Supercyclicity with Respect to a Sequence on Special Sequence Spaces

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Abstract. Let \( \{\beta(n)\}_{n=-\infty}^{\infty} \) be a sequence of positive numbers such that \( \beta(0) = 1 \) and let \( 1 < p < \infty \). We consider the space of all formal Laurent series \( f(z) = \sum_{n=-\infty}^{\infty} \hat{f}(n)z^n \) such that

\[
\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^p \beta(n)^p < \infty.
\]

We investigate the supercyclicity with respect to a sequence on the Banach spaces of formal Laurent series.

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1. Introduction

Let \( \{\beta(n)\}_{n=-\infty}^{\infty} \) be a sequence of positive numbers with \( \beta(0) = 1 \) and \( 1 < p < \infty \). Consider the space of \( f(z) = \sum_{n=-\infty}^{\infty} \hat{f}(n)z^n \) such that

\[
\|f\|_p^p = \|f\|_\beta^p = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^p \beta(n)^p < \infty.
\]
They are called formal Laurent series and the space of such formal Laurent series is denoted by $L^p(\beta)$. These are reflexive Banach spaces with the norm $\|\cdot\|_\beta$. The operator $B$ on $L^p(\beta)$ is defined by $Bf_j = f_{j-1}$ for all $j \in \mathbb{Z}$. Clearly $B$ is bounded if and only if the sequence $\{\beta(k)/\beta(k+1)\}_k$ is bounded.

Let $X$ be a complex Banach space and $B(X)$ be the set of bounded linear operators from $X$ into itself. If $T \in B(X)$, then the orbit of a vector $x \in X$ is the set

$$Orb(T, x) = \{T^n x : n \in \mathbb{N} \cup \{0\}\}.$$ 

A vector $x \in X$ is called hypercyclic for $T$ if $Orb(T, x)$ is dense in $X$. The operator $T$ is called hypercyclic if it has a hypercyclic vector. A vector $x \in X$ is said to be cyclic for an operator $T \in B(X)$ if the linear span of $Orb(T, x)$ is dense in $X$. Also a vector $x \in X$ is called a supercyclic vector for an operator $T \in B(X)$ if the set

$$\{\lambda y : y \in Orb(T, x), \lambda \in \mathbb{C}\}$$

is dense in $X$. An operator $T \in B(X)$ is cyclic (supercyclic) if it has a cyclic (a supercyclic) vector. It is evident that hypercyclicity implies supercyclicity and this, in turn, implies cyclicity.

Sources on formal series include [7, 11, 12, 14, 17]. Also, hypercyclicity and supercyclicity have been studied in several works (see [1, 2, 3, 5, 7, 8, 9, 10, 13, 15, 16, 18, 19, 20]).
We will investigate the supercyclicity with respect to a sequence on the Banach spaces of formal Laurent series.

2. Main Result

Supercyclicity was introduced by Hilden and Wallen ([6]). They showed that all unilateral backward weighted shifts are supercyclic, but there does not exist a vector that is supercyclic vector for all the unilateral backward weighted shifts. H. Salas ([10]) gives a condition for supercyclicity in Frechet spaces.

We can extend the notions to sequences of linear operators; let \(\{n_k\}\) be an increasing sequence of nonnegative integers. Then the sequence \(\{T_{n_k}\}_{k\geq0}\) of bounded linear operators from a complex Banach space \(X\) into itself is hypercyclic (supercyclic) if there exists \(x \in X\) such that the orbit \(\{T_{n_k}x\}_{k\geq0}\) (\(\{\lambda T_{n_k}x : k \in \mathbb{N} \cup \{0\}, \lambda \in \mathbb{C}\}\)) is dense in \(X\). In the special case when \(T \in B(X)\) and the sequence \(\{T^{n_k}\}_{k\geq0}\) is hypercyclic (supercyclic), we say that the operator \(T\) is hypercyclic (supercyclic) with respect to the sequence \(\{n_k\}\). Here we will investigate the supercyclicity of the operator \(B\) with respect to a sequence on the Banach spaces of formal Laurent series.

Suppose that \(B\) is bounded on \(L^p(\beta)\) and \(\{n_k\}\) is an increasing sequence of nonnegative integers. For investigation about the supercyclicity of the sequence \(\{B^{n_k}\}_k\), we need the following lemma.
Lemma 1. Let $E$ be a normed space and $T$ be a bounded linear operator on $E$. Then the sequence $\{T^{nk}\}$ is supercyclic if and only if the set

$$\{ (x, \lambda T^{nk} x) : x \in E, \lambda \in Q + iQ, k \in \mathbb{N} \}$$

is dense in $E \times E$.

Proof. The proof is similar to the proof of Theorem 1.2.2 in [4, page 11] and so we omit it. □

Theorem 2. The sequence $\{B^{ni}\}_i$ is supercyclic on $L^p(\beta)$ if and only if

$$\liminf_{i \to \infty} \max \left\{ \frac{\beta(j - n_i)\beta(k + n_i)}{\beta(j)\beta(k)} : |j| \leq n_m, |k| \leq n_m \right\} = 0$$

for all $m \in \mathbb{N}$.

Proof. Let $0 < \varepsilon < 1$ and $m \in \mathbb{N}$. Choose $\alpha > 0$ such that $\frac{\alpha}{1-\alpha} < \varepsilon^\frac{1}{2}$. Let

$$y = w = \sum_{|j| \leq n_m} f_j / \beta(j)$$

be in $L^p(\beta)$. Suppose $\{B_{ni}\}_i$ is supercyclic. Then by Lemma 1 there exists an arbitrary large $i > m$, a vector

$$x = \sum_n \hat{x}(j)f_j$$

in $L^p(\beta)$, and a complex number $\lambda$ such that $\|x - w\| < \alpha$ and $\|\lambda B^{n_i}x - w\| < \varepsilon$. □
\( \|x - w\|^p = \sum_{|j| \leq n_m} |\hat{x}(j)\beta(j) - 1|^p + \sum_{|j| > n_m} |\hat{x}(j)|^p \beta(j)^p \)
\( < \alpha^p. \)

Thus
\[
|\hat{x}(j)|\beta(j) > 1 - \alpha, \quad |j| \leq n_m \tag{1}
\]
\[
|\hat{x}(j)|\beta(j) < \alpha, \quad |j| > n_m. \tag{2}
\]

Also since
\[
\|\lambda B^nx - y\|^p = \sum_{|k| \leq n_m} |\lambda \hat{x}(k + n_i)\beta(k) - 1|^p
\]
\[
+ \sum_{|k| > n_m} |\lambda|^p |\hat{x}(k + n_i)|^p \beta(k)^p < \alpha^p,
\]

we have
\[
|\lambda \hat{x}(k + n_i)\beta(k) - 1| < \alpha, \quad |k| \leq n_m \tag{3}
\]
\[
|\lambda| |\hat{x}(k + n_i)|\beta(k) < \alpha, \quad |k| > n_m. \tag{4}
\]

Note that \( j - n_i < -n_m \) for \( |j| \leq n_m \), so by (1) and (4) we have
\[
\frac{\beta(j - n_i)}{\beta(j)} < \frac{1}{|\lambda|} \frac{\alpha}{1 - \alpha}
\]

for \( |j| \leq n_m \). Also since \( k + n_i > n_m \) for \( |k| \leq n_m \), by (2) the relation
\[
|\hat{x}(k + n_i)| < \frac{\alpha}{\beta(k + n_i)}
\]
is consistent and so by (3) we get
\[ \frac{\beta(k + n_i)}{\beta(k)} < |\lambda| \frac{\alpha}{1 - \alpha} \]
for \(|k| \leq n_m\). Therefore
\[ \frac{\beta(j - n_i)\beta(k + n_i)}{\beta(j)\beta(k)} < (\frac{\alpha}{1 - \alpha})^2 < \varepsilon \]
for all \(-n_m \leq j, k \leq n_m\) and \(i > m\) arbitrarily large enough.

Conversely suppose that \(\varepsilon > 0\) is given and consider
\[ y = \sum_{|j| \leq n_m} \hat{g}(j)f_j \]
and
\[ w = \sum_{|j| \leq n_m} \hat{w}(j)f_j \]
in \(L^p(\beta)\) such that both are different from zero. By Lemma 1, it is sufficient to find \(x \in L^p(\beta)\) and \(i \in \mathbb{N}\) such that \(\|x - y\| \leq \varepsilon\) and \(\|\lambda B^{n_i}x - w\| \leq \varepsilon\) for some \(\lambda \in \mathbb{C}\). Let
\[ S^{n_i}w = \sum_{|k| \leq n_m} \hat{w}(k)f_{k + n_i}, \]
where \(i \in \mathbb{N}\). Also let
\[ x = y + \frac{1}{\lambda} S^{n_i}w \]
with \(i\) to be determined but \(\|\frac{1}{\lambda} S^{n_i}w\| = \varepsilon\). Note that
\[ B^{n_i}x = B^{n_i}y + \frac{1}{\lambda}w. \]
Thus it suffices to find $i$ such that $\|\lambda B^{n_i} y\| < \varepsilon$. We have
\[
\|\lambda B^{n_i} x - w\|^p = \|\lambda B^{n_i} y\|^p
\]
\[
= \|B^{n_i} y\|^p \|S^{n_i} w\|^p / \varepsilon^p
\]
\[
= \| \sum_{|j| \leq n_m} \hat{y}(j) f_{j-n_i} \|^p \| \sum_{|k| \leq n_m} \hat{w}(k) f_{k+n_i} \|^p / \varepsilon^p
\]
\[
= \left( \sum_{|j| \leq n_m} |\hat{y}(j)|^p \beta(j-n_i)^p \right)
\times \left( \sum_{|k| \leq n_m} |\hat{w}(k)|^p \beta(k+n_i)^p \right) / \varepsilon^p.
\]
So we get
\[
\|\lambda B^{n_i} x - w\| \leq \max \left\{ \frac{\beta(j-n_i) \beta(k+n_i)}{\beta(j) \beta(k)} : |k| \leq n_m, |j| \leq n_m \right\}
\times \| y \| \| w \| / \varepsilon
\]
and consequently by our hypothesis there exists $i$ large enough such that $\|\lambda B^{n_i} x - w\| < \varepsilon$. This completes the proof. □

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