}(K, I) \leq \varepsilon(s) \leq \frac{\kappa' - \kappa}{L},\end{aligned}$$

where $\prod_{i=1}^{m} h_i = (t_{1,j_1+1} - t_{1,j_1}) \cdots (t_{m,j_m+1} - t_{m,j_m})$. Using theorem 2.2 and $\sum_{i_1,\dots,i_m=0}^{1} = 2^m$, for all $k \in \mathbb{N} \cup \{0\}, s \in I$ we get

$$\begin{aligned}&|x_{k+1}(s) - u_{k+1}(s)| \\ &\quad \leq |x_{k+1}(s) - \widehat{x_k}(s)| + |\widehat{x_k}(s) - u_{k+1}(s)| \\ &\quad \leq \frac{1}{2^m} \sum_{i_1,\dots,i_m=0}^{1} \Bigg[\Bigg| \int_I K(s,t)h(t, x_k(t))dt\end{aligned}$$

$$
\begin{aligned}
&- \sum_{j_1=0}^{n_1-1} \cdots \sum_{j_m=0}^{n_m-1} \prod_{i=1}^{m} h_i K(s, t_{1,j_1+i_1}, \ldots, t_{m,j_m+i_m}) \\
&\quad h(t_{1,j_1+i_1}, \ldots, t_{m,j_m+i_m}, x_k(t_{1,j_1+i_1}, \ldots, t_{m,j_m+i_m}) \Big| \\
&+ \sum_{j_1=0}^{n_1-1} \cdots \sum_{j_m=0}^{n_m-1} \prod_{i=1}^{m} h_i \Big| K(s, t_{1,j_1+i_1}, \ldots, t_{m,j_m+i_m}) \\
&\quad \big[h(t_{1,j_1+i_1}, \ldots, t_{m,j_m+i_m}, x_k(t_{1,j_1+i_1}, \ldots, t_{m,j_m+i_m})) \\
&\qquad - h(t_{1,j_1+i_1}, \ldots, t_{m,j_m+i_m}, u_k(t_{1,j_1+i_1}, \ldots, t_{m,j_m+i_m})) \big] \Big| \Big] \\
&\le \frac{1}{2^m} \sum_{i_1,\ldots,i_m=0}^{1} \Big[\big(|I| \omega_\delta^1(s)(Khx_k) \\
&+ L \left(\sum_{j_1=0}^{n_1-1} \cdots \sum_{j_m=0}^{n_m-1} \prod_{i=1}^{m} h_i K(s, t_{1,j_1+i_1}, \ldots, t_{m,j_m+i_m}) \right) \parallel x_k - u_k \parallel_u \Big] \\
&\le |I| \omega_\delta^1(s)(Khx_k) + L \left(\int_I |K(s,t)| dt + \frac{\kappa' - \kappa}{L} \right) \parallel x_k - u_k \parallel_u \\
&\le |I| \sup_{s \in I} \omega_\delta^1(s)(Khx_k) + \kappa' \parallel x_k - u_k \parallel_u .
\end{aligned}
$$

Thus, we get

$$
\parallel x_{k+1} - u_{k+1} \parallel_u \le |I| \sup_{s \in I} \omega_\delta^1(s)(Khx_k) + \kappa' \parallel x_k - u_k \parallel_u . \tag{10}
$$

Take the limsup as $\delta \to 0, k \to +\infty$ from both sides of (10) and from Lemma 3.1-(e) and $\kappa' < 1$, it is concluded that $\parallel x_k - u_k \parallel \to 0$. So from Theorem 3.2 and $\parallel u^* - u_k \parallel_u \le \parallel u^* - x_k \parallel_u + \parallel x_k - u_k \parallel_u$, it is concluded that u_k convergence uniformly to u^*. □ Since the above proof can be apply for all $B^n, n \in \mathbb{N}$ and remark 3.3 implies that Picard sequence $x_k = B(x_{k-1})$ converges to u^*. Similar to remark 3.3, in case (b) and (c) the condition (IV)° is superfluous and it can be removed from theorem 3.4.

3.2 Algorithm of the Approach

Here, we propose an algorithm to carry out the method.

Initial step:

Set $\varepsilon^{'} > 0, n \in \mathbb{N}, k = 1$ and for any partition $\{I_i\}_{i=1}^{r} \prec I$, denote $I_0 = \{(t_{1,j}, \dots, t_{m,j}), j = 0, \dots, n\}$, where $t_{i,j}$ is given by equation (3). It can be choose as initial value, $u_0(s) = 0, s \in I_0$ (it is arbitrary), and go to the main steps.

Main steps:

Step 1: Compute $u_k(s)$ by (5), for all $s \in I_0$ and go to Step 2.

Step 2: Compute $M_k = \max\{\mid u_k(s) - u_{k-1}(s) \mid, s \in I_0\}$ and go to Step 3.

Step 3: If $M_k < \varepsilon^{'}$, print $u_k(s).s \in I_0$, Stop. Otherwise, set $k := k+1$ and go to Step 1.

3.3 Rate of Convergence

Let $n_1, \dots, n_m \in \mathbb{N}$ be fixed numbers, and $n = \max\{n_1, \dots, n_m\}$. The trapezoidal rule provides the approximate value

$$S_{j_1,\dots,j_m}(x) := \frac{\prod_{i=1}^{m} h_i}{2^m} \sum_{i_1,\dots,i_m=0}^{1} x(t_{1,j_1+i_1}, \dots, t_{m,j_m+i_1}),$$

in the subinterval $I_{j_1,\dots,j_m} := [t_{1,j_1}, t_{1.j_1+1}] \times \cdots \times [t_{m,j_m}, t_{m,j_m+1}]$ of the nods $t_{i,j} = a_i + j_i h$ where $h_i = \frac{b_i - a_i}{n_i}$, $i = 1, \dots, m, j_i = 0, \dots, n_i$. Let $t = (t_1, \dots, t_m) \in I_{j_1,\dots,j_m}$ and $T_0 = (t_{1,j_1}, \dots, t_{m,j_m})$.

Let $x \in C^2(\mathbb{R}^m)$, i.e., the second order partial $\frac{\partial^2 x}{\partial t_r \partial t_s}$ derivatives., for $i, j = 1, 2, \dots, m$ all exist and are continuous on $\mathbb{R}^m$, then $H(t) = \max_{r,s=1}^{m} \frac{\partial^2 x}{\partial t_r \partial t_s}(t)$ is a continuous and there exists $H_{\min} \le H_{\max}$ such that $H_{\min} \le H(t) \le H_{\max}$ on I. Use Taylor's theorem to write $x(t) = p(t) + e(t)$, where $p(t)$ and $e(t)$ have the forms

$$p(t) = x(T_0) + \sum_{r=1}^{m} \frac{\partial x}{\partial t_r}(T_0)(t_r - t_{r,j_r}),$$

and

$$e(t) = \sum_{r,s=1}^{m} \frac{\partial^2 x}{\partial t_r \partial t_s}(\xi(t))(t_r - t_{r,j_r})(t_s - t_{s,j_s}),$$

where $\xi(t)$ is on the line segment joining t and T_0. $S_{j_1,\ldots,j_m}$ is exact when applied to any and all linear functions, and since polynomials $(t_r - t_{r,j_r})(t_s - t_{s,j_s})$ and terms $(t_{r,j_r+i_r} - t_{r,j_r})(t_{s,j_s+i_s} - t_{s,j_s})$, $r, s = 1, \ldots, m$ are positive sign on $I_{j_1,\ldots,j_m}$ we get

$$\begin{aligned}
\int_{I_{j_1,\ldots,j_m}} x(t)dt - S_{j_1,\ldots,j_m}(x) &= \int_{I_{j_1,\ldots,j_m}} e(t)dt - S_{j_1,\ldots,j_m}(e) \\
&\leq H_{\max} \sum_{r,s=1}^{m} \int_{I_{j_1,\ldots,j_m}} (t_r - t_{r,j_r})(t_s - t_{s,j_s})dt \\
&\quad - H_{\min} \frac{\prod_{i=1}^{m} h_i}{2^m} \sum_{r,s=1}^{m} \sum_{i_1,\ldots,i_m=0}^{1} (t_{r,j_r+i_r} - t_{r,j_r})(t_{s,j_s+i_s} - t_{s,j_s}) \\
&= \frac{1}{4} H_{\max} \prod_{i=1}^{m} h_i \left(\sum_{r,s=1}^{m} h_r h_s \right) - H_{\min} \frac{\prod_{i=1}^{m} h_i}{2^m} \sum_{r,s=1}^{m} 2^{m-2} h_r h_s \\
&= \frac{H_{\max} - H_{\min}}{4} \prod_{i=1}^{m} h_i \left[\sum_{r,s=1}^{m} h_r h_s \right] \leq \frac{m^2 (H_{\max} - H_{\min})}{4} h^{m+2},
\end{aligned}$$

where $h = \max\{h_1, \ldots, h_m\}$. For the entire interval I, we obtain the approximation,

$$\begin{aligned}
&\left| \int_I x(t)dt - \sum_{j_1=0}^{n_1-1} \cdots \sum_{j_m=0}^{n_m-1} S_{j_1,\ldots,j_m}(x) \right| \\
&\qquad \leq \sum_{j_1=0}^{n_1-1} \cdots \sum_{j_m=0}^{n_m-1} \left| \int_{I_{j_1,\ldots,j_m}} x(t)dt - S_{j_1,\ldots,j_m}(x) \right| \\
&\qquad \leq n^m \frac{m^2 (H_{\max} - H_{\min})}{4} h^{m+2} = \frac{m^2 (H_{\max} - H_{\min}) A^m}{4} h^2,
\end{aligned}$$

where $A = nh = \max\{b_1 - a_1, \ldots, b_m - a_m\}$. So that the rate of convergence of the trapezoidal rule that is used in relation (4) to approximate the m-D integral in integral equation (1) is $O\left(\frac{1}{n^2}\right)$. Consequently, for every $x \in C^2(I)$ and $h \in C^2(I \times \mathbb{R})$ and for every $s \in I_0$ such that $K(s, \cdot) \in C^2(I)$, we have $K(s, \cdot)h(\cdot, x(\cdot)) \in C^2(I)$ and

$$|B(x)(s) - \widehat{x}(s)| = O\left(\frac{1}{n^2}\right), \tag{11}$$

suppose that $u_{k+1}(s) \simeq x_{k+1}(s) \simeq x_k(s) \simeq u^*(s), s \in I_0$ be the approximate solution for (1), where u_k, x_k, u^* be same as theorem 3.4, notice from theorem 3.2 it is concluded that the u_k converges to unique fixed point u^* and for every $s \in I_0$, from (11) we get

$$\begin{aligned} |u^*(s) - u_{k+1}(s)| &\simeq |\widehat{u^*}(s) - x_{k+1}(s)| \\ &= |\widehat{u^*}(s) - B(x_k)(s)| \simeq |\widehat{u^*}(s) - B(u^*)(s)| \\ &= O\left(\frac{1}{n^2}\right), \end{aligned}$$

thus, the rate of convergence of this method as you see in section 4 is $O\left(\frac{1}{n^2}\right)$.

4 Numerical Experiments

In this section, to show the practicability, accuracy, and efficiency of the theoretical results, we present some examples. Let $e_n(s) = |u^*(s) - u_k(s)|$ is a pointwise error function in $s \in I_0$ and $\|e_n\|_{\max}$ is a maximum absolute errors, i.e. $\|e_n\|_{\max} := \max\{e_n(s), s \in I_0\}$, where $I_0 \subseteq I$, where u^* is the exact solution and u_k is the numerical solution obtained by equation (5), which is computed by the algorithm described in Section 4, where k is the number of iterations and $n \in \mathbb{N}$ is a fixed integer. All the examples are satisfied in conditions (I)°-(III)°.

For the first example, let us consider the Voltetra type as

$$x(s) = f(s) + \int_I K(s,t)h(t,x(t))dt, \tag{12}$$

where, $I = [a_1, s_1] \times \cdots \times [a_m, s_m]$, for all $s = (s_1, \ldots, s_m) \in I$. More precisely, in the algorithm described in Section 3.2 the Voltetra trapezoidal formula becomes:

$$\begin{aligned} u_k(s) &= f(s) \\ &+ \frac{\prod_{i=1}^m h_i}{2^m} \sum_{j_1=0}^{J_1-1} \cdots \sum_{j_m=0}^{J_m-1} \sum_{i_1,\ldots,i_m=0}^{1} [K(s, t_{1,j_1+i_1}, \ldots, t_{m,j_m+i_m}) \\ &\quad h((t_{1,j_1+i_1}, \ldots, t_{m,j_m+i_m}), u_{k-1}(t_{1,j_1+i_1}, \ldots, t_{m,j_m+i_m}))], \end{aligned} \tag{13}$$

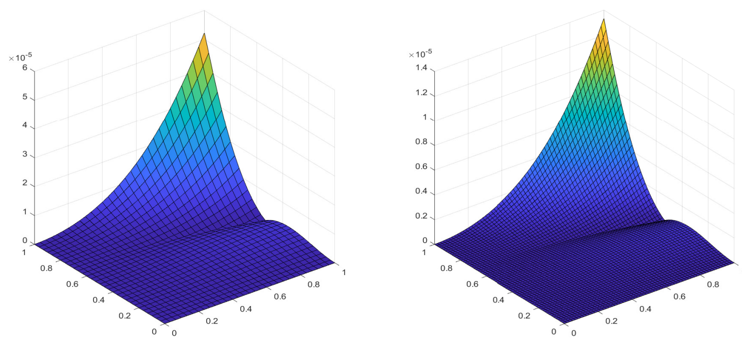

Figure 1: The graphs of error functions $e_{30}(s), s \in I_2(30)$ and $e_{60}(s), s \in I_2(60)$

for all $s = (s_1, \dots, s_m) = (t_{1,J_1}, \dots, t_{m,J_m}) \in I_0, 1 \leq J_1 \leq n_1, \dots, 1 \leq J_m \leq n_m$, where $h_i = \frac{s_i - a_i}{n_i}, i = 1, \dots, m$ and $s_i = t_{i,J_i} = a_i + J_i h_i$ also define $u_k(s) = f(s)$, for all $s = (s_1, \dots, s_m) \in I$ such that for some $0 \leq i \leq m, s_i = a_i$, since it is clear in this case that $u^*(s) = f(s)$. In what follows, for convenience, we suppose that $n_1 = n_2 = n_3 = n$ be a fixed integer.

Example 4.1 ([26])**.** Consider the following two-dimensional linear Voltetra integral equation:

$$u(s_1, s_2) = s_1 \sin(s_2) - \frac{1}{4}s_1^5 + \frac{1}{4}s_1^5 \cos(s_2) - \frac{1}{4}s_1^2 \sin^2(s_2) + \int_0^{s_1} \int_0^{s_2} (s_1 t_1^2 + \cos(t_2))u(t_1, t_2)dt_1 dt_2,$$

where $s_1, s_2 \in [0, 1]$. The exact solution is $u^*(s_1, s_2) = s_1 \sin(s_2)$.

Denote $I_1(n) = \{0, 1/n, 2/n, \dots, 1\}$ and $I_2(n) = I_1(n) \times I_1(n)$ and apply the algorithm and equation (13) for $m = 2, n = 15, 30, 60, \varepsilon' = 10^{-10}$ and $I_0 = I_2(n)$. For more details, please see Table 1 and Figure 1, where $s \in I_2(n)$ is optionally selected. The results $\|e_{15}\|_{\max}, \|e_{30}\|_{\max}, \|e_{60}\|_{\max}$ are 2.18473e-4, 5.22629e-5 and 1.33539e-5, respectively.

Nodes		$n = 15$		$n = 30$		$n = 60$	
$s := (1/l, 1/l)$	$u^*(s)$	$u_8(s)$	$e_{15}(s)$	$u_8(s)$	$e_{30}(s)$	$u_7(s)$	$e_{60}(s)$
$l = 1$	0.8415	0.8417	2.1847e-04	0.8415	5.2263e-05	0.8415	1.3354e-05
$l = 3$	0.1091	0.1091	2.1987e-06	0.1091	5.4909e-07	0.1091	1.3724e-07
$l = 5$	0.0397	0.0397	4.1058e-07	0.0397	1.0256e-07	0.0397	2.5636e-08
$l = 15$	0.0044	0.0044	6.5838e-09	0.0044	1.6447e-09	0.0044	4.1111e-10
$l = 30$	0.0011	$s \notin I_2(15)$		0.0011	1.0859e-10	0.0011	2.7144e-11
$l = 60$	2.7776e-04	$s \notin I_2(15)$		$s \notin I_2(30)$		2.7776e-04	1.7415e-12

Table 1: Summary of numerical results of Example 4.1 for $n = 15, 60, \varepsilon^{'} = 10^{-10}$

Example 4.2 ([9, 12, 14]). Consider the following two-dimensional nonlinear mixed integral equation:

$$u(s_1,s_2) = -\ln\left(1+\frac{s_1s_2}{1+{s_1}^2}\right)+\frac{s_2{s_1}^2}{8(1+s_1)(1+{s_1}^2)} + \int_0^{s_1}\int_0^1 \frac{s_2(1-{t_2}^2)}{(1+s_1)(1+{t_1}^2)}(1-\exp(-u(t_1,t_2))dt_1dt_2,$$

where $s_1, s_2 \in [0,1]$. The exact solution is given by

$$u^*(s_1,s_2) = -\ln\left(1+\frac{s_1s_2}{1+{s_1}^2}\right)$$

The 2-D mixed Voltetra-Ferdholm trapezoidal formula described in the previous section for this example is

$$u_k(s) = f(s) + \frac{\prod_{i=1}^2 h_i}{2^2}\sum_{j_1=0}^{J_1-1}\sum_{j_2=0}^{n-1}\sum_{i_1,i_2=0}^{1}[K(s,t_{1,j_1+i_1},t_{2,j_2+i_2}) h((t_{1,j_1+i_1},t_{2,j_2+i_2}),u_{k-1}(t_{1,j_1+i_1},t_{2,j_2+i_2}))]\,. \tag{14}$$

for all $s=(s_1,s_2)=(t_{1,n},t_{2,n})\in I_0$, where $h_2=h_1=\frac{1}{n}$ and $s_1 = t_{1,J_1} = a_1+J_1h_1, 1\le J_1\le n$, also define $u_k(s)=f(s)$, for all $s = (0,s_2)\in I_0, s_2\in[0,1]$, since it is clear in this case $u^*(s)=f(s)$.

Applying the algorithm and equation (14) for $n = 30, 60, 120, \varepsilon' = 10^{-10}$ and $I_0 = I_2(n)$. The results $\|e_{30}\|_{\max}, \|e_{60}\|_{\max}$ and $\|e_{120}\|_{\max}$ are 5.2302e-5 , 1.3079e-5 and 3.2699e-6 , respectively. Figure 2 illustrates the error results for this example.

Example 4.3 ([17, 18]). Consider the following three-dimensional nonlinear mixed integral equation:

$$u(s_1,s_2,s_3) = s_1^2s_2s_3 - \frac{11}{5760}s_1^6s_3(4s_2^2+3) + \frac{1}{4}\int_0^{s_1}\int_0^1\int_0^1 (s_1+t_1)(s_2^2+t_3)s_3t_2u^2(t_1,t_2,t_3)dt_1dt_2dt_3,$$

where $s_1,s_2,s_3\in[0,1]$ and the exact solution is given by $u^*(s_1,s_2,s_3) = s_1^2s_2s_3$.

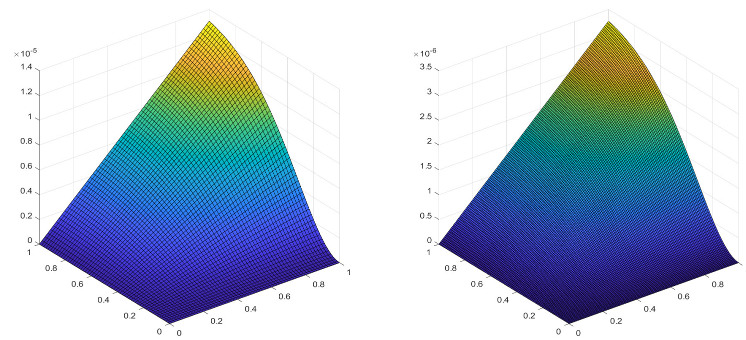

Figure 2: The graphs of error functions $e_{60}(s), s \in I_2(60)$ and $e_{120}(s), s \in I_2(120)$

Applying the algorithm and formula similar to equation (14) for three variables, and $n = 5, 10, 20, \varepsilon^{'} = 10^{-10}$. The results $\|e_5\|_{\max}, \|e_{10}\|_{\max}$ and $\|e_{20}\|_{\max}$ are 5.70978e-4 ,9.92278e-5 and 1.69753e-5, respectively. Figure 3 illustrates the error results for this example.

Example 4.4 ([6, 13, 14])**.** Consider the following two-dimensional nonlinear Fredholm integral equation:

$$u(s_1, s_2) = \frac{1}{(1+s_1+s_2)^2} - \frac{s_1}{6(1+s_2)} + \int_0^1 \int_0^1 \frac{s_1}{1+s_2}(1+t_1+t_2)u^2(t_1,t_2)dt_1dt_2,$$

where $s_1, s_2 \in [0,1]$ and the exact solution is given by $u^*(s_1, s_2) = \frac{1}{(1+s_1+s_2)^2}$.

Applying the algorithm and formula similar to equation (4) for two variables, and $n = 40, 80, \varepsilon^{'} = 10^{-50}$. The results $\|e_{40}\|_{\max}$ and $\|e_{80}\|_{\max}$ are 1.2422e-04 and 3.1047E-5, respectively. Figure 4 illustrates the error results for this example.

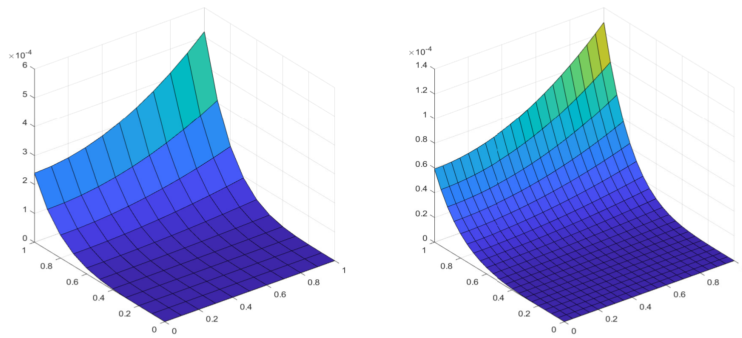

Figure 3: The graphs of error functions $e_{10}(s), s = (s_1, s_2, 1) \in I_2(10)$ and $e_{20}(s), s = (s_1, s_2, 1) \in I_2(20)$

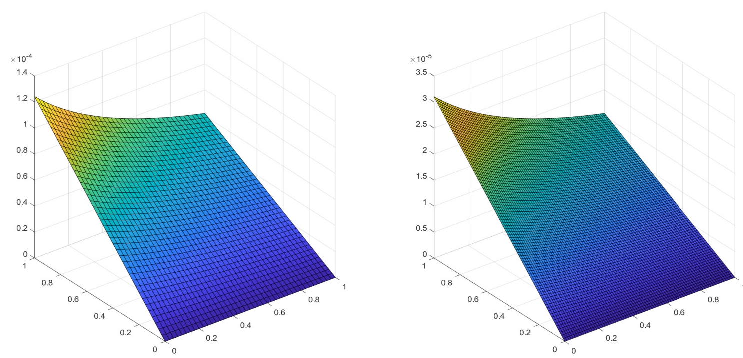

Figure 4: The graphs of error functions $e_{40}(s), s = (s_1, s_2) \in I_2(40)$ and $e_{80}(s), s = (s_1, s_2) \in I_2(80)$

5 Conclusions

In this study, we propose the (2^m-points) trapezoidal method for solving linear and nonlinear m-D integral equations of the second kind. The proposed method is very simple and accurate for obtaining the approximate solution of the integral equation. An approximation of the error bound and the rate of convergence for the proposed method are presented by proving some theorems. In section 4, some numerical examples are considered to confirm the applicability and efficiency of the method. All results are obtained using programs written in Matlab. The numerical results verify that the typical convergence rate of the method is $O(\frac{1}{n^2})$. The method is very convenient for solving higher-dimensional (linear and nonlinear) integral equations of any type, including (mixed) Volterra and Fredholm or any combination of them, such as examples considered in [1, 4, 7, 8, 11, 19, 27].

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