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Regularity Properties for Locally Lipschitz $C(T)$-Valued Functions on Hilbert Spaces

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Abstract. In this paper, we introduce several regularity properties for the locally Lipschitz $C(T)$-valued functions which are defined on a Hilbert space. The relationships with various regularity properties are investigated. All results are given in terms of the Clarke subdifferential. Non-trivial numerical examples are incorporated to demonstrate the validity of results established in this paper.

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1 Introduction

In this paper, we study the regularity property for locally Lipschitz $C(T)$-valued functions $f : \mathcal{H} \to C(T)$ where $\mathcal{H}$ is a Hilbert space and $C(T)$ denotes the set of real-valued continuous functions on a (not necessarily compact) metric space T. Note that the $C(T)$-valued function $\varphi : \mathcal{H} \to C(T)$ is said to be locally Lipschitz if for all $t \in T$ the function $\varphi(\cdot)(t) : \mathcal{H} \to \mathbb{R}$ is locally Lipschitz, i.e., for each $x_0 \in \mathcal{H}$, we can find a

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neighborhood U^{x_0} of x_0 and a positive constant number $L_{U^{x_0}} > 0$ such that

$$|\varphi(x)(t) - \varphi(y)(t)| \leq L_{U^{x_0}} \|x - y\|_{\mathcal{H}}, \quad \forall x, y \in U^{x_0},\ \forall t \in T,$$

where $\|\cdot\|_{\mathcal{H}}$ denotes the norm of $\mathcal{H}$. Note that if T is a compact space, $C(T)$ becomes a Banach space, and the above inequality is equivalent to

$$\|\varphi(x) - \varphi(y)\|_{C(T)} \leq L_{U^{x_0}} \|x - y\|_{\mathcal{H}}, \quad \forall x, y \in U^{x_0},$$

in which $\|\cdot\|_{C(T)}$ denotes the norm of $C(T)$, defined as

$$\|\vartheta\|_{C(T)} := \max\big\{\vartheta(t) \mid t \in T\big\}, \qquad \forall \vartheta \in C(T).$$

Given a locally Lipschitz $C(T)$-valued functions $\varphi : \mathcal{H} \to C(T)$, we consider the following subset of $\mathcal{H}$,

$$S := \big\{x \in \mathcal{H} \mid \varphi(x)(t) \leq 0\big\}.$$

Note that, if T is a compact space, S can be written as

$$S = \big\{x \in \mathcal{H} \mid \|\varphi(x)\| \leq 0_{C(T)}\big\},$$

where $0_{C(T)}$ denotes the zero vector of $C(T)$. According to

$$S = \bigcap_{t \in T} \{x \in \mathcal{H} \mid \varphi(x)(t) \leq 0\} = \bigcap_{t \in T} \varphi^{-1}\big((-\infty, 0]\big)(t),$$

and regarding the closedness of level sets of real-valued continuous functions, we conclude that S is an intersection of closed sets, and consequently is itself closed (Here, $\varphi^{-1}(\cdot)(t)$ denotes the inverse relation of $\varphi(\cdot)(t)$). In what follows we shall assume that $S \neq \emptyset$.

If a locally Lipschitz $C(T)$-valued function $\varphi : \mathcal{H} \to C(T)$ and a vector $\hat{x} \in S$ are given, the following condition, which is named `Basic constraint qualification`, is very important in many theoretical and applied problems (see, e.g., [1,12,14]):

$$N_{\mathcal{F}}(S, \hat{x}) \subseteq cone\Bigg(\bigcup_{t \in T(\hat{x})} \partial_c \varphi(\hat{x})(t)\Bigg),$$

where the Fréchet normal cone of S at $\hat{x}$ is denoted by $N_{\mathcal{F}}(S, \hat{x})$, the Clarke subdifferential of function $\varphi(\cdot)(t)$ at $\hat{x}$ is denoted by $\partial_c \varphi(\hat{x})(t)$, and $T(\hat{x})$ is defined as $T(\hat{x}) := \{t \in T \mid \varphi(\hat{x})(t) = 0\}$.

As a result, finding conditions from which Basic constraint qualification can be concluded are of great importance. Any property that is a sufficient condition for the Basic constraint qualification is called a `regularity property`.

It should be noted that if T is a finite set, $C(T)$ can be considered the same as $\mathbb{R}^{|T|}$, and then the $C(T)$-valued functions $f : \mathcal{H} \to C(T)$ are reduced to the vector-valued functions $f : \mathcal{H} \to \mathbb{R}^{|T|}$. The regularity properties of this special type of $C(T)$-valued functions can be seen in [2,5,11]. Also, if $\mathcal{H} = \mathbb{R}^n$ and T is a compact space, then the the $C(T)$-valued function $f : \mathbb{R}^n \to C(T)$ is said to be semi-infinite function. The regularity properties of semi-infinite functions are studied in [7] for linear case, in [4] for differentiable case, in [3,10] for convex case, and in [6,8,9] for locally Lipschitz case. Recently, it have been studied in [12,14] the cases where $\mathcal{H}$ is a Banach space and T is a compact space.

Since in this article we do not consider any of the limitations of the above papers (even compactness of T), then the results of this article can be considered as a generalization of all the above papers. To the best of our knowledge, this paper is the first to investigate the regularity properties for the locally Lipschitz $C(T)$-valued functions.

The rest of the paper unfolds as follows: Section 2 is devoted to preliminaries, and Section focuses on the defining of some regularity properties of the locally Lipschitz $C(T)$-valued functions and their relationships.

2 Preliminaries

In this section, we overview some notations and preliminary results from [1,2,13] that will be used throughout this paper.

For a non-empty subset B of the Hilbert space $\mathcal{H}$, its polar and its strictly polar sets are respectively defined as

$$B^{\odot} := \{x \in \mathcal{H} \mid \langle x, y \rangle \leq 0, \quad \forall y \in B\},$$

$$B^{\ominus} := \{x \in \mathcal{H} \mid \langle x, y\rangle < 0, \quad \forall y \in B\},$$

where, $\langle \cdot, \cdot \rangle$ denotes the inner-product of $\mathcal{H}$. With convention $\emptyset^{\odot} = \emptyset^{\ominus} = \mathcal{H}$, it is easy to see that $M^{\odot}$ is a weakly closed convex cone for each $M \subseteq \mathcal{H}$, and if M_1 and M_2 are two subsets of $\mathcal{H}$ and $M_1 \subseteq M_2$, then $M_2^{\odot} \subseteq M_1^{\odot}$ and $M_2^{\ominus} \subseteq M_1^{\ominus}$. Also, if Λ is an arbitrary index set and M_α is a subset of $\mathcal{H}$ for each $\alpha \in \Lambda$, then

$$\Big(\bigcup_{\alpha\in\Lambda} M_\alpha\Big)^{\odot} = \bigcap_{\alpha\in\Lambda} M_\alpha^{\odot} \qquad \text{and} \qquad \Big(\bigcup_{\alpha\in\Lambda} M_\alpha\Big)^{\ominus} = \bigcap_{\alpha\in\Lambda} M_\alpha^{\ominus}.$$

For a given $M \subseteq \mathcal{H}$, the weakly closure of M, the convex hull of M, and the convex cone of M are denoted by $\overline{M}^w$, $conv(M)$, and $cone(M)$, respectively. The weakly closure of the convex hull of M (resp. the convex cone of M) is denoted by $\overline{conv}^w(M)$ (resp. $\overline{cone}^w(M)$), i.e.,

$$\overline{conv}^w(M) := \overline{conv(M)}^w \qquad \text{and} \qquad \overline{cone}^w(M) := \overline{cone(M)}^w.$$

It is easy to see ([2]) that $M^{\odot}$ is always a weakly closed convex cone in $\mathcal{H}$, and

$$\begin{aligned} M^{\odot} &= \big(\overline{M}^w\big)^{\odot} = \big(conv(M)\big)^{\odot} = \big(cone(M)\big)^{\odot} \\ &= \big(\overline{conv}^w(M)\big)^{\odot} = \big(\overline{cone}^w(M)\big)^{\odot}. \end{aligned}$$

Also, we can see [13] that if $M^{\ominus} \neq \emptyset$, then $M^{\odot} = \overline{M^{\ominus}}^w$.

Theorem 2.1. *(Bipolar)[2,13] Let $M \subseteq \mathcal{H}$ be given. Then,*

$$M^{\odot\odot} := \big(M^{\odot}\big)^{\odot} = \overline{cone}^w(M).$$

Theorem 2.2. [13] *Let $M \subset \mathcal{H}$ be a weakly compact set. Then, $cone(M)$ is a weakly closed cone if $0_{\mathcal{H}} \notin conv(M)$, where $0_{\mathcal{H}}$ denotes the null vector of $\mathcal{H}$.*

Suppose that I is an arbitrary index set and $C_i \subseteq \mathcal{H}$ is a nonempty convex set for each $i \in I$. If $C := \bigcup_{i\in I} C_i$, then ([13])

$$conv(C) = \Big\{\sum_{i\in I_*} \beta_i c_i \mid \beta_i \geq 0, \sum_{i\in I_*} \beta_i = 1, c_i \in C_i, I_* \subseteq I, |I_*| < \infty\Big\}. \tag{1}$$

Assume that $M \subseteq \mathcal{H}$ and $x_0 \in \overline{M}^w$ are given. The feasible directions cone, the Bouligand tangent cone, and the attainable directions cone of M at x_0 are respectively defined by

$$\Gamma_F(M, x_0) := \big\{v \in \mathcal{H} \mid \exists \delta > 0 \ : \ x_0 + \varepsilon v \in M, \quad \forall \varepsilon \in (0, \delta)\big\},$$

$$\Gamma_B(M, x_0) := \big\{v \in \mathcal{H} \mid \exists r_n \downarrow 0, \ \exists v_n \to v \ : \ x_0 + r_n v_n \in M, \quad \forall n \in \mathbb{N}\big\},$$

$$\Gamma_A(M, x_0) := \big\{v \in \mathcal{H} \mid \forall r_n \downarrow 0, \ \exists v_n \to v \ : \ x_0 + r_n v_n \in M, \quad \forall n \in \mathbb{N}\big\}.$$

Also, the Fréchet normal cone of M at x_0 is defined as polar cone of $\Gamma_B(M, x_0)$, i.e.,

$$N_{\mathcal{F}}(M, x_0) := \big(\Gamma_B(M, x_0)\big)^{\odot}.$$

We can see ([13]) that $\Gamma_B(M, x_0)$ and $\Gamma_A(M, x_0)$ are always weakly closed (not necessarily convex) cones in $\mathcal{H}$, and

$$\overline{\Gamma_F(M, x_0)}^w \subseteq \Gamma_A(M, x_0) \subseteq \Gamma_B(M, x_0) \subseteq \overline{conv}^w\big(\Gamma_B(M, x_0)\big). \tag{2}$$

It is worth mentioning that if M is a convex subset of $\mathcal{H}$, the above inclusions increase to equality and $\big(N_{\mathcal{F}}(M, x_0)\big)^{\odot} = \Gamma_B(M, x_0)$.

Let $f : \mathcal{H} \to \mathbb{R}$ be a locally Lipschitz function, and $x_0 \in \mathcal{H}$. The Clarke subdifferential of f at x_0 is defined as

$$\partial_c f(x_0) := \{\xi \in \mathcal{H} \mid f^c(x_0; v) \geq \langle \xi, v\rangle, \quad \forall v \in \mathcal{H}\},$$

where $f^c(x_0; v)$ denotes the Clarke generalized directional derivative of f at x_0 in the direction $v \in \mathcal{H}$,

$$f^c(x_0; v) = \limsup_{y \to x_0 \ r \downarrow 0} \frac{f(y + rv) - f(y)}{r}.$$

It should be note from [1] that if $g : \mathcal{H} \to \mathbb{R}$ is continuously differentiable at x_0, then $\partial_c g(x_0) = \{\nabla g(x_0)\}$, where $\nabla g(x_0)$ denotes the standard gradient of g at x_0. Moreover, it can be shown ([1]) that if $h : \mathcal{H} \to \mathbb{R}$ is a convex function, then $\partial_c h(x_0) = \partial h(x_0)$ and $h^c(x_0; v) = h'(x_0; v)$, where $\partial h(x_0)$ and $h'(x_0; v)$ denote the convex subdifferential of h at x_0 and the standard directional derivative of h at x_0 in the direction v, are respectively defined as ([13])

$$\partial h(x_0) := \{\xi \in \mathcal{H} \mid h(x) - h(x_0) \geq \langle \xi, x - x_0\rangle, \quad \forall x \in \mathcal{H}\},$$

$$h'(x_0; v) = \lim_{r \downarrow 0} \frac{h(x_0 + rv) - h(x_0)}{r}.$$

Consequently, the sentences expressed in terms of ∂_c are generalizations of sentence that are expressed with gradient for C^1 functions and with convex subdufferential of convex functions.

The locally Lipschitz function $f : \mathcal{H} \to \mathbb{R}$ is said to be regular at $x_0 \in \mathcal{H}$ when $f'(x_0; v)$ exists for all $v \in \mathcal{H}$, and $f^c(x_0; v) = f'(x_0; v)$. The function f is said to be regular if it is regular at each $x_0 \in \mathcal{H}$. The continuously differentiable functions and the convex functions are examples for regular functions; see, e.g., [1].

Moreover, it can be shown that if φ is a locally Lipschitz function, $\partial_c \varphi(x_0)$ is a nonempty, convex and compact set. Also, $\varphi^\circ(x; v)$ is a convex function with respect to v.

As the final point of this section, in the following theorem we summarize some important properties of the Clarke directional derivative and the Clarke subdifferential from [1] which are widely used in what follows.

Theorem 2.3. *If f_1 and f_2 are locally Lipschitz functions from $\mathcal{H}$ to $\mathbb{R}$ and x_0 is a point in $\mathcal{H}$, then*

- $\partial_c f_1(x_0)$ *is a nonempty convex weakly compact subset of* $\mathcal{H}$.
- *one has* $\partial_c(f_1 + f_2)(x_0) \subseteq \partial_c f_1(x_0) + \partial_c f_2(x_0)$. *Moreover, the above inclusion increses to equality if f_1 and f_2 are regular at x_0.*
- *we have* $\partial_c(\lambda f_1)(x_0) = \lambda \partial_c f_1(x_0), \quad \forall \lambda \in \mathbb{R}$. *Moreover, if f_1 is regular at x_0 and $\lambda \geq 0$, then λf_1 is regular at x_0.*
- *one has* $f^c(x_0; v) = \max\{\langle \xi, v \rangle \mid \xi \in \partial_c f(x_0)\}$.
- *the function* $v \mapsto f^c(x_0; v)$ *is convex on* $\mathcal{H}$, *and*
$$\alpha\big(f^c(x_0; \cdot)\big)(0_{\mathcal{H}}) = \partial_c f(x_0).$$

3 Main Results

As the starting point of this section, we put

$$Ł(\hat{x}) := \bigcup_{t \in T(\hat{x})} \partial_c \varphi(\hat{x})(t),$$

where $\partial_c\varphi(\hat{x})(t)$ denotes the Clarke subdifferential of the function $\varphi(\cdot)(t)$ at $\hat{x} \in \mathcal{H}$, i.e.,

$$\partial_c\varphi(\hat{x})(t) := \partial_c\Big(\varphi(\cdot)(t)\Big)(\hat{x}), \qquad \forall t \in T.$$

Also, let the Clarke directional derivative of the function $\varphi(\cdot)(t)$ at $\hat{x}$ in the direction $d \in \mathcal{H}$ be denoted by $\varphi^c(\hat{x}; d)(t)$, i.e.,

$$\varphi^c(\hat{x}; d)(t) := \Big(\varphi(\cdot)(t)\Big)^c(\hat{x}; d), \qquad \forall t \in T.$$

Set

$$\varphi(x) := \sup_{t\in T} \varphi(x)(t), \qquad \forall x \in S.$$

Note that if T is finite, then φ is locally Lipschitz and by [1, Proposition 2.3.12] we have

$$\partial_c\varphi(\hat{x}) \subseteq conv\big(Ł(\hat{x})\big), \qquad \forall \hat{x} \in S. \tag{3}$$

In general, (3) does not hold if T is infinite (see [1,Theorem 2.8.2]), and we are thus led to the following definition.

Definition 3.1. We say that the Pshenichnyi-Levin-Valadire property (PLVP, briefly) is satisfied for $\varphi : \mathcal{H} \to C(T)$ at $\hat{x} \in S$, if φ is Lipschitz around $\hat{x}$ and (3) holds.

Note that the above definition is in agreement with [3] for convex $C(T)$-valued functions.

Remark 3.2. An interesting sufficient condition ensuring the Lipschitzian property of φ around $\hat{x}$ can be found in [1].

The following theorem characterizes the Basic constraint qualification for the locally Lipschitz $C(T)$-valued functions.

Theorem 3.3. *Suppose that the locally Lipschitz $C(T)$-valued function $\varphi : \mathcal{H} \to C(T)$ and $\hat{x} \in S$ are given.*

(i): *If the Basic constraint qualification holds at $\hat{x}$, then $Ł^{\odot}(\hat{x}) \subseteq \overline{cone}^w\big(\Gamma_B(S, \hat{x})\big)$, where $Ł^{\odot}(\hat{x}) := \big(Ł(\hat{x})\big)^{\odot}$.*

(ii): *If* $Ł^{\odot}(\hat{x}) \subseteq \overline{cone}^{w}\big(\Gamma_B(S,\hat{x})\big)$ *and* $cone\big(Ł(\hat{x})\big)$ *is a weakly closed set in* $\mathcal{H}$*, then the Basic constraint qualification holds at* $\hat{x}$*.*

Proof.
(i) If the Basic constraint qualification holds at $\hat{x}$, i.e., $N_{\mathcal{F}}(S,\hat{x}) \subseteq cone\big(Ł(\hat{x})\big)$, then

$$\big(Ł(\hat{x})\big)^{\odot} = \Big(cone\big(Ł(\hat{x})\big)\Big)^{\odot} \subseteq \big(N_{\mathcal{F}}(S,\hat{x})\big)^{\odot} = \big(\Gamma_B(S,\hat{x})\big)^{\odot\odot}.$$

This inclusion and Theorem 2.1 imply that

$$\big(Ł(\hat{x})\big)^{\odot} \subseteq \overline{cone}^{w}\big(\Gamma_B(S,\hat{x})\big)$$

, as required.
(ii) Assume that $Ł^{\odot}(\hat{x}) \subseteq \overline{cone}^{w}\big(\Gamma_B(S,\hat{x})\big)$ and $cone\big(Ł(\hat{x})\big)$ is a weakly closed set in $\mathcal{H}$. The definition of Fréchet normal cone and Theorem 2.1 conclude that

$$\begin{aligned} N_{\mathcal{F}}(S,\hat{x}) = \big(\Gamma_B(S,\hat{x})\big)^{\odot} = \Big(\overline{cone}^{w}\big(\Gamma_B(S,\hat{x})\big)\Big)^{\odot} \subseteq \big(Ł(\hat{x})\big)^{\odot\odot} \\ = \overline{cone}^{w}\big(Ł(\hat{x})\big) = cone\big(Ł(\hat{x})\big). \qquad \square \end{aligned}$$

The above theorem leads us to the some parts of the following definition, which introduces some regularity properties for locally Lipschitz $C(T)$-valued functions.

Definition 3.4. Let $\varphi : \mathcal{H} \to C(T)$ be a given locally Lipschitz $C(T)$-valued function and $\hat{x} \in S$. We say that φ satisfies

(a): the Abadie regularity property (ARP briefly) at $\hat{x}$ if

$$Ł^{\odot}(\hat{x}) \subseteq \Gamma_B(S,\hat{x}).$$

(b): the Guignard regularity property (GRP briefly) at $\hat{x}$ if

$$Ł^{\odot}(\hat{x}) \subseteq \overline{conv}^{w}\big(\Gamma_B(S,\hat{x})\big).$$

(c): the Zangwill regularity property (ZRP briefly) at $\hat{x}$ if

$$Ł^{\odot}(\hat{x}) \subseteq \overline{\Gamma_F(S,\hat{x})}^{w}.$$

(d): the Tucker regularity property (TRP briefly) at $\hat{x}$ if

$$Ł^{\odot}(\hat{x}) \subseteq \Gamma_A(P, \bar{x}).$$

(e): the Kuhn-Tucker regularity property (KTRP briefly) at $\hat{x}$ if

$$\{d \in \mathcal{H} \mid \varphi^c(\hat{x}; d) \leq 0\} \subseteq \Gamma_A(S, \hat{x}).$$

(f): the Cottle regularity property (CRP briefly) at $\hat{x}$ if $Ł^{\ominus}(\hat{x}) \neq \emptyset$, where $Ł^{\ominus}(\hat{x}) := \big(Ł(\hat{x})\big)^{\ominus}$.

It should be noted that the ARP, GRP, ZRP, TRP, and KTRP, which are based on the concepts of Bouligand, feasible directions, and attainable directions tangent cones, are named geometric regularity properties, and the CRP, which is not based on any tangent cone, is an algebraic regularity property.

The following theorem shows that if the $C(T)$-valued function is regular at $\hat{x}$, then the GRP is equivalent to the equation

$$Ł^{\odot}(\hat{x}) = \overline{conv}^w\big(\Gamma_B(S, \hat{x})\big).$$

Theorem 3.5. *Suppose that the locally Lipschitz $C(T)$-valued function $\varphi : \mathcal{H} \to C(T)$ are given. If the function $\varphi(\cdot)(t)$ is regular at $\hat{x} \in S$ for all $t \in T(\hat{x})$, then*

$$\overline{conv}^w\big(\Gamma_B(S, \hat{x})\big) \subseteq Ł^{\odot}(\hat{x}).$$

Proof. Suppose that $d \in \Gamma_B(S, \hat{x})$ is given arbitrarily. Then, we can find some sequences $r_k \downarrow 0$ and $d_k \Gamma d$ such that $\hat{x} + r_k d_k \in S$, and so

$$\varphi(\hat{x} + r_k d_k)(t) \leq 0, \qquad \forall t \in T.$$

Let $t_0 \in T(\hat{x})$ be given. Thus, $\varphi(\hat{x})(t_0) = 0$, and so

$$\frac{\varphi(\hat{x} + r_k d_k)(t_0) - \varphi(\hat{x} + r_k d)(t_0)}{r_k} + \frac{\varphi(\hat{x} + r_k d)(t_0) - \varphi(\hat{x})(t_0)}{r_k} =$$

$$\frac{\varphi(\hat{x} + r_k d_k)(t_0) - \varphi(\hat{x})(t_0)}{r_k} = \frac{\varphi(\hat{x} + r_k d_k)(t_0)}{r_k} \leq 0. \tag{4}$$

If the Lipschitz constant of $\varphi(\cdot)(t)$ near to $\hat{x}$ is denoted by L_t, for sufficiently small r_k we have

$$\left|\frac{\varphi(\hat{x}+r_kd_k)(t_0)-\varphi(\hat{x}+r_kd)(t_0)}{r_k}\right| \leq L_{t_0}\|d_k-d\|\gamma 0 \quad \text{(when } r_k\gamma 0).$$

Taking into consideration the regularity of $\varphi(\cdot)(t_0)$ at $\hat{x}$, (4) , and the definition of directional derivative we conclude

$$\varphi^c(\hat{x};d)(t_0)=\varphi'(\hat{x};d)(t_0)=\lim_{r_k\gamma 0}\frac{\varphi(\hat{x}+r_kd)(t_0)-\varphi(\hat{x})(t_0)}{r_k}=$$

$$\lim_{r_k\gamma 0}\frac{\varphi(\hat{x}+r_kd_k)(t_0)-\varphi(\hat{x}+r_kd)(t_0)}{r_k}+\lim_{r_k\gamma 0}\frac{\varphi(\hat{x}+r_kd)(t_0)-\varphi(\hat{x})}{r_k}\leq 0.$$

Hence, with regard to Theorem 2.3 , for each $\xi_*\in\partial_c\varphi(\hat{x})(t_0)$ we have

$$\langle\xi_*,d\rangle\leq\max\big\{\langle\xi,d\rangle\mid\xi\in\partial_c\varphi(\hat{x})(t_0)\big\}=\varphi^c(\hat{x};d)(t_0)\leq 0,$$

and so $d\in\Big(\partial_c\varphi(\hat{x})(t_0)\Big)^{\odot}$. Since t_0 is an arbitrary index in $T(\hat{x})$, then

$$d\in\bigcap_{t\in T(\hat{x})}\Big(\partial_c\varphi(\hat{x})(t)\Big)^{\odot}=\bigg(\bigcup_{t\in T(\hat{x})}\partial_c\varphi(\hat{x})(t)\bigg)^{\odot}=Ł^{\odot}(\hat{x}).$$

The above inclusion and the arbitrary nature of the member d in $\Gamma_B(S,\hat{x})$ imply that

$$\Gamma_B(S,\hat{x})\subseteq Ł^{\odot}(\hat{x}).$$

Owing to the convexity and weakly closedness of $Ł^{\odot}(\hat{x})$, we have

$$\overline{conv}^w\big(\Gamma_B(S,\hat{x})\big)\subseteq\overline{conv}^w\Big(Ł^{\odot}(\hat{x})\Big)=Ł^{\odot}(\hat{x}),$$

and the proof is complete. □

Observe that, there is no relation of implication between the defined regularity properties in Definition 3.4 and the PLVP. Indeed, for any finite T the PLVP is trivially true, but it may not satisfy each of the regularity properties; while in the following example the $C(T)$-valued function actually satisfies all the regularity properties at $\hat{x}=0$, but the PLVP does not hold at this point.

Example 3.6. Let $T = \mathbb{N} \cup \{0\}$, $\mathcal{H} = \mathbb{R}$, $\hat{x} = 0$, and

$$\left| \begin{array}{ll} \varphi(x)(0) = 2x, & \\ \varphi(x)(2k+1) = x - \frac{1}{k+1}, & k = 0, 1, 2, ..., \\ \varphi(x)(2k) = 3x - \frac{1}{k}, & k = 1, 2, ... \end{array} \right.$$

It is easy to see that

$$S = \overline{\Gamma_F(S, \hat{x})}^w = \Gamma_A(S, \hat{x}) = \Gamma_B(S, \hat{x}) = (-\infty, 0],$$

$$T(\hat{x}) = \{0\}, \qquad conv\big(Ł(\hat{x})\big) = Ł(\hat{x}) = \partial_c \varphi(\hat{x})(0) = \{2\},$$

$$Ł^{\odot}(\hat{x}) = (-\infty, 0], \qquad Ł^{\ominus}(\hat{x}) = (-\infty, 0),$$

$$\varphi(x) = \sup_{t \in \mathbb{N}} \{\varphi(x)(0), \varphi(x)(t)\} = \begin{cases} x, & \text{if} \quad x < 0, \\ 3x, & \text{if} \quad x \geq 0, \end{cases}$$

$$\varphi'(\hat{x}; d) = \begin{cases} -1, & \text{if} \quad d < 0, \\ 0, & \text{if} \quad d = 0, \\ 3, & \text{if} \quad d > 0 \end{cases}, \qquad \partial \varphi(\hat{x}) = [1, 3].$$

Therefore, PLVP is not satisfied for φ at $\hat{x}$ but CRP, GRP, ARP, ZRP, TRP, and KTRP are satisfied at $\hat{x}$.

The relationships between the defined geometric regularity properties are given in the following theorem.

Theorem 3.7. *Assume that the locally Lipschitz $C(T)$-valued function $\varphi : \mathcal{H} \to C(T)$ and $\hat{x} \in S$ are given.*

(i): *The following implications hold at $\hat{x}$:*

$$\text{ZRP} \Longrightarrow \text{TRP} \Longrightarrow \text{ARP} \Longrightarrow \text{GRP}.$$

(ii): *If* KTRP *and* PLVP *are satisfied at $\hat{x}$, then* TRP *holds at $\hat{x}$.*

Proof.
(i) The result is immediately concluded from(2).

(ii) According to Theorem 2.3, we have

$$\begin{aligned} d^* \quad & \in \{d \in \mathcal{H} \mid \varphi^c(\hat{x}; d) \leq 0\} \\ & \Longleftrightarrow \varphi^c(\hat{x}; d^*) \leq 0 \\ & \Longleftrightarrow \max\left\{\langle \xi, d^* \rangle \mid \xi \in \partial_c \varphi(\hat{x})\right\} \leq 0 \\ & \Longleftrightarrow \langle \xi, d^* \rangle \leq 0, \quad \forall \xi \in \partial_c \varphi(\hat{x}) \\ & \Longleftrightarrow d^* \in (\partial_c \varphi(\hat{x}))^{\odot}. \end{aligned}$$

Thus,

$$\{d \in \mathcal{H} \mid \varphi^c(\hat{x}; d) \leq 0\} = (\partial_c \varphi(\hat{x}))^{\odot}.$$

This inclusion and the PLVP assumption imply that

$$Ł^{\odot}(\hat{x}) = \Big(conv\big(Ł(\hat{x})\big)\Big)^{\odot} \subseteq \big(\partial_c \varphi(\hat{x})\big)^{\odot}$$

$$= \big\{d \in \mathcal{H} \mid \varphi^c(\hat{x}; d) \leq 0\big\},$$

which, together with KTRP, yields

$$Ł^{\odot}(\hat{x}) \subseteq \big\{d \in \mathcal{H} \mid \varphi^c(\hat{x}; d) \leq 0\big\} \subseteq \Gamma_A(S, \hat{x}),$$

as required. □

The following example shows that the GRP is strictly weaker than other introduced geometric regularity properties.

Example 3.8. Let $T = \mathbb{N}$, $\mathcal{H} = \mathbb{R}^2$, $\hat{x} = (0, 0)$, and for all $x = (x_1, x_2) \in \mathbb{R}^2$ the function $\varphi(x)(t)$ be defined as

$$\varphi(x)(t) = \begin{cases} -x_1, & \text{if} \quad t = 1, \\ -x_2, & \text{if} \quad t = 2, \\ x_1 x_2 - \dfrac{1}{t} & \text{if} \quad t \in \{3, 4, ...\}. \end{cases}$$

It is easy to see that

$$\begin{aligned}
&S = \Big(\{0\} \times [0, +\infty)\Big) \cup \Big([0, +\infty) \times \{0\}\Big),\\
&T(\hat{x}) = \{1, 2\},\\
&Ł(\hat{x}) = \big\{(-1, 0), (0, -1)\big\},\\
&Ł^{\odot}(\hat{x}) = [0, +\infty) \times [0, +\infty),\\
&Ł^{\ominus}(\hat{x}) = (0, +\infty) \times (0, +\infty),\\
&\Gamma_A(S, \hat{x}) = \overline{\Gamma_F(S, \hat{x})}^{w} = \Gamma_B(S, \hat{x}) = S,\\
&\overline{conv}^{w}\big(\Gamma_B(S, \hat{x})\big) = [0, +\infty) \times [0, +\infty).
\end{aligned}$$

Thus, φ satisfies the GRP at $\hat{x}$, but does not satisfy the ARP, ZRP, TRP, and KTRP at this point.

The following theorems presents the relashionship between the algebraic regularity property CRP and the geometric regularity property ZRP.

Theorem 3.9. *Suppose that the locally Lipschitz $C(T)$-valued function $\varphi : \mathcal{H} \to C(T)$ satisfies* PLVP *at $\hat{x}$. Then, the* CRP *implies the* ZRP *at $\hat{x}$.*

Proof. Let $d \in Ł^{\ominus}(\hat{x})$. If $\xi \in conv\big(Ł(\hat{x})\big)$ is arbirarily given, (1) implies that there exist some real numbers $\alpha_1, \ldots, \alpha_m$ in $[0, 1]$ and some vectores $\xi_1, \ldots, \xi_m$ in $Ł(\hat{x})$ such that

$$\xi = \sum_{i=1}^{m} \alpha_i \xi_i, \qquad \sum_{i=1}^{m} \alpha_i = 1.$$

Thus,

$$\langle d, \xi \rangle = \Big\langle d, \sum_{i=1}^{m} \alpha_i \xi_i \Big\rangle = \sum_{i=1}^{m} \alpha_i \overbrace{\langle d, \xi_i \rangle}^{<0} < 0,$$

and so $d \in \Big(conv\big(Ł(\hat{x})\big)\Big)^{\ominus}$. This inclusion and the PLVP assumption at $\hat{x}$ conclude that $d \in \big(\partial_c \varphi(\hat{x})\big)^{\ominus}$. So

$$\varphi^c(\hat{x}; d) = \max\big\{\langle \xi, d \rangle \mid \xi \in \partial_c \varphi(\hat{x})\big\} < 0.$$

Consequently, there exists a scalar $\delta > 0$ such that

$$\varphi(\hat{x}+\varepsilon d) - \varphi(\hat{x}) < 0, \qquad \forall\, \varepsilon \in (0,\delta),$$

and so

$$\varphi(\hat{x}+\varepsilon d) < \varphi(\hat{x}) \leq 0, \qquad \forall\, \varepsilon \in (0,\delta).$$

The above inequality and the definition of φ deduce that

$$\varphi(\hat{x}+\varepsilon d)(t) \leq \varphi(\hat{x}+\varepsilon d) < 0, \qquad \forall \varepsilon \in (0,\delta),\ \forall t \in T.$$

Therefore, for all $\varepsilon \in (0,\delta)$ we have $\hat{x}+\varepsilon d \in S$, which implies $d \in \Gamma_F(S,\hat{x})$. Thus, we have proved $Ł^{\ominus}(\hat{x}) \subseteq \Gamma_F(S,\hat{x})$, and hence

$$Ł^{\odot}(\hat{x}) = \overline{Ł^{\ominus}(\hat{x})}^{w} \subseteq \overline{\Gamma_F(S,\hat{x})}^{w},$$

and the proof is complete. □

Mutivating by (3), we introduce the following algebraic regularity property.

Definition 3.10. We say that the locally Lipschitz $C(T)$-valued function $\varphi : \mathcal{H} \to C(T)$ satisfies the Slater regularity property (SRP), if

- for all $t \in T$, the function $\varphi(\cdot)(t) : \mathcal{H} \to \mathbb{R}$ is convex,
- T is a compact metric space,
- φ is a continuous function in $\mathcal{H}$,
- there is a vector $x_0 \in \mathcal{H}$, named Slater point, such that $\varphi(x_0)(t) < 0$ for all $t \in T$.

Theorem 3.11. *Suppose that the locally Lipschitz $C(T)$-valued function $\varphi : \mathcal{H} \to C(T)$ satisfies the* SRP *and $\hat{x} \in S$ are given. Then,*

(i) *φ satisfies the* CRP *at $\hat{x}$.*

(ii) *$Ł(\hat{x})$ is a weakly compact set and the* PLVP *is satisfied for φ at $\hat{x}$.*

(iii) *$cone\big(Ł(\hat{x})\big)$ is a weakly closed cone in $\mathcal{H}$.*

Proof.
(i) By the definition of SRP, there is an x_0 such that

$$\varphi(x_0)(t) < 0, \qquad \text{for all } t \in T.$$

Let $t_0 \in T(\hat{x})$ and $\xi_{t_0} \in \partial\varphi(\hat{x})(t_0)$. Then, the definition of convex subdifferential implies than

$$\langle \xi_{t_0}, x_0 - \hat{x} \rangle \leq \overbrace{\varphi(x_0)(t_0)}^{<0} - \overbrace{\varphi(\hat{x})(t_0)}^{=0} < 0.$$

This inequality implies that $x_0 - \hat{x} \in \Big(\partial\varphi(\hat{x})(t_0)\Big)^{\ominus}$, and since $t_0 \in T(\hat{x})$ was chosen arbitrarily, we get

$$x_0 - \hat{x} \in \bigcap_{t \in T(\hat{x})} \Big(\partial\varphi(\hat{x})(t)\Big)^{\ominus} = \Bigg(\bigcup_{t \in T(\hat{x})} \partial\varphi(\hat{x})(t)\Bigg)^{\ominus} = \text{Ł}^{\ominus}(\hat{x}).$$

This means that $\text{Ł}^{\ominus}(\hat{x}) \neq \emptyset$, as required.
(ii) It is Ioffe-Tikhomirov Theorem [13, Theorem 2.4.18]
(iii) If $d \in \text{Ł}^{\ominus}(\hat{x})$, by the proof of Theorem 3.9 we have

$$d \in \Big(conv\big(\text{Ł}(\hat{x})\big)\Big)^{\ominus}. \tag{5}$$

We claim that

$$0_{\mathcal{H}} \notin conv\big(\text{Ł}(\hat{x})\big).$$

Otherwise, we have $\Big(conv\big(\text{Ł}(\hat{x})\big)\Big)^{\ominus} = \emptyset$, which contradits (5). Thus, the above claim is true. Therefore, the weakly compactness of $\text{Ł}(\hat{x})$ and Theorem 2.2 conclude the result. □

Theorem 3.12. *Suppose that the locally Lipschitz $C(T)$-valued function $\varphi : \mathcal{H} \to C(T)$ is given. If the* CRP *and* PLVP *are satisfied at $\hat{x} \in S$, then* KTRP *holds at $\hat{x}$.*

Proof. If $d^* \in \text{Ł}^{\ominus}(\hat{x})$, then $d^* \in \Big(conv\big(\text{Ł}(\hat{x})\big)\Big)^{\ominus}$ by proof of Theorem 3.9. Thus, the PLVP assumption at $\hat{x}$ implies that $d^* \in \big(\partial_c\psi(\hat{x})\big)^{\ominus}$, which concludes that $\langle \xi, d^* \rangle < 0$ for all $\xi \in \partial_c \psi(\hat{x})$. From this and Theorem 2.3 we obtain that $\varphi^c(\hat{x}; d^*) < 0$, and so

$$\big\{d \in \mathcal{H} \mid \varphi^c(\hat{x}; d) < 0\big\} \neq \emptyset.$$

If $\hat{d} \in \big\{d \in \mathcal{H} \mid \varphi^c(\hat{x}; d) < 0\big\}$ is given arbitrarily, then $\varphi^c(\hat{x}; \hat{d}) < 0$. By the definition of Clarke directional derivative and the definition of sup-function φ, we obtain a $\delta > 0$ such that for all $\varepsilon \in (0, \delta)$ one has

$$\varphi(\hat{x} + \varepsilon\hat{d})(t) \leq \varphi(\hat{x} + \varepsilon\hat{d}) < \varphi(\hat{x}) \leq 0, \qquad \forall t \in T.$$

Thus, $\hat{x} + \varepsilon\hat{d} \in S$ for all $\varepsilon \in (0, \delta)$, which yiels $\hat{d} \in \Gamma_F(S, \hat{x})$. In summary, we have shown that

$$\{d \in \mathcal{H} \mid \varphi^c(\hat{x}; d) < 0\} \subseteq \Gamma_F(S, \hat{x}).$$

This inclusion together with the continuity of $\varphi^c(\hat{x}; \cdot)$ implies that

$$\{d \in \mathcal{H} \mid \varphi^c(\hat{x}; d) \leq 0\} = \overline{\{d \in \mathcal{H} \mid \varphi^c(\hat{x}; d) < 0\}}^w \subseteq \overline{\Gamma_F(S, \hat{x})}^w \subseteq \Gamma_A(S, \hat{x}),$$

and the result is proved. $\square$

The following corollary collects Theorems3.7, 3.9, 3.11, and 3.12 in one diagram.

Corollary 3.13. *Suppose that the locally Lipschitz $C(T)$-valued function $\varphi : \mathcal{H} \to C(T)$ is given. Then, the implications of the following diagram hold true at each $\hat{x} \in S$:*

$$\begin{array}{ccccccc}
\text{SRP} & & & & & & \\
\Downarrow & & & & & & \\
[\text{CRP} \wedge \text{PLVP}] & \Longrightarrow & [\text{KTRP} \wedge \text{PLVP}] & & & & \\
\Downarrow & & \Downarrow & & & & \\
\text{ZRP} & \Longrightarrow & \text{TRP} & \Longrightarrow & \text{ARP} & \Longrightarrow & \text{GRP}
\end{array}$$

Example 3.14. Considering Example 3.6, we have $cone\big(\text{Ł}(\hat{x})\big) = [0, +\infty)$. So, the converse of the following implications are not true, even when $cone\big(\text{Ł}(\hat{x})\big)$ is weakly closed:

$$[\text{KTRP} \wedge \ \text{PLVP}] \Longrightarrow \text{TRP} \qquad \text{and} \qquad [\text{CRP} \wedge \ \text{PLVP}] \Longrightarrow \text{ZRP}.$$

This example, also, shows that the compactness condition of T is necessary in the following implication

$$\text{SRP} \Longrightarrow [\text{CRP} \wedge \ \text{PLVP}].$$

The following theorem shows that the Basic constraint qualification is equivalent to $N_{\mathcal{F}}(S,\hat{x}) = cone\big(Ł(\hat{x})\big)$ for convex $C(T)$-valued function. This equlity is named "locally Farkas-Minkowski constraint qualificatin" in [3].

Theorem 3.15. *Assume that the locally Lipschitz $C(T)$-valued function $\varphi : \mathcal{H} \to C(T)$ and $\hat{x} \in S$ are given. If the real-valued function $\varphi(\cdot)(t) : \mathcal{H} \to \mathbb{R}$ is convex for all $t \in T$, then*

$$cone\big(Ł(\hat{x})\big) \subseteq N_{\mathcal{F}}(S,\hat{x}).$$

Proof. At the first, according to

$$S = \bigcap_{t\in T} \{x \in \mathcal{H} \mid \varphi(x)(t) \leq 0\},$$

and regarding the convexity of level sets of real-valued convex functions [13, p. 41], we conclude that S is an intersection of convex sets, and consequently is itself convex. Thus, $\overline{\Gamma_F(S,\hat{x})}^w = \Gamma_B(S,\hat{x})$.

Now, suppose that $\xi \in Ł(\hat{x})$ is given. Thus, $\xi \in \partial_c\varphi(\hat{x})(t_0)$ for some $t_0 \in T(\hat{x})$, and so $\varphi(\hat{x})(t_0) = 0$. If $d \in \Gamma_F(S,\hat{x})$, then $\hat{x} + \delta d \in S$ for some $\delta > 0$, and hence $\varphi(\hat{x} + \delta d)(t_0) \leq 0$. Hence, by the definition of convex subdifferential we get

$$0 \geq \frac{1}{\delta}\Big(\varphi(\hat{x} + \delta d)(t_0) - \varphi(\hat{x})(t_0)\Big) \geq \frac{1}{\delta}\langle \xi, \delta d\rangle = \langle \xi, d\rangle.$$

Since the above inequality holds for all $d \in \Gamma_F(S,\hat{x})$, we have $\xi \in \big(\Gamma_F(S,\hat{x})\big)^{\odot}$. This inclusion and the fact that

$$\big(\Gamma_F(S,\hat{x})\big)^{\odot} = \big(\overline{\Gamma_F(S,\hat{x})}^w\big)^{\odot} = (\Gamma_B(S,\hat{x}))^{\odot} = N_{\mathcal{F}}(S,\hat{x}),$$

conclude that $\xi \in N_{\mathcal{F}}(S,\hat{x})$, and so

$$Ł(\hat{x}) \subseteq N_{\mathcal{F}}(S,\hat{x}).$$

Taking convex cone in both sides of the above inclusion, we get

$$cone\big(Ł(\hat{x})\big) \subseteq cone\big(N_{\mathcal{F}}(S,\hat{x})\big) = N_{\mathcal{F}}(S,\hat{x}),$$

and the proof is complete. $\square$

As an immediate consequence of Theorem 3.11(ii), Corrolary 3.13, and Theorems 2.2 and 3.15, we can obtain the following Corollary which was proved in [3] for special case of $\mathcal{H} = \mathbb{R}^n$.

Corollary 3.16. *Suppose that the locally Lipschitz $C(T)$-valued function $\varphi : \mathcal{H} \to C(T)$ is given. If the* SRP *holds at $\hat{x} \in S$, then*

$$N_{\mathcal{F}}(S, \hat{x}) = cone\big(Ł(\hat{x})\big).$$

References

[1] Clarke, F.H.: Optimization and nonsmooth analysis. *Wiley. Interscience* (1983).

[2] Dumford, N., Schwartz, J.T.: Linear Operators: General Theory. Chichester. *Wiley* (1988).

[3] Goberna, M.A., Kanzi, N.: Optimality conditions in convex multiobjective *SIP. Math. Programm.* **164**, 167–191 (2017).

[4] Hettich, R., Kortanek, O.: Semi-infinite programming: theory, methods, and applications. *SIAM Rev.* **35**, 380–429 (1993).

[5] Hiriart-Urruty, J.B., Lemarechal, C.: Convex Analysis and Minimization Algorithms. I., Fundamentals. *Springer. Berlin* (1993).

[6] Kanzi, N.: Karush–Kuhn–Tucker types optimality conditions for non-smooth semi-infinite vector optimization problems. *Journal of Mathematical Extension.* **9**, 45-56 (2015).

[7] Kanzi, N., Shaker Ardekani, J., Caristi, G.: Optimality, scalarization and duality in linear vector semi-infinite programming. *Optimization.* **67**, 523–536 (2018).

[8] Kanzi, N.: Necessary and sufficient conditions for (weakly) efficient of nondifferentiable multi-objective semi-infinite programming. *Iran. J. Sci. Technol. Trans. A: Sci.* **42**, 1537–1544 (2017).

[9] Kanzi, N.: Constraint qualifications in semi-infinite systems and their applications in nonsmooth semi-infinite problems with mixed constraints. *SIAM Journal on Optimization.* **24**, 559–572 (2014).

[10] Li, W., Nahak, C., Singer, I.: Constraint qualifications in semi-infinite systems of convex inequalities. *SIAM Journal of Optimization.* textbf11, 31-52 (2000).

[11] Rockafellar, R.T.: Convex Analysis. Princeton University Press. Princeton. (1970).

[12] Winkler, K.A.: Characterization of efficient and weakly efficient points in convex vector optimization. *SIAM Journal on Optimization.* **19**, 756-765 (2008).

[13] Zalinescu, C.: Convex Analysis in General Vector Spaces. World Scientific. Singapore. (2002).

[14] Zhao, K.Q., Yang, X.M.: Characterizations of efficiency in vector optimization with $C(T)$-valued mappings. *Optimization Letters.* **9**, 391-401 (2015).

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