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Original Research Paper

A Quantile Approach to the Kullback-Leibler Divergence on Type-I Censored Variable and Record Values

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Abstract. This study introduces a novel quantile-based approach to measuring Kullback-Leibler divergence, a key statistical distance metric. The method allows characterization of KL divergence for general data, Type-I censored variable, and record value data. Furthermore, it establishes connections between the quantile-based KL divergence and various stochastic orderings, providing insights into how probability distributions differ.

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1 Introduction

The Kullback-Leibler (KL) divergence (also called relative entropy) [9] and quantile function (Qf) of the continuous random variable (r.v) play

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key roles in information theory, and they are widely used in many fields. is a measure of the difference between two probability distributions. It quantifies how one distribution diverges from another. In statistical analysis, the KL divergence is commonly used to compare two probability distributions, such as comparing an estimated distribution with a true distribution. This measure is a fundamental quantity of uncertainty theorems and Qf is more convenient in approximation of the distributions. There are various issues about the quantile functions and its relations to KL divergence. For example Mansourvar (2022) introduces a new way to measure the difference between two probability distributions. This new method is based on something called the "quantile function," which is a way of describing a probability distribution. The author shows that this new method is related to other well-known methods, and that it has some nice properties [10].

Sunoj and Saranya (2024) investigated the quantile-based cumulative Kullback-Leibler divergence within the context of past lifetimes, exploring its properties and demonstrating its applications [20]. Joseph and Mathew (2025) introduced a quantile-based version of Matusita's measure, specifically for residual lifetimes [8]. Al-Labadi et al. (2021) explored the use of Kullback-Leibler divergence for Bayesian nonparametric model checking, published in the Journal of the Korean Statistical Society [3]. Al-Labadi and Tahir (2022) explored the estimation of entropy and extropy for right-censored data using a Bayesian non-parametric approach, published in Monte Carlo Methods and Applications [4]. Al-Labadi, Fazeli-Asl, and Ly (2024) developed a Bayesian non-parametric approach, based on Kullback-Leibler divergence estimation, to evaluate model fit for type II censored data, as published in Communications in Statistics-Simulation and Computation [5]. For more details, we can review [6, 14, 15, 16, 17, 19, 22].

Letting X and Y be two absolutely continuous, non-negative r.v's, with the probability distribution functions, (pdf)'s, $g(x)$ and $f(x)$, also cumulative distribution functions (cdf)'s $G(x)$ and $F(x)$, and survival functions (sf)'s $\bar{G}(x)$ and $\bar{F}(x)$, respectively, the mutual information between two continuous r.v's can be indicated in terms of the KL divergence as follows:

$$D_{KL}(X\|Y) = \int_{-\infty}^{\infty} g(x) \log \frac{g(x)}{f(x)} \mathrm{d}x, \tag{1}$$

where, $g(x)$ represents the probability distribution function precisely measured and $f(x)$ represents instead a description or an approximation of $g(x)$.

The quantile approach to the KL divergence involves estimating the quantiles of the censored distributions and then computing the KL divergence based on these estimated quantiles. This approach allows for a more accurate comparison of censored distributions by taking into account the uncertainty introduced by censoring.

The quantile function of a probability distribution can be specified in terms of the distribution function $G(x)$ as follow:

$$Q_1(u) = G^{-1}(u) = inf\{x|G(x) \geq u\}, \, 0 \leq u \leq 1. \tag{2}$$

Supporting the definition of quantile function $Q_1(u)$ in Equation (2), we can get $G(Q_1(u)) = u$, where $G(x)$ is increasing function of continuous variable X. Besides, the differentiation of $G(Q_1(u))$ function implies $g(Q_1(u)).q_1(u) = 1$, where $g(Q_1(u))$ is the density quantile function and $q_1(u) = \frac{\partial}{\partial u}Q_1(u)$, is the quantile density function.

Similarly, $F(Q_1(u)) = F(G^{-1}(u))$. Hence, by differentiating of $F(G^{-1}(u))$ with respect to u we have

$$\frac{\partial}{\partial u}F(G^{-1}(u)) = \frac{\partial}{\partial u}F(Q_1(u)) = q_1(u).f(Q_1(u)). \tag{3}$$

Then, by taking $Q_3(u) = F(Q_1(u))$ we can get

$$q_3(u) = \frac{\partial}{\partial u}Q_3(u) = \frac{\partial}{\partial u}Q_2^{-1}(Q_1(u)) = \frac{\partial}{\partial u}F(Q_1(u)). \tag{4}$$

Therefore, by taking $q_2(u) = f(Q_1(u))$, hence Equations (3) and (4) can would yield

$$q_3(u) = q_2(u)q_1(u) = \frac{f(Q_1(u))}{g(Q_1(u))}. \tag{5}$$

Thus, according to (1) the quantile-based KL divergence measure is defined as [16]:

$$D_{KL}(q_1\|q_2) = -\int_0^1 \ln(q_3(u))\, \mathrm{d}u. \tag{6}$$

The hazard rate function of the non negative continuous r.v X is defined by [15]:

$$H_3(u) = \frac{1}{(1-u)q_3(u)}. \tag{7}$$

where $0 < u < 1$ as well as $H_3(u)$ stands the quantile based hazard rate function.
In a recent paper we have obtained a quantile-based KL divergence on typical data. Section 2, proposes the KL measure on Type-I censored variable. In this Section we prove some characterization results based on the quantile-based Type-I censored KL divergence. In Section 3 we introduce a quantile version of the KL divergence measure for the distribution of the upper and lower record values. Furthermore, in our studies, we use some quantile based stochastic orders such as stochastic dominance, likelihood ratio, the hazard and reversed hazard. Finally, Section 4 provides a numerical solutions for the models of proportional hazards and proportional reversed hazards for general data, Type-I censored variable and record values.

2 Quantile Based Type-I Censored KL Divergence

Type-I censoring is a common technique used in survival analysis when the exact failure times are not observed but only known to be larger than a certain threshold. In this type of censoring, the data is right-censored, meaning that the failure times are only known to be greater than a specific value. In this case, the KL divergence between two censored distributions measures how much information is lost when one distribution is used to approximate another.
The quantile approach to the KL divergence on censored data has been applied in various fields, including survival analysis and reliability engineering. It provides a useful tool for comparing censored distributions and assessing the similarity or dissimilarity between them.
Assume that X and Y are two absolutely continuous r.v's. The density

function for a Type-I censored variable is defined as

$$g_C(x) = \begin{cases} g(x) & \text{if } x < C \\ \bar{G}(C) & \text{if } x = C \\ 0 & \text{if } x > C. \end{cases}$$

Where C is the censoring point that is supposed to be a constant [11, 12, 13].
KL divergence between two r.v's X and Y on Type-I censored variable can be introduced by

$$D_{KL}^{C-I}(g||f) = \int_{-\infty}^{C} g(x) \ln \frac{g(x)}{f(x)} dx + \bar{G}(C) \ln \frac{\bar{G}(C)}{\bar{F}(C)}.$$

In addition, the quantile Type-I censored KL divergence is defined as

$$D_{KL}^{C-I}(q_1 \| q_2) = -\int_0^P \ln q_3(u) \mathrm{d}u - (1-P) \ln(\frac{1-Q_3(P)}{1-P}), \tag{8}$$

where $P = G(C)$.
Also, using (7) we have

$$D_{KL}^{C-I}(q_1 \| q_2) = \int_0^P \ln H_3(u) \mathrm{d}u - (1-P) \ln(1-Q_3(P)) - P,$$

Besides, We know that Type-I censored Shannon Entropy ($H_P(u)$) is defined by

$$H_P(u) = \int_0^P \ln q_1(u) du - (1-P) \ln(1-P). \tag{9}$$

Remark 2.1. $\frac{\partial}{\partial P} D_{KL}^{C-I}(q_1 \| q_2) = A(P) - \ln A(P) - 1$, where $A(P) = \frac{(1-P)q_3(P)}{1-Q_3(P)} \geq 0$, that yields the quantile KL divergence on Type-I censored variable is a monotonous increasing function of P. Hence, Equation (8) tends to (6) when P tends to 1.

One of the quantile functions that can have useful mathematical properties is the $Q_3(u) = 1 - (1-u)^\theta$ function. In the following, we will examine its characteristics in relation to KL divergence on Type-I censored variable.

Proposition 2.2. *Suppose that $Q_3(u) = 1-(1-u)^{\theta}$. Then, the quantile-based Type-I censored KL divergence is obtained as*

$$D_{KL}^{C-I}(q_1\|q_2) = -P\ln\theta + (\theta-1)\big[\Gamma(\Lambda,2,1) - (1-P)\ln(1-P)\big],$$

and the quantile-based KL divergence for general data is given by

$$D_{KL}(q_1\|q_2) = \theta - \ln\theta - 1.$$

where $\Gamma(\Lambda,2,1)$ is uncompleted Gamma distribution and $\Lambda = -\ln(1-P)$.

Proof. Obviously, by substituting $\vartheta = -\ln(1-u)$ and $\Lambda = -\ln(1-P)$ it can be transformed to Gamma distribution form and we can yield the results. □

Proposition 2.3. *Let $Q_3(u) = 1-(1-u)^{\theta}$. Then*

$$-\int_0^P \ln q_3(u)\mathrm{d}u \leq D_{KL}^{C-I}(q_1\|q_2) \leq -\int_0^1 \ln q_3(u)\mathrm{d}u$$

Proof. Clearly, from Remark 2.1 the upper bound for KL divergence on Type-I censored variable is KL divergence on general scheme ((6)). Furthermore, $Q_3(P)-P = (1-P)(1-(1-P)^{\theta-1}) > 0,\ 0<u<1$. Then $-(1-P)\ln\big(\frac{1-Q_3(P)}{1-P}\big) > -(1-P)\ln\big(\frac{1-P}{1-P}\big) = 0$. Thus, this complete the proof. □ In Table (1), we show some well known lifetime distributions with quantile function $Q_3(u) = 1-(1-u)^{\theta}$.

Proposition 2.4. $q_3^X(u) \leq (\geq) q_3^Y(u) \Rightarrow D_{KL}^{C-I}(q_1, q_2^X) \geq (\leq) D_{KL}^{C-I}(q_1, q_2^Y)$.

Proof. Letting $q_3^X(u) \leq (\geq) q_3^Y(u)$ we would have

$$\int_0^P \ln q_3^X(u)du \leq (\geq) \int_0^P \ln q_3^Y(u)du.$$

Moreover, it yields $1-Q_3^X(u) \geq (\leq) 1-Q_3^Y(u)$. So we would have

$$(1-P)\ln\big(\frac{1-P}{1-Q_3^X(u)}\big) \leq (\geq)(1-P)\ln\big(\frac{1-P}{1-Q_3^Y(u)}\big).$$

Then, from Equation (8) we can get $D_{KL}^{C-I}(q_1, q_2^X) \geq (\leq) D_{KL}^{C-I}(q_1, q_2^Y)$. □

Table 1: Some Quantile functions distributions

Distribution	QFs	
Pareto I	$Q_1(u)=(1-u)^{-\frac{1}{\lambda_1}};\lambda_1>0$	$Q_3(u)=1-(1-u)^{\theta};\ \theta=\frac{\lambda_2}{\lambda_1}$
	$Q_2(u)=(1-u)^{-\frac{1}{\lambda_2}};\lambda_2>0$	$q_3(u)=\theta(1-u)^{\theta-1}$
Pareto II	$Q_1(u)=\alpha\big((1-u)^{-\frac{1}{\lambda_1}}\big);\alpha>0,\lambda_1>0$	$Q_3(u)=1-(1-u)^{\theta};\ \theta=\frac{\lambda_2}{\lambda_1}$
	$Q_2(u)=\alpha\big((1-u)^{-\frac{1}{\lambda_2}}\big);\alpha>0,\lambda_2>0$	$q_3(u)=\theta(1-u)^{\theta-1}$
Gompertz	$Q_1(u)=\frac{1}{\ln c}\big(1-\frac{\ln c.\ln(1-u)}{\lambda_1}\big);c>0,\lambda_1>0$	$Q_3(u)=1-(1-u)^{\theta};\ \theta=\frac{\lambda_2}{\lambda_1}$
	$Q_2(u)=\frac{1}{\ln c}\big(1-\frac{\ln c.\ln(1-u)}{\lambda_2}\big);c>0,\lambda_2>0$	$q_3(u)=\theta(1-u)^{\theta-1}$
Finite Range	$Q_1(u)=b\big(1-(1-u)^{\frac{1}{\lambda_1}}\big);\lambda_1>0,b>0$	$Q_3(u)=1-(1-u)^{\theta};\ \theta=\frac{\lambda_2}{\lambda_1}$
	$Q_2(u)=b\big(1-(1-u)^{\frac{1}{\lambda_2}}\big);\lambda_2>0,b>0$	$q_3(u)=\theta(1-u)^{\theta-1}$
Exponential	$Q_1(u)=-\frac{1}{\lambda_1}\ln(1-u);\lambda_1>0$	$Q_3(u)=1-(1-u)^{\theta};\ \theta=\frac{\lambda_2}{\lambda_1}$
	$Q_2(u)=-\frac{1}{\lambda_2}\ln(1-u);\lambda_2>0$	$q_3(u)=\theta(1-u)^{\theta-1}$

Example 2.5. We could have an illustration of Proposition 2.4 in the well-known distributions of Table 1. Furthermore the following proof is an extension sample for exponential distribution. Suppose that $Q_1(u)=-\frac{1}{\theta}\ln(1-u);\ \theta>0$, and X and Y represent two exponential r.v's with QFs respectively by $Q_2^X(u)=-\frac{1}{\lambda_1}\ln(1-u);\ \lambda_1>0$, $Q_2^Y(u)=-\frac{1}{\lambda_2}\ln(1-u);\ \lambda_2>0$, where $\lambda_1>\lambda_2$. Then $Q_3^X(u)=1-(1-u)^{\frac{\lambda_1}{\theta}}$, $Q_3^Y(u)=1-(1-u)^{\frac{\lambda_2}{\theta}}$. Hence $\ln q_3^X(u)=\ln\frac{\lambda_1}{\theta}+\frac{\lambda_1-\theta}{\theta}$ and $\ln q_3^Y(u)=\ln\frac{\lambda_2}{\theta}+\frac{\lambda_2-\theta}{\theta}$ that yields $q_3^X(u)>q_3^Y(u)$ and implies $D_{KL}^{C-I}(q_1,q_2^X)<D_{KL}^{C-I}(q_1,q_2^Y)$.

Example 2.6. Let $Q_2(u)=u;\ 0<u<1$. Furthermore
(a) $q_1(u)=ku^{\nu}(1-u)^{A+\nu};\ k>0$. Then $Q_3(u)=k.B(u,\nu+1,A+\nu+1)$, where $B(P,\nu+1,A+\nu+1)$ is the Beta incomplete function.
(b) $q_1(u)=k(1-u)^{A}\big(-\ln(1-u)\big)^{\nu}$. Then $Q_3(u)=k.\Gamma(u,\nu+1,A+1)$, where $\Gamma(u,\nu+1,A+1)$ is the Gamma incomplete function.
Using (a) in (8) we would have

$$D_{KL}^{C-I}(q_1,q_2)=-\int_0^P \ln\big(ku^{\nu}(1-u)^{A+\nu}\big)du-(1-P)\ln\big(\frac{1-kB(P,\nu+1,A+\nu+1)}{1-P}\big)$$
$$=P(\nu-\ln k-1)-\nu P\ln P-(A+\nu+1)-(1-P)\ln\big(\frac{1-k.B(P,\nu+1,A+\nu+1)}{1-P}\big),$$

Similarity, using (b) in (8) we have

$$
\begin{aligned}
D_{KL}^{C-I}(q_1,q_2) &= -\int_0^P \ln\big(k(1-u)^A\big(-\ln(1-u)\big)^\nu\big)du-(1-P)\ln\big(1-k\frac{\Gamma(u,\nu+1,A+1)}{1-P}\big),\\
&= -k\frac{W^\nu\big(W(A+1)\big)^{-\nu}\big(\Gamma(\nu+1)-\Gamma(W,\nu+1,(A+1))\big)}{A+1}\\
&-(1-P)\ln\big(\frac{1-k\Gamma(u,\nu+1,A+1)}{1-P}\big).
\end{aligned}
$$

where $W=-\ln(1-P)$.

Definition 2.7. Let X and Y be two non negative r.v's such that $H_1(u)\geq H_2(u)$ or equivalently $q_1(u)\leq q_2(u)$ for all $u\in(0,1)$. Then X is said to be smaller than Y in hazard quantile function order denoted by $X\overset{HQ}{\leq}Y$.

Proposition 2.8. *Let $X\overset{HQ}{\leq}Y$. Then*

$$
D_{KL}^{C-I}(q_1\|q_2)\leq -2\int_0^P \ln q_1(u)du \tag{10}
$$

$$
D_{KL}^{C-I}(q_1\|q_2)\leq -2\big(H_P(u)-(1-P)\ln(1-P)\big). \tag{11}
$$

Proof. Let us $H_1(P)\geq H_2(P)$ for all $P\in(0,1)$ that is $\frac{1}{(1-u)q_1(u)}\geq\frac{1}{(1-u)q_2(u)}$ and implies respectively $q_1(u)\leq q_2(u)$ and $Q_1(u)\leq Q_2(u)$. Therefore $Q_3(u)\leq u$. That yields $(1-P)\ln\big(\frac{1-Q_3(P)}{1-P}\big)\geq 0$. Besides, from (5) we have $q_3(u)=q_2(u)q_1(u)$. This implies $q_3(u)\geq q_1^2(u)$ and $\int_0^P\ln q_3(u)du\geq 2\int_0^P\ln q_1(u)du$. So, we can obtain (10). Furthermore, combining Relations (10) and (9) can obtained (11) and the proof is completed. □

Proposition 2.9. *If $X\overset{st}{\leq}Y$ then $D_{KL}^{C-I}(q_1\|q_2)\leq-\int_0^P\ln q_3(u)du$.*

Proof. If $X\overset{st}{\leq}Y$ then $Q_1(u)\leq Q_2(u)$ for all $u\in(o,1)$ [18], that implies $Q_3(P)\leq P$. The proof is complete. □

Proposition 2.10. *Let us X and Y be two different continuous r.v's. Then the quantile KL divergence on Type-I censored variable is not a constant.*

Proof. Suppose that $D_{KL}^{C-I}(q_1\|q_2) = k$. Then by differentiating both sides of this equation we would have $A(P) - \ln\big(A(P)\big) - 1 = 0$, that implies $A(P) = 1$. Hence $\frac{q_3(P)}{1-Q_3(P)} = \frac{1}{(1-P)}$, or equivalently

$$\frac{\partial \ln(1-Q_3(P))}{\partial P} = \frac{\partial \ln(1-P)}{\partial P}. \tag{12}$$

Integrating (12) with respect to P between the limits 0 and P, we have $Q_3(P) = P$. Therefore we can get $F(Q_1(P)) = G(Q_1(P)) = P$, that yields $k = 0$, which completes the proof of the proposition. □

Proposition 2.11. *Let X and Y are continuous r.v's, also a and b are constant, then $D_{KL}^{C-I}(q_1\|q_2) = a + b.P$, if and only if X and Y stand ph model.*

Proof. Suppose that $D_{KL}^{C-I}(q_1\|q_2) = a + b.P$. Then by differentiating both sides of this equation and by taking $k = b+1$ we would have $A(P) - \ln\big(A(P)\big) = k$. So $A(P) = \theta$ is a constant that implies $\dfrac{(1-P)q_3(P)}{1-Q_3(P)} = \theta$. This means that $Q_3(u) = 1-(1-u)^{\theta}$, that yields $\bar{F}(x) = (\bar{G}(x))^{\theta}$ which completes the proof of the proposition. For reverse proof is clear. Hence, it is omitted for brevity. □

3 Quantile Based Record Value KL Divergence

Quantile-based record value KL divergence is a statistical measure used to compare two pdf's by quantifying the difference between their quantiles. The quantile-based record value KL divergence is particularly useful when dealing with heavy-tailed distributions or when comparing extreme values. It provides a way to assess the dissimilarity between two distributions in terms of their extreme values rather than their overall shape. This measure has applications in various fields such as finance, environmental sciences, and risk analysis. It can be used to compare extreme value distributions, assess tail dependence in multivariate models, and evaluate risk measures for rare events.

Record values obtained from a sequence of r.v's are closely related to order statistics. It was first proposed by Chandler[7] and has a major contribution when obtaining observation is difficult or observations are

being destroyed in an experimental test [21]. See Ahsanullah[1, 2] for more details.
Let $X_1, X_2, \cdots, X_n$, $n \geq 1$, is a sequence of independent and identically distributed r.v with a non-increasing survival function, $Y_n = \max\{X_1, X_2, \cdots, X_n\}$ and $Z_n = \min\{X_1, X_2, \cdots, X_n\}$. Then $X_{U(i)}$ is the i^{th} upper record statistic of $X_1, X_2, \cdots, X_n$ if $Y_i \geq Y_{i-1}$, $i \geq 1$. In a similar way $X_{L(i)}$ is the i^{th} lower record statistic of $X_1, X_2, \cdots, X_n$ if $Z_i \leq Z_{i-1}$, $i \geq 1$.
So the pdf functions of the $X_{U(i)}$ and $X_{L(i)}$ in respectively are given by

$$f_{X_{U(i)}}(x) = \frac{[-\ln(\bar{F}_X(x))]^{i-1} f_X(x)}{\Gamma(i)}, \quad i = 1, 2, \cdots, n, \tag{13}$$

$$f_{X_{L(i)}}(x) = \frac{[-\ln(F_X(x))]^{i-1} f_X(x)}{\Gamma(i)}, \quad i = 1, 2, \cdots, n,$$

where $\Gamma(i) = (i-1)!$.
Then the quantile KL divergence function between the upper (lower) record values respectively is defined by

$$D_{KL}(q_1^{(u_i)} \| q_2^{(u_i)}) = -\int_0^1 \frac{[-\ln(1-u)]^{i-1}}{\Gamma(i)} \ln(q_3^{(u_i)}(u)) \mathrm{d}u, \tag{14}$$

$$D_{KL}(q_1^{(l_i)} \| q_2^{(l_i)}) = -\int_0^1 \frac{[-\ln(u)]^{i-1}}{\Gamma(i)} \ln(q_3^{(l_i)}(u)) \mathrm{d}u,$$

where

$$q_3^{(u_i)}(u) = \frac{[-\ln(1-Q_3(u))]^{i-1} q_3(u)}{[-\ln(1-u)]^{i-1}}, \text{ and } q_3^{(l_i)}(u) = \frac{[-\ln(Q_3(u))]^{i-1} q_3(u)}{[-\ln(u)]^{i-1}}.$$

Proposition 3.1. *If* $Q_3(u) = 1 - (1-u)^\theta$. *Then,*

$$q_3^{(u_i)}(u) = \frac{[-\theta \ln(1-u)]^{i-1}}{[-\ln(1-u)]^{i-1}} \theta (1-u)^{\theta-1} = \big(\theta\big)^i (1-u)^{\theta-1}.$$

Hence, from (14) and substituting $-\ln(1-u)$ *with* φ *the quantile KL divergence function between the upper record value is obtained by*

$$\begin{aligned} D_{KL}(q_1^{(u_i)} \| q_2^{(u_i)}) &= -\int_0^1 \frac{[-\ln(1-u)]^{i-1}}{\Gamma(i)} \ln\big((\theta)^i (1-u)^{\theta-1}\big) \mathrm{d}u \\ &= -\int_0^\infty \frac{\varphi^{i-1}}{\Gamma(i)} \ln\big((\theta)^i e^{-\varphi(\theta-1)}\big) e^{-\varphi} \mathrm{d}\varphi = i\big(\theta - \ln(\theta) - 1\big). \end{aligned}$$

Remark 3.2. If $Q_3(u) = 1 - (1-u)^\theta$. Then,

- $D_{KL}(q_1^{(u_i)} \| q_2^{(u_i)})$ is a monotonous increasing function of i, $\forall\, i = 1, 2, ..., n$.
- $D_{KL}(q_1^{(u_i)} \| q_2^{(u_i)})$ is a monotonous increasing function of θ, $\forall\, \theta > 1$.
- $D_{KL}(q_1^{(u_i)} \| q_2^{(u_i)})$ is a monotonous decreasing function of θ, $\forall\, 0 < \theta < 1$.

Example 3.3. Assume that X and Y follow Cox ph model. Then under the assumption of relation (14), we have

$$q_3^{(u_i)}(u) = \frac{[-\ln(1-u)^\theta]^{i-1}\theta(1-u)^{\theta-1}}{[-\ln(1-u)]^{i-1}} = \theta^i(1-u)^{\theta-1},$$

and

$$q_3^{(l_i)}(u) = \frac{[-\ln(1-(1-u)^\theta]^{i-1}\theta(1-u)^{\theta-1}}{[-\ln(u)]^{i-1}}.$$

Then

$$\begin{aligned} D_{KL}(q_1^{(u_i)} \| q_2^{(u_i)}) &= -\int_0^1 \frac{[-\ln(1-u)]^{i-1}}{\Gamma(i)} \ln(\theta^i(1-u)^{\theta-1})\mathrm{d}u \\ &= -\frac{(-1)^{i-1}(\theta-1)}{\Gamma(i)} \int_0^1 \big(\ln(1-u)\big)^i \mathrm{d}u - i\ln(\theta). \end{aligned}$$

Furthermore, from $\int_0^1 \big(\ln(1-u)\big)^i \mathrm{d}u = i(-1)^i\Gamma(i)$, we would have

$$D_{KL}(q_1^{(u_i)} \| q_2^{(u_i)}) = -(-1)^{2i-1}(\theta-1) - i\ln(\theta) = i(\theta - 1 - \ln(\theta)). \quad (15)$$

Besides,

$$D_{KL}(q_1^{(l_i)} \| q_2^{(l_i)}) = -\int_0^1 \frac{[-\ln(u)]^{i-1}}{\Gamma(i)} \ln(q_3^{(l_i)}(u))\mathrm{d}u.$$

Example 3.4. In the case of prh model, we have

$$q_3^{(u_i)}(u) = \frac{[-\ln(1-u^\theta)]^{i-1}\theta u^{\theta-1}}{[-\ln(1-u)]^{i-1}},$$

and

$$q_3^{(l_i)}(u) = \frac{[-\ln(u^\theta)]^{i-1}\theta u^{\theta-1}}{[-\ln(u)]^{i-1}} = \theta^i u^{\theta-1}.$$

Then,

$$D_{KL}(q_1^{(u_i)} \| q_2^{(u_i)}) = -\int_0^1 \frac{[-\ln(1-u)]^{i-1}}{\Gamma(i)} \ln(q_3^{(u_i)}(u))\mathrm{d}u,$$

and

$$\begin{aligned} D_{KL}(q_1^{(l_i)} \| q_2^{(l_i)}) &= -\int_0^1 \frac{[-\ln(u)]^{i-1}}{\Gamma(i)} \ln(\theta^i u^{\theta-1})\mathrm{d}u \\ &= -\frac{(-1)^{i-1}(\theta-1)}{\Gamma(i)} \int_0^1 \big(\ln(u)\big)^i \mathrm{d}u - i\ln(\theta), \end{aligned}$$

where from $\int_0^1 \big(\ln(1-u)\big)^i \mathrm{d}u = i(-1)^i\Gamma(i)$ and similar to (15) we can get

$$D_{KL}(q_1^{(l_i)} \| q_2^{(l_i)}) = i(\theta - 1 - \ln(\theta)).$$

Proposition 3.5. *The KL divergence measure between i^{th} and j^{th} record values from an arbitrary distribution is given by*

$$D_{KL}(q_1^{(u_j)} \| q_1^{(u_i)}) = -\int_0^1 \frac{[-\ln(1-u)]^{j-1}}{\Gamma(j)} \ln(\frac{\Gamma(j)}{\Gamma(i)}\big[-\ln(1-u)\big]^{i-j})\mathrm{d}u,$$

$$D_{KL}(q_1^{(l_j)} \| q_1^{(l_i)}) = -\int_0^1 \frac{[-\ln(u)]^{j-1}}{\Gamma(j)} \ln(\frac{\Gamma(j)}{\Gamma(i)}\big[-\ln(u)\big]^{i-j})\mathrm{d}u,$$

Proof. The proof is clear. □

4 Application

In this section, we illustrate the previous examples of general data, Type-I censored variable and record values based on ph and prh models. Example 4.1 shows the KL measure on the rate values in ph and prh models for general data. Furthermore, Example 4.2 compares the KL divergence for the quantile censored points and the rate values in ph and prh models. Finally, Example 4.3 presents KL measure in ph and prh models on the upper and lower record values.

Example 4.1. Let X and Y are continuous r.v and θ=0.25, 0.75,...,4.25. Then, considering ph, and prh models, KL measure is calculated in Table 2. From the results of Table 2 it can be concluded that KL value is equal for upper and lower record values under the various values of θ.

Table 2: KL measure for ph and prh models with the various θ in range 0.25-4.2, with steps 0.45, corresponding to Example 4.1

θ	**0.25**	**0.70**	**1.15**	**1.50**	**1.95**	**2.40**	**2.85**	**3.30**	**3.75**	**4.20**
PH, PRH	0.6363	0.0567	0.0102	0.0945	0.2822	0.5245	0.8027	1.1061	1.4282	1.7649

Example 4.2. Table 3 shows KL measure for proportional hazard and proportional reversed hazard models on Type-I censored variable with the censored quantile points $P = 0.1, 0.3, ..., 0.9$ and taking $\theta = 0.25, 0.5, 2, 3$. Table 3 shows that KL value increases as the censored quantile point increases. Comparing the values of θ, it is suggested that KL measure increases whenever the distance between θ and 1 increases.

Table 3: LW divergence measure for ph and prh models of $\theta = 0.25, 0.5, 2, 3$, on Type-I censored variable with the censored quantile points $P = 0.1, 0.3, ..., 0.9$ in Example 4.2

Quantile Point:	$P = 0.1$	$P = 0.3$	$P = 0.5$	$P = 0.7$	$P = 0.9$
θ			**PH model**		
0.25	0.0636	0.1909	0.3181	0.4459	0.5727
0.50	0.0193	0.0579	0.0966	0.1352	0.1738
2	0.0307	0.0921	0.1534	0.2148	0.2762
3	0.0901	0.2704	0.4507	0.6310	0.8112
θ			**PRH model**		
0.25	0.5398	0.6135	0.6307	0.6354	0.6363
0.50	0.1515	0.1831	0.1907	0.1928	0.1931
2	0.1752	0.2696	0.2973	0.3053	0.3068
3	0.4567	0.7623	0.8640	0.8951	0.9012

Example 4.3. Let X be a continuous r.v from a distribution with c.d.f., $G(x)$, and pdf, $g(x)$. According to Examples 3.3 and 3.4 on ph and prh

models respectively, the KL divergence measure can be expressed by Table 4. As this table suggests, there is a direct relationship between the KL measure and the record values. We expect the KL value to increase as the distance between θ and 1 increases.

Table 4: LW divergence measure for upper and lower record values based on ph and prh models of $\theta = 0.25, 0.5, 2, 3$, in Example 4.3

		Upper record value		**Lower record value**	
i^{th} record value	θ	ph model	PRH model	ph model	PRH model
1	0.25	0.6363	0.6363	0.6363	0.6363
	0.50	0.1931	0.1931	0.1931	0.1931
	2	0.3069	0.3069	0.3069	0.3069
	3	0.9014	0.9014	0.9014	0.9014
2	0.25	1.2726	0.4628	0.4628	1.2726
	0.50	0.3863	0.1380	0.1380	0.3863
	2	0.6137	0.2132	0.2132	0.6137
	3	1.8028	0.6239	0.6239	1.8028
3	0.25	1.9089	0.3492	0.3492	1.9089
	0.50	0.5794	0.1017	0.1017	0.5794
	2	0.9206	0.1487	0.1487	0.9206
	3	2.7042	0.4280	0.4280	2.7042
4	0.25	2.5452	0.2735	0.2735	2.5452
	0.50	0.7726	0.0779	0.0779	0.7726
	2	1.2274	0.1078	0.1078	1.2274
	3	3.6056	0.3041	0.3041	3.6056

Remark 4.4. The KL integral for record data after value 4 can be calculated divergently.

5 Concluding Remarks

This article proposed an alternative approach to Kullback-Leibler divergence measurement using the quantile function on general type data, Type-I censored variable, and record values. This study presents the quantile Kullback-Leibler divergence value for the various well-known models such as hazard rate and proportional hazard rate models. Quantile function applications were examined for characteristic results on

Type-I censored variable and the distribution of record values. Various properties and boundaries of the quantile based Kullback-Leibler divergence have been derived. For this purpose, we focused on stochastic orders including stochastic dominance, likelihood ratio order, hazard rate and dispersive orders as well as hazard rate and reversed hazard rate situations on general type data and Type-I censored variable. We have shown that the quantile based Kullback-Leibler measure is a monotonously increasing function of probability P corresponding to the censored point and the distribution of the i^th upper and lower record values.

The proposed quantile-based Kullback-Leibler divergence approach has several limitations, including its complexity in implementation, reliance on specific data distribution assumptions, sensitivity to outliers, limited applicability to non-continuous data, and inadequate handling of various censoring types. Future research could focus on extending the method to other censoring types, developing robust versions to mitigate outliers sensitivity, adapting it for discrete data, conducting comparative studies with other divergence measures and creating efficient numerical estimation methods for large datasets.

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