Journal of Mathematical Extension
Vol. 1, N0. 2, (2007), 117-125

# Supercyclicity with Respect to a Sequence on Special Sequence Spaces

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**Abstract.** Let $\{\beta(n)\}_{n=-\infty}^{\infty}$ be a sequence of positive numbers such that $\beta(0) = 1$ and let $1 < p < \infty$. We consider the space of all formal Laurent series $f(z) = \sum_{n=-\infty}^{\infty} \hat{f}(n)z^n$ such that

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^p \beta(n)^p < \infty.$$

We investigate the supercyclicity with respect to a sequence on the Banach spaces of formal Laurent series.

**AMS Subject Classification:** 47B37; 47A16.
**Keywords and Phrases:** Banach space of Laurent series associated with a sequence $\beta$, supercyclic vector, shift operator.

## 1. Introduction

Let $\{\beta(n)\}_{n=-\infty}^{\infty}$ be a sequence of positive numbers with $\beta(0) = 1$ and $1 < p < \infty$. Consider the space of $f(z) = \sum_{n=-\infty}^{\infty} \hat{f}(n)z^n$ such that

$$\|f\|^p = \|f\|_\beta^p = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^p \beta(n)^p < \infty.$$

They are called formal Laurent series and the space of such formal Laurent series is denoted by $L^p(\beta)$. These are reflexive Banach spaces with the norm $\|\cdot\|_\beta$. The operator $B$ on $L^p(\beta)$ is defined by $Bf_j = f_{j-1}$ for all $j \in \mathbb{Z}$. Clearly $B$ is bounded if and only if the sequence $\{\beta(k)/\beta(k+1)\}_k$ is bounded.

Let $X$ be a complex Banach space and $B(X)$ be the set of bounded linear operators from $X$ into itself. If $T \in B(X)$, then the orbit of a vector $x \in X$ is the set

$$Orb(T, x) = \{T^n x : \; n \in \mathbb{N} \cup \{0\}\}.$$

A vector $x \in X$ is called hypercyclic for $T$ if $Orb(T, x)$ is dense in $X$. The operator $T$ is called hypercyclic if it has a hypercyclic vector. A vector $x \in X$ is said to be cyclic for an operator $T \in B(X)$ if the linear span of $Orb(T, x)$ is dense in $X$. Also a vector $x \in X$ is called a supercyclic vector for an operator $T \in B(X)$ if the set

$$\{\lambda y : y \in Orb(T, x), \; \lambda \in \mathbb{C}\}$$

is dense in $X$. An operator $T \in B(X)$ is cyclic (supercyclic) if it has a cyclic (a supercyclic) vector. It is evident that hypercyclicity implies supercyclicity and this, in turn, implies cyclicity.

Sources on formal series include [7, 11, 12, 14, 17]. Also, hypercyclicity and supercyclicity have been studied in several works (see [1, 2, 3, 5, 7, 8, 9, 10, 13, 15, 16, 18, 19, 20]).

We will investigate the supercyclicity with respect to a sequence on the Banach spaces of formal Laurent series.

## 2. Main Result

Supercyclicity was introduced by Hilden and Wallen ([6]). They showed that all unilateral backward weighted shifts are supercyclic, but there does not exist a vector that is supercyclic vector for all the unilateral backward weighted shifts. H. Salas ([10]) gives a condition for supercyclicity in Frechet spaces.

We can extend the notions to sequences of linear operators; let $\{n_k\}$ be an increasing sequence of nonnegative integers. Then the sequence $\{T_{n_k}\}_{k\geqslant 0}$ of bounded linear operators from a complex Banach space $X$ into itself is hypercyclic (supercyclic) if there exists $x \in X$ such that the orbit $\{T_{n_k}x\}_{k\geqslant 0}$ ($\{\lambda T_{n_k}x : k \in \mathbb{N}\cup\{0\},\ \lambda \in \mathbb{C}\}$) is dense in $X$. In the special case when $T \in B(X)$ and the sequence $\{T^{n_k}\}_{k\geqslant 0}$ is hypercyclic (supercyclic), we say that the operator $T$ is hypercyclic (supercyclic) with respect to the sequence $\{n_k\}$. Here we will investigate the supercyclicity of the operator $B$ with respect to a sequence on the Banach spaces of formal Laurent series.

Suppose that $B$ is bounded on $L^p(\beta)$ and $\{n_k\}$ is an increasing sequence of nonnegative integers. For investigation about the supercyclicity of the sequence $\{B^{n_k}\}_k$, we need the following lemma.

**Lemma 1.** *Let $E$ be a normed space and $T$ be a bounded linear operator on $E$. Then the sequence $\{T^{n_k}\}$ is supercyclic if and only if the set*

$$\{(x, \lambda T^{n_k} x) : x \in E, \lambda \in Q + iQ,\ k \in \mathbb{N}\}$$

*is dense in $E \times E$ .*

**Proof.** The proof is similar to the proof of Theorem 1.2.2 in [4, page 11] and so we omit it. □

**Theorem 2.** *The sequence $\{B^{n_i}\}_i$ is supercyclic on $L^p(\beta)$ if and only if*

$$\liminf_{i\to\infty} \max\left\{\frac{\beta(j-n_i)\beta(k+n_i)}{\beta(j)\beta(k)} : |j| \leqslant n_m, |k| \leqslant n_m\right\} = 0$$

*for all $m \in \mathbb{N}$.*

**Proof.** Let $0 < \varepsilon < 1$ and $m \in \mathbb{N}$. Choose $\alpha > 0$ such that $\frac{\alpha}{1-\alpha} < \varepsilon^{\frac{1}{2}}$. Let

$$y = w = \sum_{|j|\leqslant n_m} f_j/\beta(j)$$

be in $L^p(\beta)$. Suppose $\{B_{n_i}\}_i$ is supercyclic. Then by Lemma 1 there exists an arbitrary large $i > m$, a vector

$$x = \sum_n \hat{x}(j) f_j$$

in $L^p(\beta)$, and a complex number $\lambda$ such that $\|x - w\| < \alpha$ and $\|\lambda B^{n_i} x -$

$y\| < \alpha$. Note that $\lambda \neq 0$. Therefore,

$$\begin{aligned}\|x - w\|^p &= \sum_{|j| \leqslant n_m} |\hat{x}(j)\beta(j) - 1|^p + \sum_{|j| > n_m} |\hat{x}(j)|^p \beta(j)^p \\ &< \alpha^p.\end{aligned}$$

Thus

$$|\hat{x}(j)|\beta(j) > 1 - \alpha, \qquad |j| \leqslant n_m \tag{1}$$

$$|\hat{x}(j)|\beta(j) < \alpha, \qquad |j| > n_m. \tag{2}$$

Also since

$$\begin{aligned}\|\lambda B^{n_i} x - y\|^p &= \sum_{|k| \leqslant n_m} |\lambda \hat{x}(k + n_i)\beta(k) - 1|^p \\ &\quad + \sum_{|k| > n_m} |\lambda|^p |\hat{x}(k + n_i)|^p \beta(k)^p < \alpha^p,\end{aligned}$$

we have

$$|\lambda \hat{x}(k + n_i)\beta(k) - 1| < \alpha, \qquad |k| \leqslant n_m \tag{3}$$

$$|\lambda||\hat{x}(k + n_i)|\beta(k) < \alpha, \qquad |k| > n_m. \tag{4}$$

Note that $j - n_i < -n_m$ for $|j| \leqslant n_m$, so by (1) and (4) we have

$$\frac{\beta(j - n_i)}{\beta(j)} < \frac{1}{|\lambda|} \frac{\alpha}{1 - \alpha}$$

for $|j| \leqslant n_m$. Also since $k + n_i > n_m$ for $|k| \leqslant n_m$, by (2) the relation

$$|\hat{x}(k + n_i)| < \frac{\alpha}{\beta(k + n_i)}$$

is consistent and so by (3) we get

$$\frac{\beta(k+n_i)}{\beta(k)} < |\lambda|\frac{\alpha}{1-\alpha}$$

for $|k| \leqslant n_m$. Therefore

$$\frac{\beta(j-n_i)\beta(k+n_i)}{\beta(j)\beta(k)} < (\frac{\alpha}{1-\alpha})^2 < \varepsilon$$

for all $-n_m \leqslant j, k \leqslant n_m$ and $i > m$ arbitrarily large enough.

Conversely suppose that $\varepsilon > 0$ is given and consider

$$y = \sum_{|j| \leqslant n_m} \hat{y}(j) f_j$$

and

$$w = \sum_{|j| \leqslant n_m} \hat{w}(j) f_j$$

in $L^p(\beta)$ such that both are different from zero. By Lemma 1, it is sufficient to find $x \in L^p(\beta)$ and $i \in \mathbb{N}$ such that $\|x - y\| \leqslant \varepsilon$ and $\|\lambda B^{n_i} x - w\| \leqslant \varepsilon$ for some $\lambda \in \mathbb{C}$. Let

$$S^{n_i} w = \sum_{|k| \leqslant n_m} \hat{w}(k) f_{k+n_i},$$

where $i \in \mathbb{N}$. Also let

$$x = y + \frac{1}{\lambda} S^{n_i} w$$

with $i$ to be determined but $\|\frac{1}{\lambda} S^{n_i} w\| = \varepsilon$. Note that

$$B^{n_i} x = B^{n_i} y + \frac{1}{\lambda} w.$$

Thus it suffices to find $i$ such that $\|\lambda B^{n_i} y\| < \varepsilon$. We have

$$\begin{aligned}
\|\lambda B^{n_i}x - w\|^p &= \|\lambda B^{n_i}y\|^p \\
&= \|B^{n_i}y\|^p \|S^{n_i}w\|^p/\varepsilon^p \\
&= \| \sum_{|j|\leqslant n_m} \hat{y}(j) f_{j-n_i}\|^p . \, \| \sum_{|k|\leqslant n_m} \hat{w}(k) f_{k+n_i}\|^p/\varepsilon^p \\
&= \left( \sum_{|j|\leqslant n_m} |\hat{y}(j)|^p \beta(j-n_i)^p \right) \\
&\quad \times \left( \sum_{|k|\leqslant n_m} |\hat{w}(k)|^p \beta(k+n_i)^p \right) /\varepsilon^p .
\end{aligned}$$

So we get

$$\begin{aligned}
\|\lambda B^{n_i}x - w\| &\leqslant \max\left\{ \frac{\beta(j-n_i)\beta(k+n_i)}{\beta(j)\beta(k)} : |k| \leqslant n_m, |j| \leqslant n_m \right\} \\
&\quad . \|y\| \ \|w\|/\varepsilon
\end{aligned}$$

and consequently by our hypothesis there exists $i$ large enough such that $\|\lambda B^{n_i}x - w\| < \varepsilon$. This completes the proof. □

## Acknowledgment

The third author, thanks the Research Council of Islamic Azad University-Shiraz Branch.

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