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Weak Dunford-Pettis Property for a Closed Sublattice of Compact Operators

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Abstract. For several Banach lattices E and F, if $K(E,F)$ denotes the space of all compact operators from E to F, it is shown that a necessary and sufficient condition for a closed subspace $\mathcal{M}$ of $K(E,F)$ to have the weak Dunford–Pettis property is that all evaluation operators $\varphi_x : \mathcal{M} \to F$ and $\psi_{y^*} : \mathcal{M} \to E^*$ are almost Dunford–Pettis operators, where $\varphi_x(T) = Tx$ and $\psi_{y^*}(T) = T^*y^*$ for $x \in E$, $y^* \in F^*$ and $T \in \mathcal{M}$.

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1. Introduction

A Banach space X has the Dunford–Pettis (DP) property, if for any Banach space Y, every weakly compact operator from X to Y is completely continuous, that is it maps weakly compact subsets of X onto norm compact subsets of Y. The DP property was introduced by Grothendieck who also showed that a Banach space X has the DP property, if and only if, for every weakly null sequences (x_n) in X and (f_n) in X^*, we have $f_n(x_n) \to 0$. We refer the reader to [1] for valuable results on DP property.

Brown and Ulger (see [4, 12]) have studied the DP property for subspaces of the compact operators on an arbitrary Hilbert space. Indeed, if $\mathcal{M}$ is a

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closed subspace of the compact operators in a Hilbert space H, then $\mathcal{M}$ has the DP property (or equivalently $\mathcal{M}^*$ has the Schur property, i.e., weak and norm convergence of sequences in $\mathcal{M}^*$ coincide), if and only if, all evaluation operators $\varphi_x : \mathcal{M} \to H$ and $\psi_x : \mathcal{M} \to H$ are compact operators, if and only if, all evaluation operators are completely continuous, where $\varphi_x(T) = Tx$ and $\psi_x(T) = T^*x$, for all $x \in H$ and $T \in \mathcal{M}$. The same conclusion has obtained by Saksman and Tylli in [10] for closed subspaces of the compact operators in $\ell_p (1 < p < \infty)$. Saksman and Tylli also showed that if $\mathcal{M}$ is a closed subspace of the compact operators in a reflexive Banach space X having the DP, then for every $x \in X$ the evaluation operators are completely continuous.

Recently In [8], Moshtaghioun and Zafarani extend these conclusions to closed subspaces of several operator ideals. They proved that for a large class of Banach spaces X and Y, the Schur property of the dual $\mathcal{M}^*$ of a closed subspace $\mathcal{M}$ of an arbitrary operator ideal $\mathcal{U}(X, Y)$ is a sufficient condition for compactness and so complete continuity of all evaluation operators φ_x and ψ_{y^*}. On the opposite direction, they have shown that for several Banach spaces X and Y with Schauder decompositions, if $\mathcal{M}$ is a closed subspace of $K(X, Y)$ or $K_{w^*}(X^*, Y)$, then the Schur property of $\mathcal{M}^*$ is a necessary condition for compactness of all point evaluations, where $K(X, Y)$ denotes all compact linear operator from X to Y and $K_{w^*}(X^*, Y)$ is the space of all compact weak*-weak continuous operators from X^* to Y.

In [3] the authors study the DP1 property, which has introduced by [6], for closed subspaces of $K(X, Y)$, where X and Y admit Schauder basis and the basis of X is shrinking; and they proved some necessary and sufficient conditions for the DP1 property of suitable subspaces of $K(X, Y)$. Some results of [3] has extended in [9], for a suitable class of closed subspaces of some operator ideals.

Here, we will show that similar consequences of [9] remain valid for the weak DP property and a suitable class of closed sublattices of compact operators, where in this case the evaluation operators must be assumed almost DP operators. Almost DP operators are introduced by Sanchez in his thesis [11] and used later by Wnuk in his paper [14] and define the weak DP property that is a weaker notion than the DP property. A Banach lattice E has the weak DP property if every weakly compact operator defined on E is almost DP.

It is evident that if E is a Banach lattice, then its dual E^*, endowed with the dual norm and pointwise order, is also a Banach lattice. The norm $\|.\|$ of a Banach lattice E is order continuous if for each generalized net (x_α) such that $x_\alpha \downarrow 0$ in E, (x_α) converges to 0 for the norm $\|.\|$, where the notation $x_\alpha \downarrow 0$ means that the net (x_α) is decreasing, its infimum exists and $\inf(x_\alpha) = 0$. A subset A of E is called solid if $|x| \leqslant |y|$ for some $y \in A$ implies that $x \in A$. Every solid subspace I of E is called an ideal (an order ideal) in E. An ideal B of E is called a band if $\sup(A) \in B$ for every subset $A \subseteq B$ which has a supremum in E. A band B in E that satisfies $E = B \oplus B^{\perp}$, where $B^{\perp} = \{x \in E : |x| \wedge |y| = 0, \forall y \in B\}$ is referred to as a projection band and so every vector $x \in E$ has a unique decomposition $x = x_1 + x_2$, where $x_1 \in B$ and $x_2 \in B^{\perp}$. Then it is easy to see that a projection $p_B : E \to E$ is defined via the formula $p_B(x) := x_1$. Any projection of the form p_B is called an order projection (or a band projection) and $p_{B^{\perp}}$ is the band projection onto $B^{\perp}$. Thus, the band projections are associated with the projection bands in a one-to-one fashion. Every band projection p_B is continuous and $\|p_B\| = 1$. An operator $T : E \to F$ between two Riesz spaces is said to be a lattice homomorphism, whenever it preserves the lattice operations, that is, whenever $T(x \vee y) = T(x) \vee T(y)$ holds for all $x, y \in E$. It is positive if $T(x) \geqslant 0$ in F, whenever $x \geqslant 0$ in E. Throughout this article, X and Y denote the arbitrary Banach spaces. Also E and F denote arbitrary Banach lattices and $E^{+} = \{x \in E : x \geqslant 0\}$ refers to the positive cone of the Banach lattice E. If x is an elemeny of a Banach lattice E, then the absolute value of x is represented by $|x|$. For unexplained notation the reader is referred to [1] and [7].

2. Main Results

Recall from [11] that, a continuous operator T from a Banach lattice E to a Banach space Y is called almost DP if $\|Tx_n\| \to 0$ for every weakly null sequence (x_n) in E consisting of pairwise disjoint elements (a sequence (x_n) is (pairwise) disjoint, if for each $i \neq j$, $|x_i| \wedge |x_j| = 0$). The following Lemma, which deals with almost DP operators and disjoint sequences in Banach lattice, is due to the author [14] and is needed in the rest of this article.

Lemma 2.1. *An operator T from a Banach lattice E into a Banach space X is almost DP, if and only if, $\|Tx_n\| \to 0$, for every weakly null disjoint sequence (x_n) in E^{+}.*

For the following theorem, see [14].

Theorem 2.2. *For a Banach lattice E, the following are equivalent:*

- *(a) E has the weak DP property,*
- *(b) For every Banach lattice F, every weakly compact operator T from E into F is an almost DP operator,*
- *(c) For every reflexive Banach lattice F, every operator T from E into F is an almost DP operator,*
- *(d) Every weakly compact operator T from E into c_0 is an almost DP operator.*

Proof. $(a) \Rightarrow (b)$. It is just the definition of the weak DP property.
$(b) \Rightarrow (c)$. Suppose that F is reflexive and T is an operator from E to F. From [1], T is weakly compact and by the hypothesis (b), T is an almost DP operator.
$(c) \Rightarrow (d)$. If $T : E \to c_0$ is a weakly compact operator, since c_0 has order continuous norm, by [1, Theorem 5.38], T factors through a reflexive Banach lattice. Hence by the hypothesis (c), T is an almost DP operator.
$(d) \Rightarrow (a)$. From [14, Proposition 1], we show that $f_n(x_n) \to 0$ for every disjoint weakly null sequence $(x_n) \in E^+$ and for all weakly null sequence $(f_n) \in E^*$. By [1, Theorem 5.26], the map $T : E \to c_0$ defined by $Tx = (f_n(x))$ is a weakly compact operator and by the hypothesis (d), it is almost DP and so Lemma 2.1 implies that, $\|Tx_n\| \to 0$, that is $sup_i|f_n(x_i)| \to 0$. Now $|f_n(x_n)| \to 0$ and therefore $f_n(x_n) \to 0$. □

As immediate consequences of the preceding Theorem we have the following Corollaries.

Corollary 2.3. *Let E, F be two reflexive Banach lattices and $\mathcal{M} \subset L(E, F)$ be a Banach lattice with the weak DP property. Then the evaluation operators φ_x and ψ_{y^*} are almost DP operators, for all $x \in E$ and $y^* \in F^*$.*

Proof. Since E and F are reflexive and $\mathcal{M}$ has the weak DP property, by Theorem 2.2, the evaluation operators φ_x and ψ_{y^*} are almost DP operators, for all $x \in E$ and $y^* \in F^*$. □

Corollary 2.4. *Let $\mathcal{M}$ be a Banach lattice and X be a Banach space, such that $\mathcal{M}^*$ has order continuous norm. If c_0 is not embeddable in X, and $\mathcal{M}$ has the weak DP property, then each continuous operator $T : \mathcal{M} \to X$ is a almost DP operator.*

Proof. By [1, Theorem 5.27], each continuous operator $T : \mathcal{M} \to X$ is weakly compact and by Theorem 2.2, the weak DP property of $\mathcal{M}$ implies that T is an almost DP operator. □

From [1, Theorem 4.61], a Banach lattice E is weakly sequentially complete (wsc), if and only if, c_0 is not embeddable in E, if and only if, c_0 is not lattice embeddable in E. So we have the following Corollary.

Corollary 2.5. *Let E^* and F be wsc and $\mathcal{M} \subset L(E, F)$ be a Banach lattice such that $\mathcal{M}^*$ has order continuous norm. If $\mathcal{M}$ has the weak DP property, then for all $x \in E$ and $y^* \in F^*$ the evaluation operators φ_x and ψ_{y^*} are almost DP.*

Proof. Since E^* and F are wsc, from [1, Theorem 4.61], c_0 is not embeddable in E^* and F. Hence by Corollary 2.4, the evaluation operators φ_x and ψ_{y^*} are almost DP operators for all $x \in E$ and $y^* \in F^*$. $\square$

From [13] a Banach lattice E has the positive Schur property if every weakly null sequence with positive terms in E is norm null or equivalently, every weakly null sequence of pairwise disjoint elements of E is norm null. If Banach lattice E has the Schur property then it has the positive Schur property, but the converse of this assertion, in general, is false. For example $L^1[0,1]$ is a Banach lattice with positive Schur and without Schur property and so its identity operator $Id_{L^1[0,1]} : L^1[0,1] \to L^1[0,1]$ is almost DP, but it is not DP because $L^1[0,1]$ does not have the Schur property. So we have the evident Proposition.

Proposition 2.6. *Let $\mathcal{M} \subset L(X,Y)$ be a Banach lattice with the positive Schur property. Then the evaluation operators φ_x and ψ_{y^*} are almost DP operators for all $x \in X$ and $y^* \in Y^*$.*

Proof. Since $\mathcal{M}$ has the positive Schur property, then every weakly null sequence of pairwise disjoint elements in $\mathcal{M}$ is norm null and by the definition of almost DP operators in [11], the evaluation operators φ_x and ψ_{y^*} are almost DP operators for all $x \in X$ and $y^* \in Y^*$. $\square$

As in [7] a discrete element of a Riesz space E is a nonzero element u of E whose generating ideal A_u equals that vector subspace generated by u in E and A_u is a projection band . A complete disjoint system $\{e_i\}_{i\in I}$ of a Riesz space E is a pairwise disjoint collection of element of E^+ such that if $u \wedge e_i = 0$ holds for all $i \in I$, then $u = 0$. A discrete Riesz space is a Riesz space E having a complete disjoint system consisting of discrete elements of E. For example, the classical Banach lattices c_0 and ℓ_p, where $1 \leqslant p < \infty$ are discrete with order continuous norm and ℓ_∞ is discrete without order continuous norm. Now, we have the following Proposition.

Proposition 2.7. *Suppose that E^* and F are discrete Banach lattices with order continuous norm and $\mathcal{M} \subset L(E,F)$ is a Banach lattice with the positive Schur property, then the evaluation operators φ_x and ψ_{y^*} are DP operators for all $x \in E^+$ and $y^* \in (F^*)^+$.*

Proof. Since $\mathcal{M}$ has the positive Schur property by proposition 2.6, the evaluation operators φ_x and ψ_{y^*} are positive almost DP operators and by [2, Theorem 2.2], they are Dunford-pettis operators, for all $x \in E^+$ and $y^* \in (F^*)^+$. $\square$

In order to prove the final Theorem, we give the following Lemma similar to [3, Remark 2.3].

Lemma 2.8. *Suppose that $\mathcal{M} \subset L(E,F)$ is a Banach lattice and evaluation* *operators φ_x and ψ_{y^*} are almost DP, for all $x \in E$ and $y^* \in F^*$. Then the mappings $M \to K(E,F)$ and $M \to K(F^*,E^*)$ defined by $T \to TS$ and $T \to T^*K$ are almost DP operators, for all $S \in K(E)$ and $K \in K(F^*)$.*

Proof. Let (T_n) be a disjoint weakly null sequence in $\mathcal{M}^+$. By Lemma 2.1, we show that $\|T_nS\| \to 0$ for all $S \in K(E)$. Since for every $x \in E$, the evaluation operator φ_x is almost DP, then $\|T_nx\| \to 0$, that is, (T_n) converges to 0 strongly. Since the sequence (T_n) is bounded, T_n converges uniformly to 0 on compact subsets. Hence, T_nS converges in norm to 0, for every $S \in K(E)$. Similarly, T_n^*K converges uniformly to 0, for every $K \in K(F^*)$. Therefore, for every $S \in K(E)$ and $K \in K(F^*)$, the mappings $M \to K(E,F)$ and $M \to K(F^*,E^*)$ defined by $T \to TS$ and $T \to T^*K$ are almost DP operators. □

Recall from [1], the vector space of all compact operators from a Banach lattice E to an AM-space F (whenever, its norm is M–norm, i.e., if $x \wedge y = 0$ in F implies $\|x \vee y\| = max\{\|x\|, \|y\|\}$) is a Banach lattice and also AM-spaces with order continuous norm are discrete (see the proof of [15, Theorem 1.4]).
Now, let E be discrete with order continuous norm and F be an AM-space with order continuous norm such that E and F have complete disjoint systems consisting of discrete elements $\{e_i\}_{i\in I}$ and $\{u_i\}_{i\in I}$, respectively and I_{e_i} and I_{u_i} be ideals generated by $\{e_i\}_{i\in I}$ and $\{u_i\}_{i\in I}$, respectively. If $\mathcal{M} \subset K(E,F)$ is a closed sublattice, then for all integers m and n and every operator $T, S \in \mathcal{M}$,

$$\|P_WTP_V + P_{W^\perp}SP_{V^\perp}\| = max\{\|P_WTP_V\|, \|P_{W^\perp}SP_{V^\perp}\|\},$$

where $V = \sum_{i=1}^{m} I_{e_i}$ and $W = \sum_{i=1}^{n} I_{u_i}$.

Theorem 2.9. *Let E be discrete with order continuous norm, F be an AM-space with order continuous norm and assume that $\mathcal{M} \subseteq K_{w^*}(E^*,F)$ is a closed subslattice. If all of the evaluation operators φ_{x^*} and ψ_{y^*} are almost DP operators, then $\mathcal{M}$ has the weak Dpp.*

Proof. We will argue by conradiction, so we assume that $\mathcal{M}$ does not satisfy the weak DP property. Therefore by [**?**], there are $r > 0$, disjoint weakly null sequence $(K_n) \subset \mathcal{M}^*$ and a weakly null sequence $(\Gamma_n) \subset \mathcal{M}^*$ satisfying $|\Gamma_n(K_n)| \geqslant r, (i = 1, 2, ..., n)$.
Since E is discrete with order continuous norm, then it has a complete disjoint system consisting of discrete elements $(e_i)_{i\in I}$, such that for a fixed $m_0 \in N$, $U = \sum_{i=1}^{m_0} I_{e_i}$ is a (principal) projection band and $P_U : E \to E$ is a band projection onto U. Every $x \in E$ is of the form $x = \sum_i t_i(x)e_i$, where numbers $t_i(x)$ are uniquely determined. Functionals f_i defined by $f_i(x) = t_i(x)$ are homomorphisms, and so they are discrete in E^*. It is easy to see that $(f_i)_{i\in I}$ is a complete disjoint system in E^*. Let $V = \sum_{i=1}^{m_0} I_{f_i}$, then V ia a projection

band in E^*. Now $P_V : E^* \to E^*$ is a band projection onto V and $P_V = P_U^*$. Similarly, Since F is discrete with order continuous norm, then it has a complete disjoint system consisting of discrete elements $(u_i)_{i \in I}$, such that for a fixed $n_0 \in N$, $W = \sum_{i=1}^{n_0} I_{u_i}$ is a (principal) projection band and $P_W : F \to F$ is a band projection onto W. We recall that P_V and P_W are continuous and $\|P_V\| = \|P_W\| = 1$. We use the technique of [9, Theorem 2.4].
Now let (ϵ_n) be a sequence of positive numbers such that $\sum n\epsilon_n < \infty$. We shall construct by induction, subsequencs (Λ_n) of (Γ_n) and (S_n) of (K_n) such that for all n, there exist finite dimensional projection bands $V \subset E^*$ and $W \subset F$, satisfying the following properties:

$$\begin{aligned}
&\|S_i P_{V^\perp}\| \leqslant \epsilon_{n+1} \quad \textit{and} \quad \|P_{W^\perp} S_i\| \leqslant \epsilon_{n+1}, && \text{for all i= 1,2,...,n,} \\
&|\langle S_i, \Lambda_{n+1}\rangle| < \frac{r}{2^{(n+1)}} && \text{for all i= 1,2,...,n,} \\
&|\langle S_{n+1}, \Lambda_{n+1}\rangle| > r \quad \textit{and} \quad |\langle S_{n+1}, \Lambda_i\rangle| \leqslant \frac{r}{2^{(n+1)}} && \text{for all i= 1,2,...,n,} \\
&\|S_{n+1} P_V\| \leqslant \epsilon_{n+1} \quad \textit{and} \quad \|P_W S_{n+1}\| \leqslant \epsilon_{n+1}.
\end{aligned} \tag{$*$}$$

Suppose that $\Lambda_1 = \Gamma_1$ and $S_1 = K_1$ and inductively, suppose that $\Lambda_1, ..., \Lambda_n \in (\Gamma_i)$ and $S_1, ..., S_n \in (K_i)$ have been chosen. To obtain Λ_{n+1} and S_{n+1}, similarly, as already explained and by [9, 8], we fined finite dimensional projection bands $V \subset E^*$ and $W \subset F$, such that

$$\|S_i P_{V^\perp}\| \leqslant \epsilon_{n+1} \quad \textit{and} \quad \|P_{W^\perp} S_i\| \leqslant \epsilon_{n+1},$$

for all $i = 1, 2, ..., n$. By Lemma 2.8, the operators $K \to KP_V$ and $K \to P_W K$ from $\mathcal{M}$ into $K_{w^*}(E^*, F)$ are almost DP. Thus by hypothesis on (K_n) we have

$$\|K_n P_V\| \to 0 \quad \textit{and} \quad \|P_W K_n\| \to 0.$$

So there exists an integer $N_1 > 0$ such that for all $j > N_1$:

$$\|K_j P_V\| \leqslant \epsilon_{n+1}, \|P_W K_j\| \leqslant \epsilon_{n+1}.$$

By the weak nullity of the sequences (K_n) and (Γ_n) there exists two integers N_2, N_3 such that

$$\begin{aligned}
&|\langle K_j, \Lambda_i\rangle| \leqslant \frac{r}{2^{(n+1)}} && \text{for all i= 1,2,...,n , } j \geqslant N_2, \\
&|\langle S_i, \Gamma_j\rangle| \leqslant \frac{r}{2^{(n+1)}} && \text{for all i= 1,2,...,n , } j \geqslant N_3.
\end{aligned}$$

Now select an integer j_0 bigger than N_1, N_2 and N_3 and set $\Lambda_{n+1} = \Gamma_{j_0}$ and $S_{n+1} = K_{j_0}$. This finishes the induction process. We have constructed a subsequence (Λ_n) of (Γ_n) and a subsequence (S_n) of (K_n) such that for all integer n, there are finite dimensional projection bands $W \subset F$ and $V \subset E^*$ respectively, that satisfy all conditions of $(*)$. These properties, as shown in [4, 8], yield the following inequalities:

$$\|P_W \sum_{i=1}^{n} S_i P_V - \sum_{i=1}^{n} S_i\| \leqslant 3n\epsilon_{n+1}\ ,\ \|P_{W^\perp} S_{n+1} P_{V^\perp} - S_{n+1}\| \leqslant 3\epsilon_{n+1}.$$

Since F is an AM-space, we obtain:

$$\begin{aligned}
\|\sum_{i=1}^{n+1} S_i\| &\leqslant \|\sum_{i=1}^{n} S_i - P_W \sum_{i=1}^{n} S_i P_V\| + \|S_{n+1} - P_{W^\perp} S_{n+1} P_{V^\perp}\| \\
&\quad + \|P_W \sum_{i=1}^{n} S_i P_V + P_{W^\perp} S_{n+1} P_{V^\perp}\| \\
&\leqslant 3n\epsilon_{n+1} + 3\epsilon_{n+1} + \max\{\|P_W \sum_{i=1}^{n} S_i P_V\|, \|P_{W^\perp} S_{n+1} P_{V^\perp}\|\} \\
&\leqslant 3(n+1)\epsilon_{n+1} + \max\{\|\sum_{i=1}^{n} S_i\|, 1\}.
\end{aligned}$$

This shows that the sequence $T_n = \sum_{i=1}^{n} S_i$ is bounded and so has a weak* limit point $T \in \mathcal{M}^{**}$. For each j, choose an integer $n > j$ such that $|\langle T - T_n, \Lambda_j\rangle| < 1/2^j$. Therefore,

$$\begin{aligned}
|\langle T, \Lambda_j\rangle| &\geqslant |\langle T_n, \Lambda_j\rangle| - |\langle T - T_n, \Lambda_j\rangle| \\
&\geqslant |\sum_{i=1}^{n} \langle S_i, \Lambda_j\rangle| - \frac{r}{2^j} \\
&\geqslant |\langle S_j, \Lambda_j\rangle| - \sum_{i=1}^{j-1} |\langle S_i, \Lambda_j\rangle| - \sum_{i=j+1}^{n} |\langle S_i, \Lambda_j\rangle| - \frac{r}{2^j} \\
&\geqslant r - \frac{r}{2} - \frac{r}{2^j} > \frac{r}{3} > 0
\end{aligned}$$

for sufficiently large j. Hence $\langle T, \Lambda_j\rangle$ and so $\langle T, \Gamma_j\rangle$ does not tend to zero. Thus the sequence (Γ_j) does not converge weakly to zero, which gives a contradiction and hence $\mathcal{M}$ has the weak DP property. □

However, under the same assumptions on E and F, a similar result can be inferred for closed sublattices of $K(E, F)$:

Corollary 2.10. *Let E be discrete with order continuous norm, F be an AM-space with order continuous norm and assume that $\mathcal{M} \subset K(E, F)$ is a closed* *subslattice. If all of the evaluation operators φ_x and ψ_{y^*} are almost DP operators, then $\mathcal{M}$ has the weak DP property.*

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