Journal of Mathematical Extension
Vol. 18, No. 11, (2024) (6) 1-14
ISSN: 1735-8299
URL: http://doi.org/10.30495/JME.2024.3109
Original Research Paper

A Power Mapping and its Fixed Points

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Abstract. In this paper, we show that for a nonempty finite set U and a power mapping $T : U \longrightarrow \mathcal{P}^*(U)$, where $\mathcal{P}^*(U)$ is set of all nonempty subsets of U, there exists a nonempty subset F of U such that $T'(F) = F$ and for any $K \subseteq U$, $T'(K) = \bigcup_{x \in K} T(x)$. Also for a power mapping $T : U \longrightarrow \mathcal{P}(U)$, we get an equivalent condition for having a nonempty fixed points set. Finally, we present a method to obtain all of fixed points of T.

AMS Subject Classification: 05C09, 05C15, 05C76.
Keywords and Phrases: Power mapping, Fixed point, Fower set

1 Introduction

In mathematics, a fixed-point theorem is a result saying that a function F will have at least one fixed point (a point x for which $F(x) = x$), under some conditions on F that can be stated in general terms. In the other words, a fixed point, also known as an invariant point, is a value that does not change under a given transformation. Any set of fixed points of a transformation is also an invariant set. There exist various types of fixed point theorems, such as Brouwer fixed point theorem, Atiyah-Bott fixed point theorem, Banach fixed point theorem, etc [3, 2, 11, 9, 10].

Received: July 2024; Accepted: February 2025
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A power mapping is a mathematical function that maps elements from one set, the domain of the function, to subsets of another set. Power mappings are used in a variety of mathematical fields, including optimization, control theory and game theory [1, 8, 14, 15].

Let U is a nonempty finite set and $T : U \longrightarrow \mathcal{P}^*(U)$ is a power mapping, then there exists a nonempty subset F of U such that $T'(F) = F$. Also for a power mapping $T : U \longrightarrow \mathcal{P}(U)$, we get a necessary and sufficient condition for having a nonempty fixed point. We know that it is very hard to calculate fixed points by direct method, for example if $|U| = 100$, then to determine all the fixed points necessary to check the number of $|\mathcal{P}^*(U)| = 2^{100} - 1 \approx 1.26765 \times 10^{30}$ subsets of U. In this paper, we present a new method to obtain all fixed points of the power mapping $T : U \longrightarrow \mathcal{P}(U)$, where U is a finite set. In addition, we state and investigate the three main fixed point theorems on the power mappings.

2 Basic Concepts

In this paper, we introduce a new concept of a power mapping. Let U and W be sets and $T : U \longrightarrow \mathcal{P}(W)$ be a mapping, then we call T a *power mapping*, where $\mathcal{P}(W)$ is set of all subsets of W. We denote the set of all power mapping from U in to $\mathcal{P}(W)$ by $\langle U, W\rangle_0 = \mathcal{M}ap(U, \mathcal{P}(W))$. Also, we set $\langle U, W\rangle = \mathcal{M}ap(U, \mathcal{P}^*(W))$, where $\mathcal{P}^*(W)$ is set of all non-empty subsets of W. We write for simplicity $\langle U, U\rangle_0 = \langle U\rangle_0$ and $\langle U, U\rangle = \langle U\rangle$. This type of mappings, appear in many mathematical theories, such as algebraic hyperstructures theory, T-rough sets theory numbers theory and graph theory [6, 5, 12, 13, 7, 4, 5].

In this section, we present several examples about fixed points of finite algebraic hyperstructures.

Example 2.1.

1. Let G be a graph and g is a vertex of G, then $T : G \longrightarrow \mathcal{P}(G)$ is a power mapping, where $T(g) = V_g$ and $V_g = \{x : g \text{and} x \text{are adjacent vertices}\}$.
2. Let $\mathbb{N} = \{1, 2, 3, \cdots\}$ be the set of natural numbers. Then the mapping $T : \mathbb{N} \longrightarrow \mathcal{P}^*(\mathbb{N})$, where $T(n) = \{d \in \mathbb{N} \ : \ d|n\}$ is a power mapping.
3. The mapping $\Phi : \mathbb{N} \longrightarrow \mathcal{P}^*(\mathbb{N})$, where $\Phi(n) = \{m \in \mathbb{N} : m \leqslant n \text{ and} (m, n) = 1\}$ is a power mapping.
4. Let $(A, \cdot)$ be a finite algebraic structure. For any $u \in A$ define $T_u(a) = \{x \in A \ : \ a \cdot x = u\}$. Then T is a power mapping.
5. Let $(H, \star)$ be a finite hyperstructure and $h \in H$. Then $T_h : H \longrightarrow \mathcal{P}^*(H)$ is a power mapping, where $T_h(a) = a \star h$.
6. Let (X, τ) be a topological space and $u \in X$. Suppose that for any $x \in X$, $T_u(x)$ is an open subset, such as $O_x \in \tau$, such that $x \in O_x$ and $y \notin O_x$. Then T_u is a power mapping.

Now, we show, if U is a nonempty finite set and $T : U \longrightarrow \mathcal{P}^*(U)$, where $\mathcal{P}^*(U)$ is set of all non-empty subsets of U is a power mapping, then there exists a nonempty subset F of U such that $T'(F) = F$.

Definition 2.2. Let U and W be sets, and $T : U \longrightarrow \mathcal{P}(W)$ be a power mapping. We define $T' : \mathcal{P}(U) \longrightarrow \mathcal{P}(W)$, such that $T'(K) = \bigcup_{x \in K} T(x)$.

It is clear that $T'(\emptyset) = \emptyset$, and $T_1 = T_2$ if and only if $T_1' = T_2'$.

Definition 2.3. Let U be a set, $T \in \langle U \rangle_0$ and K be a subset of U.

1. A subset F of U is called a fixed point of T, if and only if $T'(F) = F$.
2. $fix_K(T) = \{K_0 \subseteq K \mid K_0 \neq \emptyset \text{ and } \quad T'(K_0) = K_0\}$.
3. $Fix_K(T) = \{K_0 \subseteq K \mid T'(K_0) = K_0\}$.
4. $fix(T) = fix_U(T)$.
5. $Fix(T) = Fix_U(T)$.
6. If $F \subseteq U$ is a fixed point of T and for any $x \in F$, $T(x) \neq \emptyset$, then F is called a *normal fixed point* of T.
7. If F is not a normal fixed point of T, then F is called an *abnormal fixed point*.

Example 2.4. Let $U = \{x, y, z, u, v, w, t\}$ and

$$T : \begin{cases} x \to \{y\}, \\ y \to \{x, w\}, \\ z \to \{u, w\}, \\ u \to \{z\}, \\ v \to \{w\}, \\ w \to \{y\}, \\ t \to \{z\}. \end{cases}$$

Then $T'(\{x, y, w\}) = \{x, y, w\}$, so $\{x, y, w\}$ is a fixed point of T.

Definition 2.5. Let $T \in \langle U \rangle$ and F be a fixed point of T.

F is called a *minimal fixed point*, if $F \neq \emptyset$ and if F' is a nonempty fixed point of T and $F' \subseteq F$, then $F' = F$. F is called a *maximal fixed point* of T if there is not a fixed point such as F' such that $F \subset F'$.

Definition 2.6. Let $T : U \longrightarrow \mathcal{P}(U)$ be a power mapping and $K \subseteq U$. Then

(1) $(T')^0(K) = K$.

(2) $(T')^1(K) = T'(K)$.

(3) $(T')^{n+1}(K) = T'((T')^n(K))$ for $n \in \mathbb{N}_0$. We write for simplicity $(T')^n$ by T'^n and $T'^n(\{x_1, x_2, \cdots, x_n\})$ by $T'^n(x_1, x_2, \cdots, x_n)$ for any $x_1, x_2, \cdots, x_n \in U$.

3 Fixed Point Theorem

In this section, we will prove the main theorem 1 on a power mapping.

Lemma 3.1. *Let $T \in \langle U \rangle_0$.*

(1) If $K \subseteq K'$ and $n \in \mathbb{N}_0$, then $T'^n(K) \subseteq T'^n(K')$.

(2) If for any $\alpha \in I$, $K_\alpha \subseteq U$ and $n \in \mathbb{N}_0$, then $T'^n(\bigcup_{\alpha \in I} K_\alpha) = \bigcup_{\alpha \in I} T'^n(K_\alpha)$.

Proof. It is straitforward. □

Lemma 3.2. *Let $T \in \langle U \rangle_0$ and $K \subseteq U$. Then the $Fix_K(T)$ is closed under operation $\bigcup$.*

Proof. It follows from Lemma 3.1. □

Example 3.3. Suppose that $U = \{x, y, z, t\}$ and $T \in \langle U \rangle$, such that

$$T(u) = \begin{cases} \{x, y\} & u = x, \\ \{z\} & u = y, \\ \{z\} & u = z, \\ \{y, t\} & u = t. \end{cases} \tag{1}$$

Then $\{x, y, z\}$ and $\{y, z, t\}$ are fixed points, so $U = \{x, y, z\} \bigcup \{y, z, t\}$ is a fixed point by Lemma 3.2, but $\{y, z\} = \{z, y, z\} \bigcap \{y, z, t\}$ is not a fixed point.

Lemma 3.4. *Let U be a nonempty set and $T \in \langle U \rangle$. If there exists a nonempty finite subset K of U such that $K \supseteq T'(K)$, then there exists a nonempty subset K_0 of K such that $T'(K_0) = K_0$.*

Proof. Since $T'(K) \subseteq K$, therefore

$$K \supseteq T'(K) \supseteq (T')^2(K) \supseteq \cdots (T')^n(K) \supseteq \cdots \supset \emptyset, \tag{2}$$

by Lemma 3.1. Since K is finite, so in (2), there exists a $n \in \mathbb{N}_0$ such that $(T')^n(K) = (T')^{n+1}(K)$. Set $K_0 = (T')^n(K)$, then K_0 is a nonempty subset of K such that $T'(K_0) = K_0$. □

Lemma 3.5. *Let U be a nonempty finite set and $T \in \langle U \rangle_0$. If there exists a nonempty subset $K \subseteq U$ such that $K \subseteq T'(K)$, then there exists a subset K_1 of U such that $K_1 \supseteq K$ and $T'(K_1) = K_1$.*

Proof. We have

$$K \subseteq T'(K) \subseteq (T')^2(K) \subseteq \cdots \subseteq U, \tag{3}$$

by Lemma 3.1. Since U is a finite set, hence the chain (3) has a finite length, hence there exists a $n \in \mathbb{N}_0$, such that $(T')^n(K) = (T')^{n+1}(K)$. Set $K_1 = (T')^n(K)$, then $T'(K_1) = K_1$. □

Theorem 3.6 (Main Theorem 1: Fixed Point Theorem)**.** *If U is a nonempty finite set and $T \in \langle U \rangle$, then there exists a nonempty subset F of U such that $T'(F) = F$.*

Proof. Since $T'(U) \subseteq U$, hence the claim follows from Lemma 3.4. $\square$

If the conditions of Fixed Point Theorem are not established, then it is not necessary that the assertion to be established. See the following examples.

Example 3.7.
(1) Suppose that $U = \{x, y\}$, $T(x) = \{y\}$ and $T(y) = \emptyset$. Then T does not have any nonempty fixed point.
(2) Let $\mathbb{N}$ be the set of natural numbers, and $T : \mathbb{N} \longrightarrow \mathcal{P}^*(\mathbb{N})$ be a power mapping, where $T(n) = \{n + 1\}$. Then T does not have any nonempty fixed point.

Example 3.8. Suppose that $(H, \star)$ is a finite algebraic hyperstructure. for any $h \in H$ define $T_h(x) = h \star x$. Then for any $h \in H$, there exists a nonempty subset F_h of H, such that $T'_h(F_h) = F_h$, by Fixed Point Theorem.

4 Obtaining the Fixed Points

Suppose that $T : U \longrightarrow \mathcal{P}^*(U)$ is a power mapping, therefore $T' : \mathcal{P}^*(U) \longrightarrow \mathcal{P}^*(U)$ is a mapping. If we want to obtain all of fixed points, directly, then it means that we have to test $2^{|U|} - 1$ members of $\mathcal{P}^*(U)$, this is a very difficult task.

Definition 4.1. Let $T \in \langle U \rangle_0$ and K be a subset of U. K is called a *T-small subset* (or a *small subset* of U), if $K \subseteq T'(K)$. K is called a *T-big subset* (or a *big subset of* U), if $K \supseteq T'(K)$. The subset K of U is called a *T-normal subset* (or a *normal subset of* U) if K is a T-small subset or a T-big subset of U.

Example 4.2. Suppose that $U = \{x, y, z, u, v, w, t\}$ and

$$T = \begin{cases} x \longrightarrow \{y\}, \\ y \longrightarrow \{y, z\}, \\ z \longrightarrow \{z\}, \\ u \longrightarrow \{u, v\}, \\ v \longrightarrow \{u\}, \\ w \longrightarrow \{u\}, \\ t \longrightarrow \emptyset. \end{cases}$$

We have $T'(\{x, y, z\}) = \{y, z\}$, hence $\{x, y, z\}$ is a big subset of U. Since $T'(\{u\}) = \{u, v\}$, hence $\{u\}$ is a small subset of U. Therefore the sets $\{x, y, z\}$ and $\{u\}$ are normal subsets of U. Also $\{z\}$ is a normal subset of U. The sets $\{v\}$, $\{w\}$, $\{v, w\}$, and $\{v, w, t\}$ are not a normal subset of U.

According to Lemma 3.4, Lemma 3.5, and Definition 4.1, we have:

Corollary 4.3. *Let $T \in \langle U \rangle$, then*

(1) *any nonempty finite big subset of U, contains a nonempty fixed point;*

(2) *if U is a finite set, then any nonempty small subset is contained in a nonempty fixed point.*

Definition 4.4. Let $T \in \langle U \rangle_0$ and K be a normal subset of U. If n is the smallest number such that $(T')^n(K) = (T')^{n+1}(K)$, then we write $o_T(K) = n$.

Lemma 4.5. *Let U be a finite set, $T \in \langle U \rangle_0$ and K be a subset of U, then*

(1) *if K is a big subset of U, then $o_T(K) \leqslant |K|$;*

(2) *if K is a small subset of U, then $o_T(K) \leqslant |K^c|$.*

Proof. It is obvious. □

Lemma 4.6. *Let U be a nonempty finite set and K be a normal subset of U.*

(1) *If $T \in \langle U \rangle$, then $T^{o_T(K)}(K)$ is a nonempty fixed point.*

(2) *If $T \in \langle U \rangle_0$, then $T^{o_T(K)}(K)$ is a fixed point.*

Proof. It is obvious. □

Example 4.7. Let $U = \{x, y, z, u, v, w\}$. Define

$$T : \begin{cases} x \to \{y\}, \\ y \to \{x\}, \\ z \to \{z, u\}, \\ u \to \{z\}, \\ v \to \{v\}, \\ w \to \{u\}. \end{cases} \qquad \Lambda : \begin{cases} x \to \{x, y, z\}, \\ y \to \emptyset, \\ z \to \{x\}, \\ u \to \{v\}, \\ v \to \emptyset, \\ w \to \{v, w\}. \end{cases}$$

Then $\{z\}$ is a $T-$small subset. So $\{z\} \subset \{z, u\} = T'(z) = (T')^2(z)$. Therefore $o_T(\{z\}) = 1$ and $\{z, u\}$ is a fixed point by Lemma 4.6. Also $\{z.u.w\}$ is a $T-$big subset and $o_T(\{z, u, w\}) = 1$.

The subset $\{x, y\}$ is a $\Lambda-$small subset, and $o_\Lambda(\{x, y\}) = 1$. Thus $\Lambda'\{x, y\} = \{x, y, z\}$ is a nonempty fixed point, by Lemma 4.6. Also $\{u, v, w\}$ is a $\Lambda-$big subset and $o_\Lambda(\{u, v, w\}) = 1$. We have $\Lambda'\{u.v, w\} = \{v, w\}$.

Definition 4.8. Let $T \in \langle U \rangle_0$, S is a small subset and B is a big subset of U. We define $(T')^\infty(S) = \bigcup_{n \geqslant 0}(T')^n(S)$ and $(T')^\infty(B) = \bigcap_{n \geqslant 0}(T')^n(B)$.

Corollary 4.9. *Let $T \in \langle U \rangle_0$.*

(1) *If S is a small subset of U and $o_T(S) < \infty$, then $(T')^\infty(S) = (T')^{o_T(S)}(S)$.*

(2) *If B is a big subset of U and $o_T(B) < \infty$, then $(T')^\infty(B) = (T')^{o_T(B)}(B)$.*

Corollary 4.10. *Let U be a finite set and K be a normal subset of U.*

(1) *If $T \in \langle U \rangle$, then $(T')^\infty(K)$ is a nonempty fixed point.*

(2) If $T \in \langle U\rangle_0$, then $(T')^{\infty}(K)$ is a fixed point.

Lemma 4.11. *Let U be a finite set and $T \in \langle U\rangle_0$.*

(1) If K is a small subset of U, then $T'^{\infty}(K)$ is the smallest fixed point, such that contains K.

(2) If K is a big subset of U, then $T'^{\infty}(K)$ is the biggest fixed point, such that is contained in K.

Proof.

(1) If $T'(K) = K$, then it is clear. Suppose that $K \subset T'(K)$ and F is a fixed point, such that $K \subset F \subset T'^{\infty}(K)$. Then

$$T'^{\infty}(K) \subseteq T'^{\infty}(F) = F \subset T'^{\infty}(K).$$

It is a contradiction.

(2) It is similar to (1).

□

Definition 4.12. Let $T \in \langle U\rangle_0$ and K be a subset of U. We define $\overrightarrow{K} = \bigcup_{n\geqslant 0}(T')^n(K)$, and $[K] = (T')^{\infty}(\overrightarrow{K})$. For any $x_1, x_2, \cdots, x_n \in U$, we show $[\{x_1, x_2, \cdots, x_n\}]$ by $[x_1, x_2, \cdots, x_n]$.

Theorem 4.13. *Let U be a nonempty finite set and K be a nonempty subset of U. Then*

(1) $\overrightarrow{K}$ is a big subset of U.

(2) If $T \in \langle U\rangle$, then $[K]$ is a nonempty fixed point.

(3) If $T \in \langle U\rangle_0$, then $[K]$ is a fixed point of T.

Proof. By Lemma 3.1, we have

$$\begin{aligned} T'(K \cup T'(K) \cup T'^2(K) \cup \cdots) &= T'(K) \cup T'^2(K) \cup T'^3(K) \cdots \\ &\subseteq K \cup T'(K) \cup T'^2(K) \cup T'^3(K) \cdots . \end{aligned}$$

Therefore $\bigcup_{n\geqslant 0}(T')^n(K)$ is a big subset of U. Now the proof follows from Corollary 4.10. □

Example 4.14. In Example 4.7, we have

$$\overrightarrow{\{x\}} = \bigcup_{n\geqslant 0}(T')^n(x) = \{x\} \cup \{y\} = \{x, y\}.$$

Hence $\{x, y\}$ is a big subset of U by Theorem 4.13. Since $T'^{\infty}(\{x, y\}) = \{x, y\}$, therefore $\{x, y\}$ is a fixed point of T by Theorem 4.13.

Example 4.15. Assume that $U = \mathbb{N}$ and $T : \mathbb{N} \longrightarrow \mathcal{P}^*(\mathbb{N})$ is a power mapping, where

$$T(n) = \begin{cases} \{2\} & n = 1, \\ \{3\} & n = 2, 3, \\ \{1, n+1\} & n = 4, 5, 6, \cdots . \end{cases}$$

Let $K = \{1, 2, 3, 4\}$, then $\overrightarrow{K} = \mathbb{N}$ and $[K] = \{1, 2, 3\}$. It is obvious that $[K]$ is not a fixed point and $[[K]] = \{3\}$ is a fixed point of T.

Lemma 4.16. *If $T \in \langle U \rangle_0$ and $K_1 \subseteq K_2$, then $\overrightarrow{K_1} \subseteq \overrightarrow{K_2}$ and $[K_1] \subseteq [K_2]$.*

Proof. It follows from Lemma 3.1. $\square$

Lemma 4.17. *Let U be a nonempty finite set, $T \in \langle U \rangle_0$ and $F \subseteq U$. Then F is a fixed point of T if and only if $\overrightarrow{F} = [F] = F$.*

Proof. It is obvious. $\square$

Lemma 4.18. *Let U be a nonempty finite set, $T \in \langle U \rangle_0$ and $K \subseteq U$. Then $\overrightarrow{\overrightarrow{K}} = \overrightarrow{K}$ and $[\overrightarrow{K}] = \overrightarrow{[K]} = [[K]] = [K]$.*

Proof. It follows from Theorem 4.13 and Lemma 4.17. $\square$

Corollary 4.19. *Let U be a nonempty finite set, $T \in \langle U \rangle_0$ and $K_1, K_2 \subseteq U$. If $\overrightarrow{K_1} = \overrightarrow{K_2}$, then $[K_1] = [K_2]$.*

Proof. We have $[K_1] = [\overrightarrow{K_1}] = [\overrightarrow{K_2}] = [K_2]$, by Lemma 4.18. $\square$

Lemma 4.20. *Let U be a nonempty finite set and $T \in \langle U \rangle_0$. If K is a subset of U, then there exists a number $n \geqslant 0$ such that*

$$\overrightarrow{K} = K \cup T'(K) \cup \cdots (T')^n(K).$$

Proof. Assume that

$$K \subset K \cup T'(K) \subset K \cup T'(K) \cup T'^2(K) \subset \cdots \subset U.$$

This chain can not be strict, forever. Therefore there exists the smallest number $n \geqslant 0$, such that

$$\begin{aligned} &K \subset K \cup T'(K) \subset \cdots \subset K \cup T'(K) \cup \cdots (T')^n(K) \\ &= K \cup T'(K) \cup \cdots \cup (T')^n(K) \cup (T')^{n+1}(K). \end{aligned}$$

Then for any $m \geqslant n$,

$$\bigcup_{i=0}^{i=m\geqslant n} (T')^i(K) = K \cup T'(K) \cup \cdots \cup (T')^n(K).$$

$\square$

Definition 4.21. Let U be a nonempty finite set, $T \in \langle U \rangle_0$ and K be a subset of U. We write $O_T(K) = n$, if n is the smallest number, such that

$$K \cup T'(K) \cup \cdots \cup T'^n(K) = K \cup T'(K) \cup \cdots \cup T'^n(K) \cup T'^{n+1}(K).$$

Example 4.22. Let U be a nonempty finite set and $T \in \langle U \rangle_0$. Then

(1) $O_T(\emptyset) = O_T(U) = 0$.

(2) If K is a normal subset of U, then $O_T(K) \leqslant o_T(K)$.

Lemma 4.23. *Let U be a nonempty finite set, $T \in \langle U \rangle_0$ and $K_1, K_2 \subseteq U$, then*

(1) $\overrightarrow{K_1 \cup K_2} = \overrightarrow{K_1} \cup \overrightarrow{K_2}$.

(2) $[K_1 \cup K_2] = [K_1] \cup [K_2]$.

Proof.

(1) It follows from Lemma 3.1 and Lemma 4.20.

(2) It follows from (1) and Lemma 3.1.

□

Lemma 4.24. *Let U be a nonempty finite set, $T \in \langle U \rangle_0$ and K be a subset of U. Then the following statements are equivalent.*

(1) $\overrightarrow{K} \subseteq [K]$,

(2) $\overrightarrow{K} = [K]$,

(3) $\overrightarrow{K} \in Fix(T)$.

Proof.

- ($1 \Rightarrow 2$) If $K \subseteq [K]$, then $\overrightarrow{K} \subseteq \overrightarrow{[K]} = [K]$ by Lemma 4.16 and Lemma 4.17.
- ($2 \Rightarrow 3$) It is obvious.
- ($3 \Rightarrow 4$) It follows from Theorem 4.13.
- ($4 \Rightarrow 1$) It is clear that $K \subseteq \overrightarrow{K}$. Since $\overrightarrow{K}$ is a fixed point of T, hence $[\overrightarrow{K}] = \overrightarrow{K}$. Therefore $K \subseteq \overrightarrow{K} = [\overrightarrow{K}] = [K]$ by Lemma 4.18.

□

Lemma 4.25. *Let U be a nonempty finite set, $T \in \langle U \rangle_0$, $K \subseteq U$ and F be a fixed point of T. If $F \subseteq \overrightarrow{K}$ then $F \subseteq [K]$.*

Proof. It follows from Lemma 4.16 and Lemma 4.18. □

Lemma 4.26. *Let U be a nonempty finite set, $T \in \langle U \rangle$ and F be a minimal fixed point of T. If K is a nonempty subset of F, then $[K] = \overrightarrow{K} = F$.*

Proof. It follows from Lemma 4.16 and Theorem 4.13. □

Corollary 4.27. *Let U be a nonempty finite set and $T \in \langle U \rangle$. If F is a minimal fixed point and $x \in F$, then $F = [x]$.*

Definition 4.28. Let $T \in \langle U \rangle_0$ and F be a fixed point. F is called a *principal fixed point* of T if and only if there exists $x \in U$ such that $F = [x]$.

By Corollary 4.27, any minimal fixed point is a principal fixed point. But it is not necessary that a principal fixed point is a minimal fixed point. For example:

Example 4.29. Let $U = \{x, y, z, t, v, w\}$ and

$$T(u) = \begin{cases} \{x, y\} & u = x, \\ \{y, z\} & u = y, \\ \{z\} & u = z, \\ \{x, z\} & u = t, \\ \{z\} & u = v, \\ \{x, y\} & u = w. \end{cases}$$

We have $[x] = \{x, y, z\}$, $[y] = \{y, z\}$ and $[z] = \{z\}$. Therefore

$$\emptyset \subset [z] \subset [y] \subset [x] = [U].$$

Lemma 4.30. *Let U be a nonempty finite set and $T \in \langle U \rangle_0$. Then $\emptyset$ and $[U]$ are minimum and maximum fixed point, respectively.*

Proof. It is obvious. □

Lemma 4.31. *Let U be a nonempty finite set and $T \in \langle U \rangle_0$. If F is a fixed point, then there exist $x_1, x_2, \cdots x_n \in [U]$ such that $F = [x_1] \cup [x_2] \cup \cdots \cup [x_n]$.*

Proof. There exist $x_1, x_2, \cdots x_n \in [U]$, such that $F = \{x_1, x_2, \cdots, x_n\}$ by Lemma 4.16 and Lemma 4.17. Hence $F = \bigcup_{x \in F} \{x\}$. Therefore $F = [F] = [\bigcup_{x \in F} \{x\}] = \bigcup_{x \in F} [x]$ by Lemma 4.17 and Lemma 4.23. □

Theorem 4.32 (Main Theorem 2)**.** *Let U be a nonempty finite set and $T \in \langle U \rangle_0$. Set $PF(T) = \{[x] \mid x \in [U]\}$, then $Fix(T) = \{\bigcup_{[x] \in \Sigma} [x] \mid \Sigma \subseteq PF(T)\}$.*

Proof. Let $\Omega = \{\bigcup_{[x] \in \Sigma} [x] \mid \Sigma \subseteq PF(T)\}$. Then $\Omega \subseteq Fix(T)$ by Theorem 4.13 and Lemma 3.2. And we have $Fix(T) \subseteq \Omega$ by lemma 4.31. Therefore $\Omega = Fix(T)$. □

Assume that $[U] = \{x, x', x'', y, y', z, u\}$, $[x] = [x'] = [x'']$ and $[y] = [y']$. Then $PF(T) = \{[x], [y], [z], [u]\} = \{[x'], [y], [z], [u]\} = \{[x'], [y'], [z], [u]\} = \cdots$.

Proposition 4.33. *Let U be a nonempty finite set and $T \in \langle U \rangle_0$. Then $|Fix(T)| \leqslant 2^{|PF(T)|}$.*

Proof. It is obvious. □

Example 4.34. Suppose that $U = \{x, y, z, u, v, w, r, t\}$ and

$$T : \begin{cases} x \to \{y\}, \\ y \to \{z\}, \\ z \to \{u\}, \\ u \to \{z\}, \\ v \to \{v, u\}, \\ w \to \{u\}, \\ r \to \{r, t\}, \\ t \to \{r\}. \end{cases}$$

Then $[U] = \{z, u, v, r, t\}$, $[z] = [u] = \{u, z\}$, $[v] = \{u, v, z\}$ and $[r] = [t] = \{r, t\}$. Hence $PF(T) = \{[z], [v], [r]\}$. And $Fix(T) = \{\emptyset, [z], [v], [r], [z] \cup [r], [v] \cup [r], [z] \cup [v] \cup [r]\}$ by Theorem 4.32. Therefore

$$Fix(T) = \{\emptyset, \{u, z\}, \{r, t\}, \{u, v, z\}, \{u, z, r, t\}, \{u, v, z, r, t\}\}$$

and $|Fix(T)| = 6$. Also we have

(1) $x \notin [x]$.
(2) $[v]$ is not a minimal fixed point.
(3) $u \in [u]$.
(4) $[x] \subset [v]$.
(5) $[z] \cup [v] = [v]$.

Example 4.35. Assume that $U = \mathbb{Z}_{24}$, where $(\mathbb{Z}_{24}, +)$ is the group of integers modulo 24. Let $T : \mathbb{Z}_{24} \longrightarrow \mathcal{P}^*(\mathbb{Z}_{24})$ be a power mapping, such that $T(\overline{n}) = 2 \cdot \overline{n} + \overline{1}$. Since $\mathbb{Z}_{24}$ is a big subset, hence $[\mathbb{Z}_{24}] = T'^{\infty}(\mathbb{Z}_{24}) = \{7, 15, 23\}$. We have $[7] = [15] = \{7, 15\}$ and $[23] = \{23\}$. By Theorem 4.32, we have $PF(T) = \{[7], [23]\}$ and therefore $Fix(T) = \{\emptyset, [7], [23], [7] \bigcup [23]\} = \{\emptyset, \{7, 15\}, \{23\}, \{7, 15, 23\}\}$.

Example 4.36. Let $(H, \star)$ be an algebraic hyperstructure, where $H = \{a, b, c, d\}$ and:

$$\begin{cases} a \star a = \{b\}, \ \ a \star b = \{a, b\}, \ \ a \star c = \{b, c\}, \\ b \star a = \{a\}, \ \ b \star b = \{b\}, \ \ b \star c = \{a\}, \\ c \star a = \{a, b, c\}, \ \ c \star b = \{b\}, \ \ c \star c = \{c\}, \\ x \star y = \{a\} \qquad \text{if } x = d \text{ or } y = d. \end{cases}$$

For any $h \in H$ define $T_h(x) = h \star x$. Since H is a nonempty finite set and $T_h \in \langle H \rangle$, therefore there exists a nonempty subset F_h of H, such that $T'_h(F_h) = F_h$, by Fixed Point Theorem. For example:

$$T_a = \begin{cases} a \longrightarrow a \star a = \{b\}, \\ b \longrightarrow a \star b = \{a, b\}, \\ c \longrightarrow a \star c = \{b, c\} \\ d \longrightarrow a \star d = \{a\}. \end{cases}$$

Use Theorem 4.32. We have $[H] = \{a, b, c\}$. $[a] = [b] = \{a, b\}$ and $[c] = \{a, b, c\}$. Hence $PF(T_a) = \{\{a, b\}, \{a, b, c\}\}$. Therefore $Fix(T_a) = \{\emptyset, \{a, b\}, \{a, b, c\}\}$. The set $\{a, b\}$ is a minimal fixed point.

Theorem 4.37 (Main Theorem 3)**.** *Let U be a nonempty finite set and $T : U \longrightarrow \mathcal{P}(U)$ be a power mapping. Then T has a nonempty fixed point, if and only if $[U] \neq \emptyset$.*

Proof. ($\Leftarrow$) It follows from Theorem 4.13.
($\Rightarrow$) Assume that F is a nonempty fixed point. Hence $\emptyset \neq F = [F] \subseteq [U]$ by Lemma 4.16 and Lemma 4.17. $\square$

Example 4.38. Suppose that $U = \{x, y, z, t\}$ and $T : U \longrightarrow \mathcal{P}(U)$ is a power mapping, where

$$T(u) = \begin{cases} \{y, z\} & u = x, \\ \{z\} & u = y, \\ \{t\} & u = z, \\ \emptyset & u = t. \end{cases}$$

Since $[U] = \emptyset$, therefore T does not have a nonempty fixed point by Theorem 4.37.

Example 4.39. In Example 4.36, define $T(x) = c \star x - x \star c$. Hence:

$$T = \begin{cases} a \longrightarrow \{a\}, \\ b \longrightarrow \{b\}, \\ c \longrightarrow \emptyset, \\ d \longrightarrow \emptyset. \end{cases}$$

Since $[H] = \{a, b\} \neq \emptyset$, hence T has a nonempty fixed point, by Theorem 4.37.
By Theorem 4.32, we have $PF(T) = \{[a], [b]\}$, where $[a] = \{a\}$ and $[b] = \{b\}$. Therefore $Fix(T) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$.

Assume that Σ is a nonempty set, $|\Sigma| = n$, $A = \{f : \Sigma \longrightarrow \Sigma \mid f \text{ is a mapping}\}$, $B = \{f \in A \mid \exists x \in \Sigma : f(x) = x\}$ and $C = A - B$. Then $|A| = n^n$, $|B| = n^n - (n-1)^n$ and $|C| = (n-1)^n$. Therefore $\lim_{|\Sigma| \to \infty} (\frac{A}{C}) = e$, $\lim_{|\Sigma| \to \infty} \frac{A}{B} = \frac{e}{e-1}$ and $\lim_{|\Sigma| \to \infty} \frac{B}{C} = e - 1$, where e is the Napier's number.

5 Conclusion

In this paper, we introduced a new concept called of a power mapping. We showed that if U is a nonempty finite set and $T : U \longrightarrow \mathcal{P}^*(U)$ is a power mapping, then there exists a nonempty subset F of U such that $T'(F) = F$ and F is called a fixed point. We proved if U is a nonempty finite set, then the power mapping $T : U \longrightarrow \mathcal{P}(U)$ has a nonempty fixed point, if and only if $[U] \neq \emptyset$. We showed that if $PF(T) = \{[x] | x \in [U]\}$ then $Fix(T) = \{\bigcup_{[x] \in \Sigma} [x] \mid \Sigma \subseteq PF(T)\}$.

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