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d-Oresme Polynomials and Their Matrix Representations

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Abstract. In this paper, we define the concept of d-Oresme polynomials, which is a generalization of the classical Oresme polynomials. We obtain several fundamental properties for these new polynomials including the generating function, the Binet-like formula, some combinatorial identities and summation formulas. We define the matrix O_d, and show that the power of O_d generates the d-Oresme polynomials. Then, we introduce the infinite d-Oresme polynomials matrix, which is a Riordan matrix. We present two new factorizations of the infinite Pascal matrix whose entries are the d-Oresme polynomials by Riordan method.

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1 Introduction

Many researchers have been interested in number sequences and their polynomials for long years since they have many applications in nature and various fields. The Fibonacci numbers are one of the most

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well-known number sequences. Fibonacci numbers are defined by the recurrence relation

$$F_{n+1} = F_n + F_{n-1}, \quad n \geq 1$$

with initial conditions $F_0 = 0$ and $F = 1$. For $W_0, W_1, p, q \in \mathbb{Z}$, the Horadam numbers $W_n = W_n(W_0, W_1; p, q)$ are defined by

$$W_{n+1} = pW_n + qW_{n-1}, \quad n \geq 1 \tag{1.1}$$

with the initial values W_0 and W_1 (see, for example, [12] and [13]). In [14], A. F. Horadam extended the equation (1.1) considering rational numbers W_0, W_1, p, q, and provided a history of number attributed to Nicole Oresme, Oresme number sequences

$$\{O_n\}_{n\in\mathbb{N}} = \left\{W_n\left(0, \frac{1}{2}; 1, -\frac{1}{4}\right)\right\}_{n\in\mathbb{N}} = \left\{0, \frac{1}{2}, \frac{2}{4}, \frac{3}{8}, \frac{4}{16}, \frac{5}{32}, \ldots, \frac{n}{2^n}, \ldots\right\}.$$

The Oresme numbers have many interesting properties and applications in many fields of science (see, for example, [7], [8], [9] and [14]). The Oresme polynomials can be useful in specific boundary-value problems and numerical methods with certain stability requirements. In particular situations, their recursive form can be computationally advantageous. The Oresme numbers are defined by the recurrence relation

$$O_{n+1} = O_n - \frac{1}{4}O_{n-1}, \quad n \geq 1$$

with initial conditions $O_0 = 0$ and $O_1 = \frac{1}{2}$. In [5], Cerda-Morales defined the Oresme polynomials by the recurrence relation

$$O_{n+1}(x) = O_n(x) - \frac{1}{x^2}O_{n-1}(x), \quad n \geq 1$$

with initial conditions $O_0(x) = 0$ and $O_1(x) = \frac{1}{x}$.

In 1883, Fibonacci polynomials, investigated by Catalan, were defined by the recurrence relation

$$F_{n+1}(x) = xF_n(x) + F_{n-1}(x), \quad n \geq 1$$

with initial conditions $F_0(x) = 0$ and $F_1(x) = 1$. Nalli and Haukkanen [19] defined $h(x)$-Fibonacci polynomials as

$$F_{h,n+1}(x) = h(x)F_{h,n}(x) + F_{h,n-1}(x), \quad n \geq 1$$

with initial conditions $F_{h,0}(x) = 0$ and $F_{h,1}(x) = 1$. In [18], Lee and Asci introduced (p, q)-Fibonacci polynomials as

$$F_{p,q,n+1}(x) = p(x)F_{p,q,n}(x) + q(x)F_{p,q,n-1}(x), \quad n \geq 1$$

with initial conditions $F_{p,q,0}(x) = 0$ and $F_{p,q,1}(x) = 1$. Let $d \in \mathbb{Z}^+ = \{1, 2, \dots\}$ and $p_i(x)$ be a real polynomial for each $i = 1, 2, \dots, d + 1$. In [20], Sadaoui and Krelifa generalized (p, q)-Fibonacci polynomials to d-Fibonacci polynomials, which are described such that

$$F_{n+1}(x) = p_1(x)F_n(x) + p_2(x)F_{n-1}(x) + \cdots + p_{d+1}(x)F_{n-d}(x), \ n \geq 1$$

with initial conditions $F_n(x) = 0$ for $n \leq 0$ and $F_1(x) = 1$. In [10], Heydari et al. generated shifted Vieta-Fibonacci polynomials for the fractal-fractional fifth-order KdV equation. Heydari and Avazzadeh [11] have developed the collocation technique based on the Fibonacci polynomials for finding the approximate solution of variable-order space-time fractional Burgers-Huxley equation. Furthermore, Kuloğlu and Özkan [15] defined the d-Tribonacci polynomials, which is a generalization of Tribonacci polynomials, and studied these polynomials.

Lawden [16] introduced the $n \times n$ lower triangular Pascal matrix $P = (p_{i,j})$ such that

$$p_{i,j} = \begin{cases} 0, & \text{if } i < j \\ \binom{i-1}{j-1}, & \text{if } i \geq j \end{cases}$$

for $i, j = 1, 2, \dots, n$ (see, for example, [2] and [4]). The infinite Pascal matrix P is given as

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 2 & 1 & 0 & \cdots & 0 \\ 1 & 3 & 3 & 1 & \cdots & 0 \\ 1 & 4 & 6 & 4 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{1.2}$$

The Pascal matrices have several applications in probability, numerical analysis, surface reconstruction and combinatorics. In [1], [3] and [23], the authors investigated the linear algebras of the generalized Pascal functional matrix, the Pascal matrix and the generalized Pascal matrix, respectively. In [17] and [22], the authors derived two factorizations of the Pascal matrix involving the Fibonacci matrix.

In [21], Shapiro et al. introduced the Riordan group as follows.

Let $i, j \in \mathbb{N} = \{0, 1, 2, \dots\}$ and $A = (a_{i,j})$ be an infinite matrix with entries in $\mathbb{C}$. Let $k \in \mathbb{N}$ and $c_k(u) = \sum_{m=0}^{\infty} a_{m,k} u^m$ be the generating function of the kth column of A. The matrix $A = (g(u), f(u))$ is called a *Riordan matrix*, if $c_k(u) = g(u)\,[f(u)]^k$, where $g(u) = \sum_{m=0}^{\infty} g_m u^m$ and $f(u) = \sum_{m=0}^{\infty} f_m u^m$.

We denote by $\mathcal{R}$ the set of Riordan matrices. It is well-known that $\mathcal{R}$ is a group under matrix multiplication $*$, and is called by *Riordan group*. We present the following features related to Riordan group.

(i) $(g(u), f(u)) * C(u) = g(u)C(f(u))$, where $C(u)$ is a column vector (matrix multiplication $*$ with $C(u)$),

(ii) $(g(u), f(u)) * (h(u), l(u)) = (g(u)h(f(u)), l(f(u)))$ (matrix multiplication $*$),

(iii) $i_{\mathcal{R}} = (1, u)$, where $i_{\mathcal{R}}$ is the identity element of $\mathcal{R}$,

(iv) $(g(u), f(u))^{-1} = \left(\frac{1}{g\left(\bar{f}(u)\right)}, \bar{f}(u)\right)$, where $\bar{f}(u)$ is compositional inverse of $f(u)$ (inverse element).

Riordan group has many applications. The three applications of Riordan group are provided by Euler's problem of the King walks, binomial and inverse identities and a Bessel-Neumann expansion in [21]. Furthermore, Cheon et al. [6] presented a generalization of Lucas polynomial sequence by using the Riordan array which is obtained from weighted Delannoy numbers.

This study is structured as follows:

In Section 2, we introduce the d-Oresme polynomials. These polynomials are a new generalization of the known Oresme polynomials. We give a variety of conclusions of the d-Oresme polynomials including the generating function, the Binet-like formula, some combinatorial identities and summation formulas. We describe the matrix O_d, and show that the power of O_d generates the d-Oresme polynomials. In Section 3, we define the infinite d-Oresme polynomials matrix, which is a Riordan matrix. Then, we obtain two factorizations of the infinite Pascal matrix including d-Oresme polynomials.

2 d-Oresme Polynomials

In this section, we introduce a new generalization of Oresme polynomials.

Definition 2.1. Let $d \in \mathbb{Z}^+$ and $q_k(x)$ be a real polynomial for $k = 1, 2, \ldots, d+1$. Then, d-Oresme polynomials $O_n^{(d)}(x)$ are defined by the recurrence relation

$$O_{n+1}^{(d)}(x) = q_1(x)O_n^{(d)}(x) + q_2(x)O_{n-1}^{(d)}(x) + \cdots + q_{d+1}(x)O_{n-d}^{(d)}(x), \ n \geq 1 \tag{2.1}$$

with initial conditions $O_n^{(d)}(x) = 0$ for $n \leq 0$ and $O_1^{(d)}(x) = \frac{1}{x}$.

Throughout this paper, we adopt the notations $O_n^{(d)}$ and q_k instead of $O_n^{(d)}(x)$ and $q_k(x)$ in (2.1), respectively. In Table 1, we give a few terms of d-Oresme polynomials.

Table 1: Some values of d-Oresme polynomials.

n	$O_n^{(d)}$
2	$\frac{q_1}{x}$
3	$\frac{q_1^2 + q_2}{x}$
4	$\frac{q_1^3 + 2q_1q_2 + q_3}{x}$
5	$\frac{q_1^4 + 3q_1^2q_2 + 2q_1q_3 + q_2^2 + q_4}{x}$
6	$\frac{q_1^5 + 4q_1^3q_2 + 3q_1^2q_3 + 3q_1q_2^2 + 2q_1q_4 + 2q_2q_3 + q_5}{x}$

In (2.1), if we take $q_1 = 1, q_2 = -\frac{1}{x^2}$ and $q_k = 0$ for $k = 3, 4, \ldots, d+1$, so we derive $O_n^{(d)} = O_n(x)$. Therefore, d-Oresme polynomials are a generalization of the classical Oresme polynomials.

By (2.1), for d-Oresme polynomials, the characteristic equation is given by

$$w^{d+1} - q_1 w^d - \cdots - q_{d+1} = 0. \tag{2.2}$$

Theorem 2.2. *Let $n \geq d$. Then we get*

$$\begin{aligned} w^n &= O^{(d)}_{n-d+1} w^d + \left(q_2 O^{(d)}_{n-d} + \cdots + q_{d+1} O^{(d)}_{n-2d+1} \right) w^{d-1} \\ &\quad + \left(q_3 O^{(d)}_{n-d} + \cdots + q_{d+1} O^{(d)}_{n-2d+2} \right) w^{d-2} \\ &\quad + \cdots + q_{d+1} O^{(d)}_{n-d}. \end{aligned} \tag{2.3}$$

Proof. To prove the theorem, we employ mathematical induction on n. For $n = 1$, it is clear that the equation (2.3) is true. Assume that the equation (2.3) satisfies for $n = j$. We will show that the equation (2.3) is true for $n = j+1$. From (2.1) and the characteristic equation (2.2), we obtain

$$\begin{aligned} & w^{j+1} \\ = \; & w^j w \\ = \; & O^{(d)}_{j-d+1} \left(q_1 w^d + \cdots + q_{d+1} \right) + \left(q_2 O^{(d)}_{j-d} + \cdots + q_{d+1} O^{(d)}_{j-2d+1} \right) w^d \\ & + \left(q_3 O^{(d)}_{j-d} + \cdots + q_{d+1} O^{(d)}_{j-2d+2} \right) w^{d-1} + \cdots + q_{d+1} O^{(d)}_{j-d} v \\ = \; & \left(q_1 O^{(d)}_{j-d+1} + \cdots + q_{d+1} O^{(d)}_{j-2d+1} \right) w^d \\ & + \left(q_2 O^{(d)}_{j-d+1} + \cdots + q_{d+1} O^{(d)}_{j-2d+2} \right) w^{d-1} \\ & + \cdots + \left(q_d O^{(d)}_{j-d+1} + q_{d+1} O^{(d)}_{j-d} \right) v + q_{d+1} O^{(d)}_{j-d+1}. \end{aligned}$$

□

Theorem 2.3. *The generating function for d-Oresme polynomials is given by*

$$g^{(d)}(w) = \frac{w}{x\left(1 - q_1 w - q_2 w^2 - \cdots - q_{d+1} w^{d+1}\right)}.$$

Proof. We have

$$\begin{aligned} g^{(d)}(w) &= \sum_{i=0}^{\infty} O_i^{(d)} w^i \\ &= O_0^{(d)} + O_1^{(d)} w + O_2^{(d)} w^2 + \cdots + O_n^{(d)} w^n + \cdots . \end{aligned} \quad (2.4)$$

If we multiply the equation (2.4) by $q_1 w, q_2 w^2, \ldots, q_{d+1} w^{d+1}$, respectively, then we derive the following equations.

$$\begin{aligned} q_1 w g^{(d)}(w) &= q_1 O_0^{(d)} w + q_1 O_1^{(d)} w^2 + q_1 O_2^{(d)} w^3 + \cdots \\ q_2 w^2 g^{(d)}(w) &= q_2 O_0^{(d)} w^2 + q_2 O_1^{(d)} w^3 + q_2 O_2^{(d)} w^4 + \cdots \\ &\vdots \\ q_{d+1} w^{d+1} g^{(d)}(w) &= q_{d+1} O_0^{(d)} w^{d+1} + q_{d+1} O_1^{(d)} w^{d+2} + \cdots . \end{aligned}$$

If the necessary calculations are made, by the equation (2.1) we have

$$g^{(d)}(w) \left(1 - q_1 w - q_2 w^2 - \cdots - q_{d+1} w^{d+1}\right) = O_0^{(d)} + \left(O_1^{(d)} - q_1 O_0^{(d)}\right) w$$

and so

$$g^{(d)}(w) = \frac{w}{x\left(1 - q_1 w - q_2 w^2 - \cdots - q_{d+1} w^{d+1}\right)}.$$

□

Let the set of the roots of (2.2) be $\{\lambda_1, \lambda_2, \ldots, \lambda_{d+1}\}$. That is, we get

$$\frac{w}{x\left(1 - q_1 w - q_2 w^2 - \cdots - q_{d+1} w^{d+1}\right)} = \sum_{i=1}^{d+1} \frac{c_i}{1 - \lambda_i w}.$$

By Theorem 2.3, we have

$$\sum_{i=0}^{\infty} O_i^{(d)} w^i = \sum_{i=1}^{d+1} c_i \sum_{k=0}^{\infty} \lambda_i^k w^k.$$

Hence, we can obtain the Binet-like formula of $O_n^{(d)}$ in the following corollary.

Corollary 2.4. *The Binet-like formula of d-Oresme polynomials is given by*

$$O_n^{(d)} = \sum_{k=1}^{d+1} c_k \lambda_k^n. \tag{2.5}$$

The multinomial coefficients, in particular, enable us to provide an explicit version of the d-Oresme polynomials.

Theorem 2.5. *For $n \geq 0$, then we have*

$$O_{n+1}^{(d)} = \frac{1}{x}\left[\sum_{\substack{j_1,j_2,\dots,j_{d+1} \\ j_1+2j_2+\cdots+(d+1)j_{d+1}=n}} \binom{j_1+j_2+\cdots+j_{d+1}}{j_1,j_2,\dots,j_{d+1}} q_1^{j_1} q_2^{j_2} \cdots q_{d+1}^{j_{d+1}}\right].$$

Proof. By Theorem 2.3, we obtain

$$\begin{aligned}
&\sum_{j=0}^{\infty} O_{j+1}^{(d)} w^j \\
= \; & \frac{1}{x\left(1-q_1 w - q_2 w^2 - \cdots - q_{d+1} w^{d+1}\right)} \\
= \; & \frac{1}{x}\sum_{j=0}^{\infty}\left(q_1 w + q_2 w^2 + \cdots + q_{d+1} w^{d+1}\right)^j \\
= \; & \frac{1}{x}\sum_{j=0}^{\infty}\left[\sum_{j_1+j_2+\cdots+j_{d+1}=j} \binom{j}{j_1,j_2,\dots,j_{d+1}} q_1^{j_1}\cdots q_{d+1}^{j_{d+1}}\right] \\
& w^{j_1+2j_2+\cdots+(d+1)j_{d+1}} \\
= \; & \frac{1}{x}\sum_{j=0}^{\infty}\left[\sum_{\substack{j_1,j_2,\dots,j_{d+1} \\ j_1+2j_2+\cdots+(d+1)j_{d+1}=j}} \binom{j_1+j_2+\cdots+j_{d+1}}{j_1,j_2,\dots,j_{d+1}} q_1^{j_1}\cdots q_{d+1}^{j_{d+1}}\right] w^j.
\end{aligned}$$

□

Theorem 2.6. *The sum of d-Oresme polynomials is given by*

$$\sum_{i=0}^{\infty} O_i^{(d)} = \frac{1}{x\left(1-q_1-q_2-\cdots-q_{d+1}\right)}.$$

Proof. We have

$$\sum_{i=0}^{\infty} O_i^{(d)} = O_0^{(d)} + O_1^{(d)} + \cdots + O_n^{(d)} + \cdots . \tag{2.6}$$

Multiplying (2.6) by $q_1, q_2, \ldots, q_{d+1}$, respectively, then we have

$$\begin{aligned} q_1 \sum_{i=0}^{\infty} O_i^{(d)} &= q_1 O_0^{(d)} + q_1 O_1^{(d)} + \cdots + q_1 O_n^{(d)} + \cdots \\ q_2 \sum_{i=0}^{\infty} O_i &= q_2 O_0^{(d)} + q_2 O_1^{(d)} + \cdots + q_2 O_n^{(d)} + \cdots \\ &\vdots \\ q_{d+1} \sum_{i=0}^{\infty} O_i^{(d)} &= q_{d+1} O_0^{(d)} + q_{d+1} O_1^{(d)} + \cdots + q_{d+1} O_n^{(d)} + \cdots . \end{aligned}$$

If we take the necessary calculations, by (2.1) we can easily obtain

$$\sum_{i=0}^{\infty} O_i^{(d)} \left(1 - q_1 - q_2 - \cdots - q_{d+1}\right) = O_0^{(d)} + \left(O_1^{(d)} - q_1 O_0^{(d)}\right)$$

and so

$$\sum_{i=0}^{\infty} O_i^{(d)} = \frac{1}{x\left(1 - q_1 - q_2 - \cdots - q_{d+1}\right)}.$$

□

Nalli and Haukkanen [19] introduced the matrix

$$Q_h(x) = \begin{bmatrix} h(x) & 1 \\ 1 & 0 \end{bmatrix},$$

Lee and Asci [18] defined the matrix

$$Q_{p,q}(x) = \begin{bmatrix} p(x) & q(x) \\ 1 & 0 \end{bmatrix},$$

that plays the role of the Fibonacci matrix

$$Q = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

Then in [20], Sadaoui and Krelifa defined the matrix

$$Q_d = \begin{bmatrix} p_1(x) & p_2(x) & \cdots & & p_{d+1}(x) \\ 1 & 0 & & & 0 \\ 0 & \ddots & & & \\ & \ddots & \ddots & & \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}.$$

Now we define the matrix O_d such that

$$O_d = \begin{bmatrix} \frac{q_1}{x} & \frac{q_2}{x} & \cdots & & \frac{q_{d+1}}{x} \\ \frac{1}{x} & 0 & & & 0 \\ 0 & \ddots & & & \\ & \ddots & \ddots & & \\ 0 & \cdots & 0 & \frac{1}{x} & 0 \end{bmatrix}. \tag{2.7}$$

This means that the determinant of O_d is the polynomial $\frac{(-1)^d q_{d+1}}{x^{d+1}}$. We provide matrix representation of $O_n^{(d)}$ in the following theorem.

Theorem 2.7. *For $n \geq 1$, then we have*

$$O_d^n = \begin{bmatrix} O_{n+1}^{(d)} & q_2 O_n^{(d)} + \cdots + q_{d+1} O_{n-d+1}^{(d)} & \cdots & q_{d+1} O_n^{(d)} \\ O_n^{(d)} & q_2 O_{n-1}^{(d)} + \cdots + q_{d+1} O_{n-d}^{(d)} & \cdots & q_{d+1} O_{n-1}^{(d)} \\ \vdots & \vdots & \vdots & \vdots \\ O_{n-d+1}^{(d)} & q_2 O_{n-d}^{(d)} + \cdots + q_{d+1} O_{n-2d+1}^{(d)} & \cdots & q_{d+1} O_{n-d}^{(d)} \end{bmatrix}.$$

Proof. We use induction method on n to demonstrate the theorem. Let $n = 1$. From (2.1) and (2.7), we get

$$O_d = \begin{bmatrix} O_2^{(d)} & q_2 O_1^{(d)} + \cdots + q_{d+1} O_{2-d}^{(d)} & \cdots & q_{d+1} O_1^{(d)} \\ O_1^{(d)} & q_2 O_0^{(d)} + \cdots + q_{d+1} O_{1-d}^{(d)} & \cdots & q_{d+1} O_0^{(d)} \\ \vdots & \vdots & \vdots & \vdots \\ O_{2-d}^{(d)} & q_2 O_{1-d}^{(d)} + \cdots + q_{d+1} O_{2-2d}^{(d)} & \cdots & q_{d+1} O_{1-d}^{(d)} \end{bmatrix}.$$

Suppose that the hypothesis is true for $n=j$. Namely,

$$O_d^j=\begin{bmatrix} O_{j+1}^{(d)} & q_2O_j^{(d)}+\cdots+q_{d+1}O_{j-d+1}^{(d)} & \cdots & q_{d+1}O_j^{(d)} \\ O_j^{(d)} & q_2O_{j-1}^{(d)}+\cdots+q_{d+1}O_{j-d}^{(d)} & \cdots & q_{d+1}O_{j-1}^{(d)} \\ \vdots & \vdots & \vdots & \vdots \\ O_{j-d+1}^{(d)} & q_2O_{j-d}^{(d)}+\cdots+q_{d+1}O_{j-2d+1}^{(d)} & \cdots & q_{d+1}O_{j-d}^{(d)} \end{bmatrix}.$$

We show that it is true for $n=j+1$. Therefore we get

$$O_d^{j+1}=O_d^jO_d=\begin{bmatrix} O_{j+2}^{(d)} & q_2O_{j+1}^{(d)}+\cdots+q_{d+1}O_{j-d+2}^{(d)} & \cdots & q_{d+1}O_{j+1}^{(d)} \\ O_{j+1}^{(d)} & q_2O_j^{(d)}+\cdots+q_{d+1}O_{j-d+1}^{(d)} & \cdots & q_{d+1}O_j^{(d)} \\ \vdots & \vdots & \vdots & \vdots \\ O_{j-d+2}^{(d)} & q_2O_{j-d+1}^{(d)}+\cdots+q_{d+1}O_{j-2d+2}^{(d)} & \cdots & q_{d+1}O_{j-d+1}^{(d)} \end{bmatrix}.$$

□

For $n,k\geq 0$, we get $O_d^nO_d^k=O_d^{n+k}$. In addition, we know that the first entry of matrix O_d^{n+k} is the production of the first row in O_d^n by the first column of O_d^k. Hence by Theorem 2.7, we can derive the following corollary.

Corollary 2.8. *Let $n,t\geq 0$. Then we have*

$$\begin{aligned}\frac{O_{n+t+1}^{(d)}}{x} &= O_{n+1}^{(d)}O_{t+1}^{(d)}+q_2O_n^{(d)}O_t^{(d)} \\ &+q_3\left(O_{n-1}^{(d)}O_t^{(d)}+O_n^{(d)}O_{t-1}^{(d)}\right) \\ &+q_4\left(O_{n-2}^{(d)}O_t^{(d)}+O_{n-1}^{(d)}O_{t-1}^{(d)}+O_n^{(d)}O_{t-2}^{(d)}\right) \\ &\vdots \\ &+q_{d+1}\left(O_{n-d+1}^{(d)}O_t^{(d)}+\cdots+O_n^{(d)}O_{t-d+1}^{(d)}\right).\end{aligned}$$

Theorem 2.9. *For $n\geq 0$, then we have*

$$\begin{aligned}O_{(d+1)n}^{(d)} &= \sum_{\substack{j_1,j_2,\ldots,j_{d+1}\\(d+1)j_1+dj_2+\cdots+j_{d+1}=n}}\binom{j_1+j_2+\cdots+j_{d+1}}{j_1,j_2,\ldots,j_{d+1}}q_1^{j_1}q_2^{j_2} \\ &\cdots q_{d+1}^{j_{d+1}}O_{n-(j_1+j_2+\cdots+j_{d+1})}^{(d)}. \qquad (2.8)\end{aligned}$$

Proof. We denote the right hand side of (2.8) by U. By using Binet-like formula in (2.5) and the characteristic equation in (2.2), for $n \geq 2$, we find that

$$\begin{aligned} U &= \sum_{\substack{j_1,j_2,\dots,j_{d+1} \\ (d+1)j_1+dj_2+\cdots+j_{d+1}=n}} \binom{j_1+j_2+\cdots+j_{d+1}}{j_1,j_2,\dots,j_{d+1}} q_1^{j_1} q_2^{j_2}\cdots q_{d+1}^{j_{d+1}} \\ &\quad \sum_{i=1}^{d+1} c_i \lambda_i^{n-(j_1+j_2+\cdots+j_{d+1})} \\ &= \sum_{\substack{j_1,j_2,\dots,j_{d+1} \\ (d+1)j_1+dj_2+\cdots+j_{d+1}=n}} \binom{j_1+j_2+\cdots+j_{d+1}}{j_1,j_2,\dots,j_{d+1}} q_1^{j_1} q_2^{j_2}\cdots q_{d+1}^{j_{d+1}} \\ &\quad \sum_{i=1}^{d+1} c_i \lambda_i^{dj_1+(d-1)j_2+\cdots+j_d} \\ &= c_1 \sum_{\substack{j_1,j_2,\dots,j_{d+1} \\ (d+1)j_1+dj_2+\cdots+j_{d+1}=n}} \binom{j_1+j_2+\cdots+j_{d+1}}{j_1,j_2,\dots,j_{d+1}} \left(q_1\lambda_1^d\right)^{j_1} \\ &\quad \left(q_2\lambda_1^{d-1}\right)^{j_2}\cdots (q_{d+1})^{j_{d+1}} \\ &\quad + \\ &\quad \vdots \\ &\quad +c_{d+1} \sum_{\substack{j_1,j_2,\dots,j_{d+1} \\ (d+1)j_1+dj_2+\cdots+j_{d+1}=n}} \binom{j_1+j_2+\cdots+j_{d+1}}{j_1,j_2,\dots,j_{d+1}} \left(q_1\lambda_{d+1}^d\right)^{j_1} \\ &\quad \left(q_2\lambda_{d+1}^{d-1}\right)^{j_2}\cdots (q_{d+1})^{j_{d+1}} \\ &= c_1\left(q_1\lambda_1^d + q_2\lambda_1^{d-1}+\cdots+q_{d+1}\right)^n \\ &\quad +\cdots + c_{d+1}\left(q_1\lambda_{d+1}^d + q_2\lambda_{d+1}^{d-1}+\cdots+q_{d+1}\right)^n \\ &= \sum_{i=1}^{d+1} c_i\lambda_i^{(d+1)n} \\ &= O_{(d+1)n}^{(d)}. \end{aligned}$$

□

Theorem 2.10. *Let $n \geq 0$. Then we have*

$$\begin{aligned}
&\sum_{t=0}^{n} \binom{n}{t} (-2q_{d+1})^{t} O^{(d)}_{(d+1)(n-t)} \\
= &\sum_{\substack{j_1,j_2,\dots,j_{d+1} \\ (d+1)j_1+dj_2+\cdots+j_{d+1}=n}} \binom{j_1+j_2+\cdots+j_{d+1}}{j_1,j_2,\dots,j_{d+1}} q_1^{j_1} q_2^{j_2} \\
&\cdots \left(-q_{d+1}^{j_{d+1}}\right) O^{(d)}_{n-(j_1+j_2+\cdots+j_{d+1})}. \qquad (2.9)
\end{aligned}$$

Proof. We denote the right hand side of (2.9) by V. Then, considering the proof of Theorem 2.9, we obtain

$$\begin{aligned}
V &= c_1 \left(q_1\lambda_1^d + \cdots + q_d\lambda_1 - q_{d+1}\right)^n \\
&\quad + \cdots + c_{d+1} \left(q_1\lambda_{d+1}^d + \cdots + q_d\lambda_{d+1} - q_{d+1}\right)^n \\
&= \sum_{i=1}^{d+1} c_i \left(\lambda_i^{d+1} - 2q_{d+1}\right)^n \\
&= \sum_{i=1}^{d+1} c_i \sum_{t=0}^{n} \binom{n}{t} \lambda_i^{(d+1)(n-t)} (-2q_{d+1})^t \\
&= \sum_{t=0}^{n} \binom{n}{t} (-2q_{d+1})^t O^{(d)}_{(d+1)(n-t)}.
\end{aligned}$$

□

3 The Infinite d-Oresme Polynomials Matrix

In this section, we introduce a new infinite matrix called the infinite d-Oresme polynomials matrix. Then, we provide two factorizations of infinite Pascal matrix.

Definition 3.1. The infinite d-Oresme polynomials matrix is defined by

$$\mathcal{N}_d = \begin{bmatrix} \frac{1}{x} & 0 & 0 & \cdots \\ \frac{q_1}{x} & \frac{1}{x} & 0 & \cdots \\ \frac{q_1^2+q_2}{x} & \frac{q_1}{x} & \ddots & \cdots \\ \frac{q_1^3+2q_1q_2+q_3}{x} & \frac{q_1^2+q_2}{x} & \ddots & \cdots \\ \vdots & \ddots & \ddots & \cdots \end{bmatrix},$$

where $(\mathcal{N}_d)_{i,1} = O_i^{(d)}$ for $i \in \mathbb{Z}^+$.

By Definition 3.1, we can express the infinite d-Oresme polynomials matrix as

$$\mathcal{N}_d = \begin{bmatrix} O_1^{(d)} & 0 & 0 & \cdots \\ O_2^{(d)} & O_1^{(d)} & 0 & \cdots \\ O_3^{(d)} & O_2^{(d)} & \ddots & \cdots \\ O_4^{(d)} & O_3^{(d)} & \ddots & \cdots \\ \vdots & \ddots & \ddots & \cdots \end{bmatrix}.$$

Then the matrix $\mathcal{N}_d$ is a Riordan matrix. Since the first column of $\mathcal{N}_d$ is

$$\left(\frac{1}{x}, \frac{q_1}{x}, \frac{q_1^2+q_2}{x}, \frac{q_1^3+2q_1q_2+q_3}{x}, \dots\right)^T$$

by Theorem 2.3, we obtain the following corollary.

Corollary 3.2. *The generating function for the first column of the matrix $\mathcal{N}_d$ is*

$$g_{\mathcal{N}_d}(w) = \frac{1}{x\left(1 - q_1 w - q_2 w^2 - \cdots - q_{d+1} w^{d+1}\right)}.$$

In the matrix $\mathcal{N}_d$, for $n \geq 1$ and $j \in \mathbb{Z}^+$, we get

$$(\mathcal{N}_d)_{n+1,j} = q_1 (\mathcal{N}_d)_{n,j} + q_2 (\mathcal{N}_d)_{n-1,j} + \cdots + q_{d+1} (\mathcal{N}_d)_{n-d,j}$$

by Definition (2.1). Hence if we take $f_{\mathcal{N}_d}(w) = w$, for the matrix $\mathcal{N}_d$ we obtain the next corollary.

Corollary 3.3. *The infinite d-Oresme polynomials matrix $\mathcal{N}_d$ is*

$$\begin{aligned}\mathcal{N}_d &= (g_{\mathcal{N}_d}(w), f_{\mathcal{N}_d}(w)) \\ &= \left(\frac{1}{x\left(1-q_1w-q_2w^2-\cdots-q_{d+1}w^{d+1}\right)}, w\right).\end{aligned}$$

For $i, j \in \mathbb{Z}^+$, we define the infinite matrix Λ_d as

$$(\Lambda_d)_{i,j} = x\sum_{t=0}^{d+1} -q_t\binom{i-t-1}{j-1},$$

where $q_0 = -1$. Thus we have

$$\Lambda_d = \begin{bmatrix} x & 0 & 0 & 0 & \cdots \\ x\left(1-q_1\right) & x & 0 & 0 & \cdots \\ x\left(1-q_1-q_2\right) & x\left(2-q_1\right) & x & \ddots & \cdots \\ x\left(1-q_1-q_2-q_3\right) & x\left(3-2q_1-q_2\right) & x\left(3-q_1\right) & \ddots & \cdots \\ \vdots & \ddots & & \ddots & \cdots \end{bmatrix}. \quad (3.1)$$

Now we give the first factorization of the infinite Pascal matrix in the following theorem.

Theorem 3.4. *Let $\mathcal{N}_d$ be the infinite d-Oresme polynomials matrix and Λ_d be the infinite matrix as in (3.1), then we get*

$$P = \mathcal{N}_d * \Lambda_d,$$

where P is the infinite Pascal matrix as in (1.2).

Proof. From the definition of the infinite Pascal matrix, we know that

$$P = \left(\frac{1}{1-w}, \frac{w}{1-w}\right). \quad (3.2)$$

The generating function for the first column of the matrix Λ_d is

$$\begin{aligned} g_{\Lambda_d}(w) &= x + x\left(1-q_1\right)w + x\left(1-q_1-q_2\right)w^2 \\ &\quad + x\left(1-q_1-q_2-q_3\right)w^3 + \cdots \\ &= x(1+w+w^2+w^3+\cdots) - xq_1\left(w+w^2+w^3+\cdots\right) \\ &\quad - xq_2\left(w^2+w^3+w^4+\cdots\right) \\ &\quad - \cdots - xq_{d+1}\left(w^{d+1}+w^{d+2}+w^{d+3}+\cdots\right) \\ &= x\left(\frac{1}{1-w} - \frac{q_1 w}{1-w} - \frac{q_2 w^2}{1-w} - \cdots - \frac{q_{d+1}w^{d+1}}{1-w}\right) \\ &= x\left(\frac{1-q_1 w - q_2 w^2 - \cdots - q_{d+1}w^{d+1}}{1-w}\right). \end{aligned} \tag{3.3}$$

On the other hand, the generating function for the second column of the matrix Λ_d is

$$\begin{aligned} & g_{\Lambda_d}(w) f_{\Lambda_d}(w) \\ = {} & xw + x\left(2-q_1\right)w^2 + x\left(3-2q_1-q_2\right)w^3 + \cdots \\ = {} & x(w+2w^2+3w^3+\cdots) - xq_1 w(w+2w^2+3w^3+\cdots) \\ & - xq_2 w^2(w+2w^2+3w^3+\cdots) \\ & - \cdots - xq_{d+1}w^{d+1}(w+2w^2+3w^3+\cdots) \\ = {} & x(1-q_1 w - q_2 w^2 - \cdots - q_{d+1}w^{d+1})(w+2w^2+3w^3+\cdots) \\ = {} & x\left(\frac{1-q_1 w - q_2 w^2 - \cdots - q_{d+1}w^{d+1}}{1-w}\right)\left(\frac{w}{1-w}\right), \end{aligned}$$

and so by the equation (3.3), we obtain

$$f_{\Lambda_d}(w) = \frac{w}{1-w}. \tag{3.4}$$

Then from the equations (3.3) and (3.4), we can find that

$$\begin{aligned} \Lambda_d &= \left(g_{\Lambda_d}(w), f_{\Lambda_d}(w)\right) \\ &= \left(x\left(\frac{1-q_1 w - q_2 w^2 - \cdots - q_{d+1}w^{d+1}}{1-w}\right), \frac{w}{1-w}\right). \end{aligned} \tag{3.5}$$

From Corollary 3.3, (3.2) and (3.5), we can derive $P = \mathcal{N}_d * \Lambda_d$.

□

For $i, j \in \mathbb{Z}^+$, we define the infinite matrix Ω_d as

$$(\Omega_d)_{i,j} = x \sum_{t=0}^{d+1} -q_t \binom{i-1}{j+t-1},$$

where $q_0 = -1$. Hence we get

$$\Omega_d = \begin{bmatrix} x & 0 & 0 & 0 & \cdots \\ x(1-q_1) & x & 0 & 0 & \cdots \\ x(1-2q_1-q_2) & x(2-q_1) & x & \ddots & \cdots \\ x(1-3q_1-3q_2-q_3) & x(3-3q_1-q_2) & x(3-q_1) & \ddots & \cdots \\ \vdots & \ddots & & \ddots & \cdots \end{bmatrix}. \quad (3.6)$$

Now we give the second factorization of the infinite Pascal matrix in the following corollary.

Corollary 3.5. *Let $\mathcal{N}_d$ be the infinite d-Oresme polynomials matrix and Ω_d be the infinite matrix as in (3.6), then we get*

$$P = \Omega_d * \mathcal{N}_d,$$

where P is the infinite Pascal matrix as in (1.2).

We can easily derive the inverse of $\mathcal{N}_d$ in Corollary 3.3 by the definition of the reverse element for the Riordan group in the next corollary.

Corollary 3.6. *The inverse for the infinite d-Oresme polynomials matrix $\mathcal{N}_d$ is*

$$\mathcal{N}_d^{-1} = \left(x \left(1 - q_1 w - q_2 w^2 - \cdots - q_{d+1} w^{d+1} \right), w \right).$$

4 Conclusions

In this work, we generalize the classical Oresme polynomials , and call these polynomials as d-Oresme polynomials $O_n^{(d)}$. We give the generating function, Binet-like formula, combinatorial identities and summation formulas for $O_n^{(d)}$. We define the new matrix O_d, whose powers generate $O_n^{(d)}$. Futhermore, we introduce the infinite d-Oresme polynomials matrix $\mathcal{N}_d$, which is a Riordan matrix. To factorize the infinite Pascal matrix P, we use the Riordan method, and therefore obtain two factorizations of P including the matrix $\mathcal{N}_d$. Then, we present the Riordan representation of the inverse of the matrix $\mathcal{N}_d$.

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