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(P,H)-Factorable Operators on $L^p(G)$ for Non-Abelian Groups

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Abstract. For a locally compact group G and a closed subgroup H of G, we define the (p, H)-bracket product, which serves as a type of semi-inner product for $L^p(G)$. We proceed to investigate some of its properties. Additionally, we delve into the study of (p, H)-factorable operators and indicate the Riesz representation type theorem for this product, among other things.

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1 Introduction

In the realm of shift invariant systems on frames, various authors, including de Boor et al. [1], Ron and Shen [8], and Cacazza and Lammers [1], have extensively utilized the bracket product defined as

$$[f,g](x) = \sum_{\alpha\in 2\pi\mathbb{Z}^n} f(x+\alpha)\overline{g(x+\alpha)},$$

on $L^2(\mathbb{R}^n)$. Interestingly, this emerges as a special instance of the inner product on a Hilbert C^*-module, a concept effectively employed by Rieffel [4] and others in advancing results in harmonic analysis on noncommutative groups. In our paper [9], we introduce the (ϕ,p)-bracket product for a locally compact Abelian group G with a lattice L, defined by

$$\Gamma_g : L^p(G) \to L^1(G/\phi(L)),$$

such that

$$f \mapsto \Gamma_g(f) = [f,g]_{\phi,p},$$

where

$$[f,g]_{\phi,p}(x) = \sum_{k\in L} fg^{p-1}(x\phi(k^{-1})).$$

Let us outline the structure of the paper. In Section 2, we revisit essential definitions and fundamentals concerning the quotient space G/H, where H denotes a closed subgroup of a locally compact group G. Section 3 introduces the definition of the (p,H)-bracket product for $L^p(G)$ and explores some of its fundamental properties. In Section 4, we delve into the study of (p,H)-factorable operators and establish a form of the Riesz Representation Theorem for the (p,H)-bracket product. While our focus has been on closed subgroups in this paper, it's worth noting that the validity of the (p,H)-bracket product can be verified for any desired subgroup.

2 Preliminaries and Notations

Let G be a locally compact group and H be a closed subgroup of G with the Haar measures dx and dh, respectively. Consider G/H as a homogeneous space in which G acts from the left, and let μ be a Radon measure on G. For x in G and a Borel subset E of G/H, the translation μ_x of μ is defined by $\mu_x(E) = \mu(xE)$. Then μ is said to be G-invariant if $\mu_x = \mu$, for all $x \in G$. Moreover, the measure μ is called strongly quasi invariant if there is a continuous function $\lambda : G \times G/H \to (0, \infty)$ such that $d\mu_x(\dot{y}) = \lambda_x(\dot{y})d\mu(\dot{y})$ for all $x \in G$ and $\dot{y} = yH \in G/H$, where λ_x is defined by $\lambda_x(\dot{y}) = \lambda(x, \dot{y})$.

A ρ-function for the pair (G, H) is a continuous function $\rho : G \to (0, \infty)$ such that

$$\rho(x\xi) = \frac{\Delta_H(g)}{\Delta_G(\xi)}\rho(x), \quad (x \in G, \xi \in H). \tag{1}$$

By [4, Proposition 2.54] for any locally compact group G and any closed subgroup H, the pair (G, H) admits a rho-function. Assume that $dx, d\dot{x}, dh, d\mu(\dot{x})$ are chosen such that

$$\int_G f(x)dx = \int_{G/H}\int_H f(xh)dhd\mu(\dot{x}), \quad (f \in L^1(G)). \tag{2}$$

This equality is known as Weil's type of formula (for details see [4]).

Suppose again that ρ is a continuous, strictly positive function on G satisfying (1). It is well known that

$$\lambda_x(j) = \frac{d\mu}{d\mu}(j) = \frac{\rho(xy)}{\rho(y)}, \quad (x, y \in G). \tag{3}$$

Also, for a relatively invariant measure on G/H which arises for a rho-function ρ, we have

$$\rho(xy) = \frac{\rho(x)\rho(y)}{\rho(e)}, \quad (x, y \in G). \tag{4}$$

The group G acts on G/H by the action $\Lambda : G \times G/H \to G/H$ defined by

$$\Lambda_y(\dot{x}) = y^{-1}x, \quad (y \in G), \tag{5}$$

which are homeomorphisms on G/H. The measure $d\mu(\dot{x})$ on G/H defined by (2) has the property

$$\int_{G/H} F(\dot{x})d\mu(\dot{x}) = \int_{G/H} F(\lambda_y(\dot{x}))d\mu(\dot{x}), \quad (x \in G, F \in L^1(G/H)),$$

where λ_y and Λ_y are given by (3) and (5), respectively.

3 (p, H)-Bracket Product and Its Basic Properties

For $1 < p < \infty$, $(L^p(G), \|\cdot\|_p)$ stands for the Banach space of equivalence classes of Haar-measurable complex-valued functions on G whose p^{th} powers are integrable.

Let q be the conjugate exponent to p. Let f, g be in $L^p(G)$, it is clear that $|g|^{p-1}$ in $L^q(G)$. So $f|g|^{p-1}$ in $L^1(G)$ and hence by Weil's formula, we get

$$\int_{G/H} \left| \int_H \frac{g|g|^{p-1}(xh)}{\rho(xh)} dh \right| d\mu(\dot{x}) = \int_G |f|\, |g|^{p-1}(x)dx \leq \|f\|_p \|g\|_p^{p-1}.$$

Thus for almost all x in G, the integral $\int_H \frac{f|g|^{p-1}(xh)}{\rho(xh)} dh$ is absolutely convergent.

Therefore, each function $g \in L^p(G)$ induces a bounded linear map

$$\Gamma_g : L^p(G) \to L^1(G/H),$$

Let

$$f \mapsto \Gamma_g(f) = [f, g]_{p,H}$$

with $\|\Gamma_g\| = \|g\|_{p'}^{-1}$, where

$$[f, g]_{p,H}(x) := \int_H \frac{g\ |g|^{p'-1}(xh)}{\rho(xh)} dh.$$

Note that $\Gamma_g(f) = [f, g]_{p,H}$ is a periodic function on H. Indeed, for $f, g \in L^p(G)$ we have

$$[f, g]_{p,H}(x\xi) = \int_H \frac{g\ |g|^{p'-1}(x\xi h)}{\rho(x\xi h)} dh$$

$$= \int_H \frac{g\ |g|^{p'-1}(xh)}{\rho(xh)} dh$$
$$= [f,g]_{p,H}(x),$$

for all $\xi \in H$. So one may consider the (p,H)-bracket product as a mapping $[\cdot,\cdot]_{p,H} : L^p(G) \times L^{p'}(G) \to L^1(G/H)$ that for $f,g \in L^p(G)$ is defined by

$$r_g(f)(\dot{x}) = \int_H \frac{g\ |g|^{p'-1}(xh)}{\rho(xh)} dh,$$

for all $\dot{x} \in G/H$. Consequently, one may define the (p,H)-norm as follows,

$$\|f\|_{p,H} : L^p(G) \to L^p(G/H),$$
$$f \mapsto \|f\|_{p,H} = (\Gamma_{|f|}(|f|))^{1/p},$$

which is an isometry, $\||\,|f|\,|\|_{p,H} = \|f\|_p$. Indeed, by Weil's Formula for $f \in L^p(G)$, $1 < p < \infty$ we have,

$$|||f|||_{p,H}^p = \int_{G/H} \|f\|_{p,H}^p(\dot{x}) d\dot{x}$$
$$= \int_{G/H} \Gamma_{|f|}(|f|)(\dot{x}) d\dot{x}$$
$$= \int_{G/H} \int_H \frac{|f|^{p-1}(xh)}{\rho(xh)} dh d\dot{x}$$
$$= \int_G \frac{|f|^p(xh)}{\rho(xh)} dh d\dot{x}$$
$$= \int_G |f|^p(x) dx$$
$$= \|f\|_p^p.$$

The basic properties of $[\cdot,\cdot]_{p,H}$, $\|\cdot\|_{p,H}$ are gathered in the next proposition and the proof is similar to [proposition 2.7, 9] the proof for which has been omitted.

Proposition 3.1. *Let H be a closed subgroup of a locally compact group G, let $1 < p < \infty$ and q the conjugate exponent to p. Then for every $f,g \in L^p(G)$, $c \in \mathbb{C}$:*

(i) $[f+h,g]_{p,H}(\dot{x}) = [f,g]_{p,H}(\dot{x}) + [h,g]_{p,H}(\dot{x})$.

(ii) $[cf,g]_{p,H}(\dot{x}) = c[f,g]_{p,H}(\dot{x}) = [f,c^{p'-1}g]_{p,H}(\dot{x})$.

(iii) $\|f\|_{p,H} = 0 \iff f = 0$ *a.e.*

(iv) $\|cf\|_{p,H} = |c|\|f\|_{p,H}$.

(v) $\|f\|_{p,H}^{p-1} = \||f|^{p-1}\|_{q,H}$.

(vi) $\|f\|_{p,H}\|g\|_{p',H} \geq |[f,g]_{p,H}(\dot{x})|$ *(Hölder's inequality).*

(vii) $\|f+g\|_{p,H}(\dot{x}) \leq \|f\|_{p,H}(\dot{x}) + \|g\|_{p,H}(\dot{x})$ *(triangle inequality).*

(viii) $\int_{G/H}[f,g]_{p,H}(\dot{x})d\dot{x} \leq \langle f, g^{p'-1}\rangle_{L^p,L^q}$, *where* $\langle\cdot,\cdot\rangle_{L^p,L^q}$ *stands for the duality of* L^p *and* L^q.

(ix) $[f,g]_{p,H}(\dot{x}) = [g^{p'-1}, f^{p-1}]_{q,H}(\dot{x})$.

Remark 3.2. *The* (p,H)*-bracket product is linear in the first component, but it is not linear in the second component.*

Remark 3.3. *Note that Proposition 3.1 shows that* $[\cdot,\cdot]_{p,H}$ *is a type of semi-inner product on* $L^p(G)$. *More precisely, for any coset* $\dot{x}$ *in* G/H, $[\cdot,\cdot]_{p,H}(\dot{x})$ *is a semi-inner product. For more details on semi-inner product see [3].*

Recall that the definition of left translation operator $L_y : L^p(G) \to L^p(G)$ is defined by $L_y(f)(x) = f(y^{-1}x)$. Further, we also define $L_y : L^1(G/H) \to L^1(G/H)$ by $L_y\Gamma_g(f) = \Gamma_g(f)(y^{-1}x)$, for any $\dot{x}$ in G/H.

Proposition 3.4. *Let* y *in* G *and* L_y *be the left translation operator. For* f,g *in* $L^p(G)$, *we have*

$$\int_{G/H} \Gamma_g L_y f(\dot{x}) d\mu(\dot{x}) = \int_{G/H} \Gamma_{L_y^{-1}g} f(\dot{x}) d\mu(\dot{x}).$$

where μ is the Radon measure on G/H satisfying the Weil's formula (2). Moreover, when μ is the relatively invariant measure which arises from a rho-homomorphism function ρ, we have:

(i) $L_y(\Gamma_g f) = \frac{\rho(y)}{\rho(e)}\Gamma_{L_y g}(L_y f)$,

(ii) $L_y[f, L_{y^{-1}}g]_{p,H} = \frac{\rho(y)}{\rho(e)}[L_y f, g]_{p,H}$,

(iii) $\|L_y f\|^p_{p,H} = \frac{\rho(e)}{\rho(y)}\|L_y f\|^p_{p,H}$.

Proof. For f, g in $L^p(G)$, we have,

$$\begin{aligned}
\int_{G/H} \Gamma_g L_y f(\dot{x}) d\mu(\dot{x}) &= \int_{G/H}\int_H \frac{L_y f|g|^{p-1}(xh)}{\rho(xh)} dh d\mu(\dot{x}) \\
&= \int_G L_y f|g|^{p-1}(x)dx \\
&= \int_G f(y^{-1}x)|g|^{p-1}(x)dx \\
&= \int_G f(x)|g|^{p-1}(yx)dx \\
&= \int_G f(x)L_{y^{-1}}g|g|^{p-1}(x)dx \\
&= \int_{G/H}\int_H \frac{fL_{y^{-1}}g^{p-1}(xh)}{\rho(xh)} dh d\mu(\dot{x}) \\
&= \int_{G/H} \Gamma_{L_{y^{-1}}g} f(\dot{x}) d\mu(\dot{x}).
\end{aligned}$$

Now using (2.4), we get,

$$\begin{aligned}
L_y(\Gamma_g f)(\dot{x}) &= \Gamma_g f(y^{-1}\dot{x}) \\
&= \int_H \frac{f|g|^{p-1}(y^{-1}xh)}{\rho(y^{-1}xh)} dh \\
&= \int_H \frac{f(y^{-1}xh)|g|^{p-1}(y^{-1}xh)}{\rho(y^{-1})\rho(xh)} dh \\
&= \frac{\rho(y)}{\rho(e)} \Gamma_{L_y g}(L_y f)(\dot{x}).
\end{aligned}$$

So the proof (i) is completed. By (i), the proof of (ii) is obvious. For (iii), we have,

$$\begin{aligned}\|L_y f\|_{p,H}(\dot{x}) &= [L_y f, L_y f]_{p,H}(\dot{x}) \\ &= \frac{\rho(e)}{\rho(y)}\|f\|_{p,H}(\dot{x}) \\ &= \frac{\rho(e)}{\rho(y)}\|L_y f\|_{p,H}(\dot{x}).\end{aligned}$$

□

Corollary 3.5. *With the assumption as in Proposition 3.2, if G/H possesses a G-invariant measure, including when G is abelian, we have:*

(i) $L_y \Gamma_g f = \Gamma_{L_y g}(L_y f)$,

(ii) $L_y[f, L_{y^{-1}} g]_{p,H} = [L_y f, g]_{p,H}$,

(iii) $\|L_y f\|_{p,H} = \|L_y f\|_{p,H}$.

Now we consider the set of all H-periodic functions in $L^\infty(G)$,

$$B_\infty(G) = \{k \in L^\infty(G); k(xh) = k(x), \text{ for all } h \in H\}.$$

It is easy to show that $B_\infty(G)$ is a subspace of $L^\infty(G)$. In the following proposition, we mention some more properties of $B_\infty(G)$.

Proposition 3.6. *Let $f, g \in L^p(G)$, $1 < p, q < \infty$ and q is the conjugate exponent of p. Then for all $k \in B_\infty(G)$ we have,*

(i) $\Gamma_g(fk) = k(\Gamma_g f)$,

(ii) $\Gamma_g f = k^{p-1} \Gamma_g f$.

In particular, if k satisfies $k(\dot{x}) \neq 0$ a.e., then $\Gamma_g f = 0$ if and only if $\Gamma_g(fk) = 0$.

Proof. By the definition of the (p, H)-bracket product, the proof is immediate. □

Definition 3.7. *Let $f \in L^p(G)$, $g \in L^q(G)$ where $1/p + 1/q = 1$*

and $1 < p, q < \infty$. *For* $E \subseteq L^p(G)$, *the* H*-orthogonal complement of* E *is*

$$\begin{aligned} E^{\perp,H} &= \{g \in L^q(G); \Gamma_g f = 0 \text{ a.e.}\mu \text{ for all } f \in E\} \\ &= \{g \in L^q(G); \langle f, g^{p-1}\rangle_{p,L^p,H} = 0 \text{ a.e.}\mu \text{ for all } f \in E\}. \end{aligned}$$

The following proposition declares the space $E^{\perp,H}$.

Proposition 3.8. *For* $E \subseteq L^p(G)$, *we have* $E^{\perp,H} = \cap_{k\in B_\infty(G)}(kE)^{\perp,H}$.

Proof. For $g \in E^{\perp,H}$, $k \in B_\infty(G)$ and $f \in E$, by Proposition 3.6, we have

$$\begin{aligned} \langle fk, g^{p-1}\rangle_{p,L^p,H} &= \int_G (fk)(g)(x)dx \\ &= \int_{G/H}\int_H \frac{fkg(xh)}{\rho(xh)} dh d\mu(\dot{x}) \\ &= \int_{G/H} \Gamma_{g^{p-1}}(fk)(\dot{x})d\mu(\dot{x}) \\ &= \int_{G/H} k(\dot{x})(\Gamma_{g^{p-1}}f)(\dot{x})d\mu(\dot{x}) \\ &= 0. \end{aligned}$$

Hence $g \in \cap_{k\in B_\infty(G)}(kE)^{\perp,H}$. Now let $g \in \cap_{k\in B_\infty(G)}(kE)^{\perp,H}$ and $f \in E$. For $n \in \mathbb{N}$, define $k_n(\dot{x}) = (\Gamma_{g^{p-1}}f)(\dot{x})$, when $|(\Gamma_{g^{p-1}}f)(\dot{x})| \leq n$, and $k_n(\dot{x}) = 0$ otherwise. Then $k_n \in B_\infty(G)$. So we have

$$\begin{aligned} 0 &= \int_{G/H} k_n|g^{p-1}f|(\dot{x})d\mu(\dot{x}) \\ &= \int_{G/H} |k_n|^{p-1}(\dot{x})(\Gamma_{g^{p-1}}f)(\dot{x})d\mu(\dot{x}) \\ &= \int_{G/H} |k_n|^p(\dot{x})d\mu(\dot{x}). \end{aligned}$$

Therefore $|k_n(\dot{x})| = 0$, for almost all $\dot{x}$. Hence $\Gamma_{g^{p-1}}f(\dot{x}) = 0$ a.e., that is $g \in E^{\perp,H}$. □

4 (p,H)-Factorable Operator on $L^p(G)$

Let G be a locally compact abelian (LCA) group and H be a closed subgroup of G. In this section, (p,H)-factorable operators are defined. Moreover, the relation between (p,H)-factorable operators and (p,H)-bracket product is indicated. Finally, a type of Riesz Representation theorem for $L^p(G)$ with the (p,H)-bracket product is given.

Let G be a (LCA) group, then G/H admits a G-invariant measure which we denote by dx. We shall denote the dual group of G by $\hat{G}$. Let the Fourier transform

$$\hat{}: L^1(G) \to C_0(\hat{G}), \quad f \mapsto \hat{f},$$

be defined by

$$\hat{f}(\xi) = \int_G f(x)\overline{\xi(x)}dx \quad \text{for } \xi \in \hat{G}.$$

It is It is well known that if $f \in L^p(G)$ $(1 \le p \le 2)$, then $\hat{f}$ in $L^q(\hat{G})$ satisfies $\|\hat{f}\|_q \le \|f\|_p$, where q and p are conjugate exponents (see [4, Theorem 4.27]).

Definition 4.1. *Let G be a LCA group and H be a closed subgroup of G. An operator $U : L^p(G) \to L^p(E)$ that $1 < r, p < \infty$ is called (p,H)-factorable if $U(kf) = kU(f)$, for all $f \in L^p(G)$ and all H-periodic $k \in L^\infty(G)$, where $E = G/H$.*

Note that for $g \in L^p(G)$, $1 < p < \infty$, Proposition 3.6(i) shows that $[\cdot, g]_{p,H}$ is (p,H)-factorable.

In the following, some properties of the (p,H)-factorable operators are investigated, whose proofs are almost the same as the ones when H is a lattice in G, (see [Lemma 3.2, 3.3, 9]), so we omit the proofs.

Lemma 4.2. *Let $U_1, U_2 : L^p(G) \to L^1(G/H)$ be two (p,H)-factorable operators. Then $U_1 = U_2$ if and only if*

$$\int_{G/H} U_1(f)(\dot{x})d\dot{x} = \int_{G/H} U_2(f)(\dot{x})d\dot{x},$$

for every $f \in L^p(G)$.

To demonstrate the lemma, it's worth noting that if $k \in L^\infty(G)$ and $f \in L^p(G)$, then $kf \in L^1(G)$. Thus, we can utilize Weil's formula.

Lemma 4.3. *Let $k \in B_\infty(G)$ and $f \in L^p(G)$ where $1 < p < \infty$. Then*
$$\int_G |k^p f(x)| dx = \int_{G/H} |k(\dot{x})|^p \|f\|_{p,H}^p(\dot{x}) d\dot{x},$$
for $\dot{x} \in G/H$.

Proposition 4.4. *Let U be a (p,H)-factorable linear operator from $L^p(G)$ to $L^p(G/H)$, $1 < p < \infty$. Then U is bounded if and only if there is a constant $B > 0$ $(B = \|U\|)$ so that for every $f \in L^p(G)$ we have,*
$$|U(f)(\dot{x})| \leq B\|f\|_{p,H}(\dot{x}), \quad \text{for} \quad \text{a.e.}\, \dot{x} \in G/H.$$

Proof. Let $k \in B_\infty(G)$ and $f \in L^p(G)$. By Lemma 4.3,
$$\begin{aligned} \int_{G/H} |k(\dot{x})|^p |U(f)(\dot{x})|^p d\dot{x} &= \int_G |U(kf)(x)|^p dx \\ &\leq \|U(kf)\|_{L^p(G)}^p \\ &\leq \|U\|^p \|kf\|_{L^p(G)}^p \\ &= \|U\|^p \int_{G/H} |k(\dot{x})|^p \|f\|_{p,H}^p(\dot{x}) d\dot{x}. \end{aligned}$$
Therefore,
$$|U(f)(\dot{x})| \leq B\|f\|_{p,H}(\dot{x}).$$
It follows immediately that $|U(f)(\dot{x})|^p \leq \|U\|^p \|f\|_{p,H}^p(\dot{x})$, a.e. for $\dot{x} \in G/H$.

Conversely, let $f \in L^p(G)$, we have,
$$\begin{aligned} \|U(f)\|_p^p &= \int_{G/H} |U(f)(\dot{x})|^p d\dot{x} \\ &\leq \int_{G/H} B^p \|f\|_{p,H}^p(\dot{x}) d\dot{x} \\ &= B^p \int_{G/H} \|f\|_{p,H}^p(\dot{x}) d\dot{x} \\ &= B^p \|f\|_p^p. \end{aligned}$$

So, the proof is completed. □

Corollary 4.5. *If $U : L^p(G) \to L^p(G/H)$ $(1 < p < \infty)$ is a (p, H)-factorable linear operator, then U is bounded if and only if there is a constant $B > 0$ $(B = \|U\|)$ so that for every $f \in L^p(G)$,*

$$\|U(f)\|_{p,H}(\dot{x}) \leq B\|f\|_{p,H}(\dot{x}).$$

Theorems **4.6** and **4.7** serve as the main theorems in this section, representing certain types of Riesz representation theorem for the (p, H)-bracket product in $L^p(G)$.

Theorem 4.6. *The operator $U : L^p(G) \to L^1(G/H)$ is a bounded (p, H)-factorable if and only if there exists $g \in L^q(G)$ such that $U(f) = [f, g]_{p,H}$ a.e. for all $f \in L^p(G)$ in which $1 < p < \infty$, $1/p + 1/q = 1$. Moreover, $\|U\| = \|g\|_q$.*

Proof. Let $U : L^p(G) \to L^1(G/H)$ for $1 < p < \infty$ be a bounded (p, H)-factorable operator. Define the linear functional $\Psi : L^p(G) \to \mathbb{C}$ by $\Psi(f) = \int_{G/H} U(f)(\dot{x})d\dot{x}$.

The isometrically isomorphic of $(L^p(G))^* \cong L^q(G)$ implies that there exists $g \in L^q(G)$ such that $\Psi(f) = \int_G fg(x)dx$ for all $f \in L^p(G)$. Thus

$$\int_{G/H} U(f)(\dot{x})d\dot{x} = \Psi(f) = \int_G fg(x)dx = \int_{G/H} (\Gamma_{g^{p-1}} f)(\dot{x})d\dot{x}.$$

By Proposition 4.4, $U(f) = \Gamma_{g^{p-1}} f$ a.e. for all $f \in L^p(G)$. Moreover, for any $f \in L^p(G)$,

$$\begin{aligned} \|U(f)\|_{L^1(G/H)} &= \|\Gamma_{g^{p-1}} f\|_{L^1(G/H)} \\ &= \|fg\|_1 \\ &\leq \|f\|_p \|g\|_q. \end{aligned}$$

So $\|U\| \leq \|g\|_q$. Now letting $f = |g^{p-1}|$, hence

$$\begin{aligned}
\|U(|g^{p-1}|)\|_{L^1(G/H)} &= \int_{G/H} |U(|g^{p-1}|)(\dot{x})|d\dot{x} \\
&= \int_{G/H} |\Gamma_{g^{p-1}}|g^{p-1}|(\dot{x})|d\dot{x} \\
&= \int_{G/H} |\langle |g^{p-1}|, |g^{p-1}|\rangle_{p,H}(\dot{x})d\dot{x} \\
&= \int_{G/H} |g|^q_{p,H}(\dot{x})d\dot{x} \\
&= \|g\|^q_q.
\end{aligned}$$

Thus

$$\|g\|^q_q = \|U(|g^{p-1}|)\|_{L^1} \leq \|U\| \|g\|^{q-1}_q,$$

i.e., $\|g\|_q \leq \|U\|$. For the converse, according to boundedness of g, U is bounded.

Moreover, for every H-periodic $k \in L^\infty(G)$ and $f \in L^p(G)$,

$$U(kf)(\dot{x}) = \Gamma_{g^{p-1}}(kf)(\dot{x}) = k(\Gamma_{g^{p-1}}f)(\dot{x}) = kU(f)(\dot{x}),$$

where $\dot{x} \in G/H$. Therefore the proof is complete. □

Note that for $p = 2$, Theorem 4.6 is the Theorem 5.25 in [5]. We say $f \in L^p(G)$ is (p, H)-bounded if there exists $M > 0$ such that $\|f\|_{p,H} \leq M$ a.e. $\dot{x} \in G/H$.

In the next Theorem we assume that H is also a co-compact subgroup of G.

Theorem 4.7. *A linear operator $U : L^p(G) \to L^p(G/H)$ $(1 < p < \infty)$ is a bounded (p, H)-factorable if and only if there exists (p, H)-bounded $g \in L^q(G)$ such that $U(f) = \Gamma_{g^{p-1}}f$ a.e. $(\dot{x} \in G/H)$ for all $f \in L^p(G)$. Moreover,*

$$\|U\| = \operatorname{ess\,sup}_{\dot{x}\in G/H} \|g\|_{q,H}(\dot{x}),$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Let U be a bounded (p,H)-factorable operator from $L^p(G) \to L^p(G/H)$. Since G/H is compact, $L^p(G/H) \subseteq L^1(G/H)$ and so by Theorem 4.6, there exists $g \in L^q(G)$ such that $U(f) = \Gamma_{g^{p-1},H} f$, a.e. $(\dot{x} \in G/H)$, for all $f \in L^p(G)$.

Letting $f = g^{q-1}$, by Proposition 4.4 we get

$$|\Gamma_{g^{p-1}}|g|^{q-1}(\dot{x})| = |U(|g^{q-1}|)(\dot{x})|$$

$$\leq \|U\||g^{q-1}|_{p,H}(\dot{x}),$$

for $\dot{x} \in G/H$. Hence $|g^{q-1}|_{p,H} \leq \|U\|$ a.e. Thus $\|g\|_{q,H} \leq \|U\|$ a.e.

For the converse, let g be a (p,H)-bounded function and $U(f) = \Gamma_{g^{p-1}} f$ a.e. $\dot{x} \in G/H$ for some $g \in L^q(G)$. Then U is (p,H)-factorable. Now by the assumption that g is (p,H)-bounded and by Theorem 4.6, we have

$$\begin{aligned}
\|U(f)\|_p^p &= \int_{G/H} |\Gamma_{g^{p-1}} f|^p(\dot{x}) d\dot{x} \\
&\leq \int_{G/H} \|f\|_{p,H}^p(\dot{x}) \|g\|_{q,H}^p(\dot{x}) d\dot{x} \\
&\leq esssup_{\dot{x} \in G/H} \|g\|_{q,H}^p(\dot{x}) \int_{G/H} \|f\|_{p,H}^p(\dot{x}) d\dot{x} \\
&= esssup_{\dot{x} \in G/H} \|g\|_{q,H}^p(\dot{x}) \|f\|_p^p.
\end{aligned}$$

Thus, U is bounded. Now by replacing $f = g^{q-1}$ in the above, we get

$$\|U\| = esssup_{\dot{x} \in G/H} \|g\|_{q,H}(\dot{x}).$$

This result makes the proof complete. □

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