Journal of Mathematical Extension
Vol. 18, No. 9, (2024) (2)1-21
URL: https://doi.org/10.30495/JME.2024.3114
ISSN: 1735-8299
Original Research Paper

Another Approach to Generate Fuzzy Normed Spaces and Fuzzy Normed Algebras by Normed Spaces and Normed Algebras Respectively

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Abstract. In this paper, we introduce a new approach to generate a fuzzy norm using a classic norm and a continuous Archimedean t-norm (CATN). Our method involves two steps. First, we utilize a CATN to create a continuous additive generator (CAG). Then, we employ the corresponding additive generator (AG) and a classic norm to generate a fuzzy norm.

AMS Subject Classification: 46S40; 46B99
Keywords and Phrases: Fuzzy normed space, Fuzzy normed algebra, Pseudo-inverse, Continuous Archimedean t-norm, Additive generator, Multiplicative generator

1 Introduction

It is well-established that the fuzzy concepts hold significant importance in multiple disciplines such as engineering, medicine, management, and

Received: July 2024; Accepted: January 2025
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mathematics. Two topics from fuzzy theory that can be very relevant are t-norm and fuzzy norm. In this paper, we will present specific information to one of the fuzzy definitions called fuzzy norm, that is a basic notion in fuzzy functional analysis.
In 1965, Zadeh [11] introduced fuzzy sets to the world of science through a scientific paper. The fuzzy sets serve as a uncertain mathematical model in the field of analysis. First time, A .K. Katsaras, defined a concept called fuzzy norm in [4]. Also, some analytical and topological definitions using fuzzy norm are stated in [10]. In later years, T. Bag and S. K. Samanta in [1], gave a newer definition of fuzzy norm, working with that definition facilitated the procedure of proving theorems. R. Saadati and J. H. Park in [7] by using t-norms and continuous t-conorms, have reached to intuitionistic fuzzy normed spaces. O. Grigorenko, J. Minana, and O. Valero [3], introduced a method for making a fuzzy metric. T. Binzar, F. Pater, and S. Nadaban [2] investigated the relationship between fuzzy normed algebras.
The main motivation of this paper is to identify new fuzzy norms and new algebra fuzzy norms. In this paper, we state that a new CATN can be obtained from additive and multiplicative generators of a previous CATN. This method shows that countless of unknown CATN can be obtained.
The rest of the paper is organized as follows : In Section 2, we mention the definitions of t-norm, additive generator, multiplicative generator, pseudo-inverse, fuzzy norm, and etc. Some new results are stated in section 3.

2 Definitions

Definition 2.1. *[3] A triangular norm (briefly, t-norm) is a function* $\star : [0,1]^2 \longrightarrow [0,1]$ *that for each* $(\alpha,\beta,\gamma) \in [0,1]^3$ *we have :*

1) $\alpha \star \beta = \beta \star \alpha$,

2) $\alpha \star (\beta \star \gamma) = (\alpha \star \beta) \star \gamma$,

3) $\alpha \star \beta \geq \alpha \star \gamma$, *for* $\beta \geq \gamma$,

4) $\alpha \star 1 = \alpha$.

Definition 2.2. *[5] A t-norm* $\star : [0,1]^2 \longrightarrow [0,1]$ *is called Archimedean t-norm if for each* $(\alpha, \beta) \in (0,1) \times (0,1)$ *there is* $k \in \mathbb{N}$ *such that* $\alpha^{(k)} < \beta$ *where*

$$\alpha^{(k)} = \overbrace{\alpha \star \alpha \star \cdots \star \alpha}^{k-times}.$$

Also, a continuous t-norm is Archimedean iff $\alpha \star \alpha < \alpha$ *for all* $0 < \alpha < 1$ *[5].*

Example 2.3. If $\star_L : [0,1]^2 \longrightarrow [0,1]$ and $\star_p : [0,1]^2 \longrightarrow [0,1]$ are defined by $\alpha \star_L \beta = \max\{\alpha + \beta - 1, 0\}$ and $\alpha \star_p \beta = \alpha\beta$, then $\star_L$ and $\star_p$ are Archimedean t-norm.

Definition 2.4. *[5] Let* $\Psi : [0,1] \longrightarrow [0,\infty]$ *be a decreasing function. Then* $\Psi^{(-1)} : [0,\infty] \longrightarrow [0,1]$ *is pseudo-inverse of* Ψ *that is defined by*

$$\Psi^{(-1)}(\gamma) = \sup\{\zeta \in [0,1] : \Psi(\zeta) > \gamma\}$$

where we assume that $\sup \emptyset = 0$*. Also if* Ψ *is continuous and strictly decreasing, then we have :*

$$\Psi^{(-1)}(\gamma) = \begin{cases} 1, & 0 \le \gamma < \Psi(1) \\ \Psi^{-1}(\gamma), & \Psi(1) \le \gamma < \Psi(0) \\ 0, & \Psi(0) \le \gamma \le \infty \end{cases}.$$

In particular, if Ψ *is continuous, strictly decreasing, and* $\Psi(1) = 0$*, then*

$$\Psi^{(-1)}(\gamma) = \begin{cases} \Psi^{-1}(\gamma), & 0 \le \gamma < \Psi(0) \\ 0, & \Psi(0) \le \gamma \le \infty \end{cases}.$$

Definition 2.5. *[5] Let* $\Psi : [0,1] \longrightarrow [0,1]$ *be an increasing function. Then* $\Psi^{(-1)} : [0,1] \longrightarrow [0,1]$ *is pseudo-inverse of* Ψ *that is defined by*

$$\Psi^{(-1)}(\gamma) = \sup\{\zeta \in [0,1] : \Psi(\zeta) < \gamma\}$$

where we assume that $\sup \emptyset = 0$*. Also if* Ψ *is continuous and strictly increasing, then we have :*

$$\Psi^{(-1)}(\gamma) = \begin{cases} 0, & 0 \le \gamma < \Psi(0) \\ \Psi^{-1}(\gamma), & \Psi(0) \le \gamma < \Psi(1) \\ 1, & \Psi(1) \le \gamma \le 1 \end{cases}.$$

In particular, if Ψ *is continuous, strictly increasing, and* $\Psi(1) = 1$*, then*

$$\Psi^{(-1)}(\gamma) = \begin{cases} 0, & 0 \le \gamma < \Psi(0) \\ \Psi^{-1}(\gamma), & \Psi(0) \le \gamma \le 1 \end{cases}.$$

Definition 2.6. *[5] A multiplicative generator (MG) of a t-norm* $\star$ *is a strictly increasing function* $\varrho : [0,1] \longrightarrow [0,1]$ *which is right-continuous at* $\zeta = 0$ *and* $\varrho(1) = 1$ *such that for each* $(\zeta, \gamma) \in [0,1] \times [0,1]$,

$$\varrho(\zeta)\,\varrho(\gamma) \in \mathrm{Ran}(\varrho) \cup [0, \varrho(0)]$$

and

$$\zeta \star \gamma = \varrho^{(-1)}\left(\varrho(\zeta)\,\varrho(\gamma)\right).$$

Corollary 2.7. *[5] Let* $\varrho : [0,1] \longrightarrow [0,1]$ *be a strictly increasing function which is right-continuous at* $\zeta = 0$*,* $\varrho(1) = 1$ *and for each* $(\zeta, \gamma) \in [0,1] \times [0,1]$,

$$\varrho(\zeta)\,\varrho(\gamma) \in \mathrm{Ran}(\varrho) \cup [0, \varrho(0)].$$

If $\star : [0,1]^2 \longrightarrow [0,1]$ *is defined by*

$$\zeta \star \gamma = \varrho^{(-1)}\left(\varrho(\zeta)\,\varrho(\gamma)\right),$$

then $\star$ *is a t-norm.*

Corollary 2.8. *[5] Let* $\star : [0,1]^2 \longrightarrow [0,1]$ *be a t-norm. Then* $\star$ *is a CATN iff there exists a continuous multiplicative generator (CMG) of* $\star$*.*

Definition 2.9. *[3] An additive generator (AG) of a t-norm* $\star$ *is a strictly decreasing function* $\xi_\star : [0,1] \longrightarrow [0,\infty]$ *which is also right-continuous at* $\zeta = 0$ *and* $\xi_\star(1) = 0$*. Also for all* $(\zeta, \gamma) \in [0,1] \times [0,1]$*, The following are valid :*

$$\xi_\star(\zeta) + \xi_\star(\gamma) \in \mathrm{Ran}(\xi_\star) \cup [\xi_\star(0), \infty]$$

and

$$\zeta \star \gamma = \xi_\star^{(-1)}\left(\xi_\star(\zeta) + \xi_\star(\gamma)\right).$$

Proposition 2.10. *[3] Let $\xi : [0,1] \longrightarrow [0,\infty]$ be a strictly decreasing function with $\xi(1) = 0$ and*

$$\xi(\zeta) + \xi(\gamma) \in \mathrm{Ran}(\xi) \cup [\xi(0), \infty]$$

for all $(\zeta,\gamma) \in [0,1] \times [0,1]$. Then $\star : [0,1]^2 \longrightarrow [0,1]$ defined by

$$\zeta \star \gamma = \xi^{(-1)}(\xi(\zeta) + \xi(\gamma))$$

is a t-norm.

Theorem 2.11. *[3] A t-norm $\star$ is a CATN iff there exists a CAG such as $\xi_\star$ such that for each $(\zeta,\gamma) \in [0,1] \times [0,1]$,*

$$\zeta \star \gamma = \xi_\star^{(-1)}(\xi_\star(\zeta) + \xi_\star(\gamma)).$$

In the following, we will state that by using a CAG and a CMG of CATN $\star$, a new CAG and consequently a new t-norm can be obtained.

Corollary 2.12. *Let $\star$ be a CATN and $\xi_\star$ and $\varrho_\star$ be CAG and CMG of $\star$ respectively. If $\xi_\star(0) = \infty$, then $h : [0,1] \longrightarrow [0,\infty]$ where $h(\zeta) = \xi_\star(\zeta) - \varrho_\star(\zeta) + 1$ is a CAG of $\star'$ where*

$$\zeta \star' \gamma = h^{(-1)}(h(\zeta) + h(\gamma))$$

for all $(\zeta,\gamma) \in [0,1] \times [0,1]$.

Proof. Since $\xi_\star$ and $\varrho_\star$ are continuous, h is continuous. As $\xi_\star$ is strictly decreasing and $\varrho_\star$ is strictly increasing, h is strictly decreasing. Also $h(1) = 0$ and $h(0) = \infty$. Hence the continuity of h implies that $\mathrm{Ran}(h) = [0,\infty]$. Therefore $h(\zeta) + h(\gamma) \in \mathrm{Ran}(h)$ for all $\zeta,\gamma \in [0,1]$. Applying Proposition 2.10, we can conclude that h is a CAG of $\star'$. □

Definition 2.13. *[9] Let Z be a linear space and $\star$ be a t-norm. A function $\aleph : Z \times \mathbb{R} \longrightarrow [0,1]$ is named a fuzzy norm on Z if for all $\zeta,\gamma \in Z$ and all $s,t \in \mathbb{R}$, we have :*

1) $\aleph(\zeta,t) = 0$ for all $t \leq 0$.

2) $\aleph(\zeta,t) = 1$ for all $t > 0$ iff $\zeta = 0$.

3) $\aleph(\mu\zeta, t) = \aleph\left(\zeta, \frac{t}{|\mu|}\right)$ *for all* $\mu \neq 0$ *and* $t \in \mathbb{R}$*.*

4) $\aleph(\zeta + \gamma, s + t) \geq \aleph(\zeta, s) \star \aleph(\gamma, t)$ *for all* $s, t \in \mathbb{R}$*.*

5) $\aleph(\zeta, .)$ *is increasing on* $\mathbb{R}$ *and* $\lim_{t\to\infty} \aleph(\zeta, t) = 1$*.*

Considering the above definition, $(Z, \aleph, \star)$ is called a fuzzy normed space.

3 Main Results

In this section, let $\star$ be a CATN and $\xi_\star$ be the corresponding CAG of $\star$. Also let $\xi_\star^{(-1)}$ be the *pseudo-inverse* of $\xi_\star$.

Theorem 3.1. *Let* $(Z, \|\cdot\|)$ *be a normed space,* $\star$ *be a CATN and* $\xi_\star$ *be a CAG of* $\star$*. Then* $(Z, \aleph_\star, \star)$ *is a fuzzy normed space which for each* $\zeta \in Z$ *and each* $t \in \mathbb{R}$ *is defined as follows :*

$$\aleph_\star(\zeta, t) = \begin{cases} \xi_\star^{(-1)}\left(\dfrac{\|\zeta\|}{t}\right), & t > 0 \\ 0, & t \leq 0 \end{cases}.$$

Proof. If $\xi_\star(0) = \infty$, then by Definition 2.4 we have

$$\aleph_\star(\zeta, t) = \begin{cases} \xi_\star^{-1}\left(\dfrac{\|\zeta\|}{t}\right), & \dfrac{\|\zeta\|}{t} \geq 0 \\ 0, & t \leq 0 \end{cases}.$$

In this case, we only prove part 4 of Definition 2.13. Let $\zeta, \gamma \in Z$ and $s, t \in \mathbb{R}$. If $s \leq 0$ or $t \leq 0$, then $\aleph_\star(\zeta, s) = 0$ or $\aleph_\star(\gamma, t) = 0$. Hence

$$\aleph_\star(\zeta + \gamma, s + t) \geq \aleph_\star(\zeta, s) \star \aleph_\star(\gamma, t) = 0.$$

If $s > 0$ and $t > 0$, then

$$\frac{\|\zeta + \gamma\|}{s + t} \leq \frac{\|\zeta\| + \|\gamma\|}{s + t} \leq \frac{\|\zeta\|}{s} + \frac{\|\gamma\|}{t}.$$

Hence

$$\begin{aligned}
\aleph_\star(\zeta+\gamma,s+t) &= \xi_\star^{-1}\left(\frac{\|\zeta+\gamma\|}{s+t}\right)\\
&\geq \xi_\star^{-1}\left(\frac{\|\zeta\|}{s}+\frac{\|\gamma\|}{t}\right)\\
&= \xi_\star^{-1}\left(\xi_\star\left(\xi_\star^{-1}\left(\frac{\|\zeta\|}{s}\right)\right)+\xi_\star\left(\xi_\star^{-1}\left(\frac{\|\gamma\|}{t}\right)\right)\right)\\
&= \xi_\star^{(-1)}\left(\xi_\star\left(\xi_\star^{(-1)}\left(\frac{\|\zeta\|}{s}\right)\right)+\xi_\star\left(\xi_\star^{(-1)}\left(\frac{\|\gamma\|}{t}\right)\right)\right)\\
&= \xi_\star^{(-1)}\left(\frac{\|\zeta\|}{s}\right)\star\xi_\star^{(-1)}\left(\frac{\|\gamma\|}{t}\right)\\
&= \aleph_\star(\zeta,s)\star\aleph_\star(\gamma,t).
\end{aligned}$$

Now, we prove the theorem for $\xi_\star(0)<\infty$. In this case

$$\begin{aligned}
\aleph_\star(\zeta,t) &= \begin{cases}\xi_\star^{(-1)}\left(\frac{\|\zeta\|}{t}\right), & t>0\\ 0, & t\leq 0\end{cases}\\
&= \begin{cases}\xi_\star^{-1}\left(\frac{\|\zeta\|}{t}\right), & 0\leq\frac{\|\zeta\|}{t}<\xi_\star(0),\ t>0\\ 0, & \xi_\star(0)\leq\frac{\|\zeta\|}{t},\ t>0\\ 0, & t\leq 0\end{cases}.
\end{aligned}$$

For each $t\leq 0$, clearly $\aleph_\star(\zeta,t)=0$. To prove the second property, if for each $t>0$, $\aleph_\star(\zeta,t)=1$. Then $\xi_\star^{-1}\left(\frac{\|\zeta\|}{t}\right)=1$. Hence $\frac{\|\zeta\|}{t}=\xi_\star(1)=0$ for all $t>0$. Then $\zeta=0$. Conversely, if $\zeta=0$, then for each $t>0$,

$$\aleph_\star(0,t)=\xi_\star^{-1}(0)=1.$$

To prove the third part, let $\zeta \in Z$, $t \in \mathbb{R}$ and $\alpha \neq 0$. We have :

$$\begin{aligned}
\aleph_\star(\alpha\zeta, t) &= \begin{cases} \xi_\star^{-1}\left(\frac{\|\alpha\zeta\|}{t}\right), & 0 \leq \frac{\|\alpha\zeta\|}{t} < \xi_\star(0), \, t > 0 \\ 0, & \xi_\star(0) \leq \frac{\|\alpha\zeta\|}{t}, \, t > 0 \\ 0, & t \leq 0 \end{cases} \\
&= \begin{cases} \xi_\star^{-1}\left(\frac{\frac{\|\zeta\|}{t}}{|\alpha|}\right), & 0 \leq \frac{\frac{\|\zeta\|}{t}}{|\alpha|} < \xi_\star(0), \, t > 0 \\ 0, & \xi_\star(0) \leq \frac{\frac{\|\zeta\|}{t}}{|\alpha|}, \, t > 0 \\ 0, & t \leq 0 \end{cases} \\
&= \begin{cases} \xi_\star^{-1}\left(\frac{\frac{\|\zeta\|}{t}}{|\alpha|}\right), & 0 \leq \frac{\frac{\|\zeta\|}{t}}{|\alpha|} < \xi_\star(0), \, \frac{t}{|\alpha|} > 0 \\ 0, & \xi_\star(0) \leq \frac{\frac{\|\zeta\|}{t}}{|\alpha|}, \, \frac{t}{|\alpha|} > 0 \\ 0, & \frac{t}{|\alpha|} \leq 0 \end{cases} \\
&= \aleph_\star\left(\zeta, \frac{t}{|\alpha|}\right).
\end{aligned}$$

To prove the fourth part, let $\zeta, \gamma \in Z$ and $s, t \in \mathbb{R}$. If $s \leq 0$ or $t \leq 0$, then $\aleph_\star(\zeta, s) = 0$ or $\aleph_\star(\gamma, t) = 0$. Since $a \star 0 = 0 \star b = 0$ for all $a, b \in [0, 1]$,

$$\aleph_\star(\zeta + \gamma, s + t) \geq \aleph_\star(\zeta, s) \star \aleph_\star(\gamma, t) = 0.$$

Now we suppose that $s, t \in (0, \infty)$. If $\xi_\star(0) \leq \frac{\|\zeta\|}{s}$ or $\xi_\star(0) \leq \frac{\|\gamma\|}{t}$, then $\aleph_\star(\zeta, s) = 0$ or $\aleph_\star(\gamma, t) = 0$. Hence

$$\aleph_\star(\zeta + \gamma, s + t) \geq \aleph_\star(\zeta, s) \star \aleph_\star(\gamma, t) = 0.$$

If $0 \leq \frac{\|\zeta\|}{s} < \xi_\star(0)$ and $0 \leq \frac{\|\gamma\|}{t} < \xi_\star(0)$, then

$$\aleph_\star(\zeta, s) = \xi_\star^{(-1)}\left(\frac{\|\zeta\|}{s}\right) = \xi_\star^{-1}\left(\frac{\|\zeta\|}{s}\right)$$

and

$$\aleph_\star(\gamma, t) = \xi_\star^{(-1)}\left(\frac{\|\gamma\|}{t}\right) = \xi_\star^{-1}\left(\frac{\|\gamma\|}{t}\right).$$

Since $\xi_\star^{(-1)}$ is decreasing on $[0,\infty]$ and $\frac{\|\zeta+\gamma\|}{s+t} \leq \frac{\|\zeta\|}{s} + \frac{\|\gamma\|}{t}$,

$$\begin{aligned}
\aleph_\star(\zeta+\gamma, s+t) &= \xi_\star^{(-1)}\left(\frac{\|\zeta+\gamma\|}{s+t}\right) \\
&\geq \xi_\star^{(-1)}\left(\frac{\|\zeta\|}{s} + \frac{\|\gamma\|}{t}\right) \\
&= \xi_\star^{(-1)}\left(\xi_\star\left(\xi_\star^{-1}\left(\frac{\|\zeta\|}{s}\right)\right) + \xi_\star\left(\xi_\star^{-1}\left(\frac{\|\gamma\|}{t}\right)\right)\right) \\
&= \xi_\star^{(-1)}\left(\xi_\star\left(\xi_\star^{(-1)}\left(\frac{\|\zeta\|}{s}\right)\right) + \xi_\star\left(\xi_\star^{(-1)}\left(\frac{\|\gamma\|}{t}\right)\right)\right) \\
&= \xi_\star^{(-1)}\left(\frac{\|\zeta\|}{s}\right) \star \xi_\star^{(-1)}\left(\frac{\|\gamma\|}{t}\right) \\
&= \aleph_\star(\zeta, s) \star \aleph_\star(\gamma, t).
\end{aligned}$$

This shows that for each $\zeta, \gamma \in Z$ and $s, t \in \mathbb{R}$,

$$\aleph_\star(\zeta+\gamma, s+t) \geq \aleph_\star(\zeta, s) \star \aleph_\star(\gamma, t).$$

To prove the fifth part, suppose that $\zeta \in Z$, $t_1, t_2 \in \mathbb{R}$ and $t_1 < t_2$. If $t_1 \leq 0$, then

$$0 = \aleph_\star(\zeta, t_1) \leq \aleph_\star(\zeta, t_2).$$

If $t_1 > 0$, then

$$\frac{\|\zeta\|}{t_2} \leq \frac{\|\zeta\|}{t_1}.$$

In this case, if $0 \leq \frac{\|\zeta\|}{t_1} < \xi_\star(0)$, then $0 \leq \frac{\|\zeta\|}{t_2} < \xi_\star(0)$ and as a result

$$\xi_\star^{(-1)}\left(\frac{\|\zeta\|}{t_2}\right) = \xi_\star^{-1}\left(\frac{\|\zeta\|}{t_2}\right) \geq \xi_\star^{-1}\left(\frac{\|\zeta\|}{t_1}\right) = \xi_\star^{(-1)}\left(\frac{\|\zeta\|}{t_1}\right).$$

If $\xi_\star(0) \leq \dfrac{\|\zeta\|}{t_1}$, then

$$\xi_\star^{(-1)}\left(\frac{\|\zeta\|}{t_2}\right) \geq 0 = \xi_\star^{(-1)}\left(\frac{\|\zeta\|}{t_1}\right).$$

This means that $\aleph_\star(\zeta, t_2) \geq \aleph_\star(\zeta, t_1)$. Therefore, $\aleph_\star$ is increasing with respect to t. Now since $\lim_{t \longrightarrow \infty} \dfrac{\|\zeta\|}{t} = 0$, there exists an $M > 0$ such that for each $t \geq M$, $\dfrac{\|\zeta\|}{t} < \xi_\star(0)$. Hence,

$$\xi_\star^{(-1)}\left(\frac{\|\zeta\|}{t}\right) = \xi_\star^{-1}\left(\frac{\|\zeta\|}{t}\right).$$

So

$$\begin{aligned}
\lim_{t \longrightarrow \infty} \aleph_\star(\zeta, t) &= \lim_{t \longrightarrow \infty} \xi_\star^{(-1)}\left(\frac{\|\zeta\|}{t}\right) \\
&= \lim_{t \longrightarrow \infty} \xi_\star^{-1}\left(\frac{\|\zeta\|}{t}\right) \\
&= \xi_\star^{-1}\left(\lim_{t \longrightarrow \infty} \frac{\|\zeta\|}{t}\right) \\
&= \xi_\star^{-1}(0) \\
&= 1.
\end{aligned}$$

□

Example 3.2. Let $(Z, \|\cdot\|)$ be a normed space, $\alpha \star_p \beta = \alpha\beta$, $\xi_{\star_p}(\alpha) = -\mathrm{Ln}(\alpha)$ for all $\alpha, \beta \in [0, 1]$, where $\mathrm{Ln}(0) = -\infty$. If $\aleph_{\star_p} : Z \times \mathbb{R} \longrightarrow [0, 1]$ is defined by

$$\aleph_{\star_p}(\zeta, t) = \begin{cases} \mathrm{e}^{-\frac{\|\zeta\|}{t}}, & t > 0 \\ 0, & t \leq 0 \end{cases}$$

for all $\zeta \in Z$ and $t \in \mathbb{R}$, then $\left(Z, \aleph_{\star_p}, \star_p\right)$ is a fuzzy normed space.

Example 3.3. Let Z be a topological space and $C^b(Z)$ be the set of all complex valued, bounded and continuous functions on Z equipped with the norm

$$\|f\|_\infty = \sup\{|f(\zeta)|,\ \zeta \in Z\}$$

for all $f \in C^b(Z)$. If

$$\alpha \star_E \beta = \frac{\alpha\beta}{1+(1-\alpha)(1-\beta)}$$

and $\xi_{\star_E}(\alpha) = \ln\frac{2-\alpha}{\alpha}$ for each $(\alpha,\beta) \in [0,1]\times[0,1]$ such that $\xi_{\star_E}(0) = \infty$, then for each $t \in \mathbb{R}$ and $g \in C^b(Z)$,

$$\aleph_{\star_E}(g,t) = \begin{cases} \dfrac{2}{e^{\left(\frac{\|g\|_\infty}{t}\right)}+1}, & t>0 \\ 0, & t \le 0 \end{cases}$$

is a fuzzy norm on functional space $C^b(Z)$.

Theorem 3.4. *[6] Let Z be a linear space and suppose that $(Z,\aleph_1,\star_1)$ and $(Z,\aleph_2,\star_2)$ are fuzzy normed spaces. If there exist c_1, $c_2 > 0$ such that*

$$\aleph_2(c_1\zeta,t) \le \aleph_1(\zeta,t) \le \aleph_2(c_2\zeta,t)$$

for each $\zeta \in Z$ and $t \in \mathbb{R}$, then $\aleph_1$ and $\aleph_2$ are equivalent fuzzy norms on Z.

Proposition 3.5. *Let Z be a linear space and $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on Z such that $c_1\|\cdot\|_2 \le \|\cdot\|_1 \le c_2\|\cdot\|_2$. Also let $\star : [0,1]^2 \longrightarrow [0,1]$ be a CATN. Define $\aleph_\star^{(1)}$ and $\aleph_\star^{(2)}$ by*

$$\aleph_\star^{(1)}(\zeta,t) = \begin{cases} \xi_\star^{(-1)}\left(\dfrac{\|\zeta\|_1}{t}\right), & t>0 \\ 0, & t \le 0 \end{cases}$$

and

$$\aleph_\star^{(2)}(\zeta,t)=\begin{cases}\xi_\star^{(-1)}\left(\frac{\|\zeta\|_2}{t}\right), & t>0\\ 0, & t\le 0\end{cases}.$$

Then, $\aleph_\star^{(1)}$ and $\aleph_\star^{(2)}$ are equivalent fuzzy norms.

Proof. We show that

$$\aleph_\star^{(2)}(c_2\zeta,t)\le\aleph_\star^{(1)}(\zeta,t)\le\aleph_\star^{(2)}(c_1\zeta,t) \tag{1}$$

for all $\zeta\in Z$ and $t\in\mathbb{R}$. Clearly

$$\aleph_\star^{(2)}(c_1\zeta,t)=\begin{cases}\xi_\star^{(-1)}\left(\frac{c_1\|\zeta\|_2}{t}\right), & t>0\\ 0, & t\le 0\end{cases}$$

and

$$\aleph_\star^{(2)}(c_2\zeta,t)=\begin{cases}\xi_\star^{(-1)}\left(\frac{c_2\|\zeta\|_2}{t}\right), & t>0\\ 0, & t\le 0\end{cases}$$

for all $\zeta\in Z$ and $t\in\mathbb{R}$. Since for each $\zeta\in Z$, $c_1\|\zeta\|_2\le\|\zeta\|_1$, $\frac{c_1\|\zeta\|_2}{t}\le\frac{\|\zeta\|_1}{t}$ for all $t>0$. By decreasing property of $\xi_\star^{(-1)}$ we have

$$\xi_\star^{(-1)}\left(\frac{c_1\|\zeta\|_2}{t}\right)\ge\xi_\star^{(-1)}\left(\frac{\|\zeta\|_1}{t}\right),\ \ t>0.$$

Therefore,

$$\aleph_\star^{(2)}(c_1\zeta,t)\ge\aleph_\star^{(1)}(\zeta,t)$$

for all $t>0$. Similarly, it can be shown that

$$\aleph_\star^{(1)}(\zeta,t)\ge\aleph_\star^{(2)}(c_2\zeta,t)$$

for all $t>0$. Hence inequality (1) hold for all $\zeta\in Z$ and $t\in\mathbb{R}$. $\square$

Definition 3.6. *[9] Let $(Z, \aleph, \star)$ be a fuzzy normed space. A sequence $\{\zeta_n\}_{n\geq 1}$ in Z is said to be fuzzy convergent, if there exists $\zeta \in Z$ such that for each $0 < \alpha < 1$ and $t > 0$, there exists $K \in \mathbb{N}$ such that for all $n \geq K$,*

$$\aleph\left(\zeta_n - \zeta, t\right) > 1 - \alpha.$$

Definition 3.7. *[9] Let $(Z, \aleph, \star)$ be a fuzzy normed space. A sequence $\{\zeta_n\}_{n\geq 1}$ in Z is said to be a fuzzy Cauchy sequence if for each $0 < \alpha < 1$ and $t > 0$, there exists $K \in \mathbb{N}$ such that for all $n > m \geq K$,*

$$\aleph\left(\zeta_n - \zeta_m, t\right) > 1 - \alpha.$$

Definition 3.8. *[8] A fuzzy normed space $(Z, \aleph, \star)$ is said to be a fuzzy Banach space, if every fuzzy Cauchy sequence, is fuzzy convergent in Z.*

Proposition 3.9. *Let $(Z, \|\cdot\|)$ be a Banach space and $\star$ be a CATN. Then $(Z, \aleph_\star, \star)$ is a fuzzy Banach space.*

Proof. Suppose that $\{\zeta_n\}_{n\geq 1}$ is a fuzzy Cauchy sequence in Z. We show that $\{\zeta_n\}_{n\geq 1}$ is a Cauchy sequence with respect to the norm $\|\cdot\|$. Let $\epsilon > 0$ be given. Since $\xi_\star : [0, 1] \longrightarrow [0, \infty]$ is continuous at $\alpha_0 = 1$ and $\xi_\star(1) = 0$, there exists $0 < \alpha_1 < 1$ such that $\xi_\star(1 - \alpha_1) < \epsilon$. By Definition 3.7, for α_1 and $t = 1$, there exists $K \in \mathbb{N}$ such that for all $n > m \geq K$

$$\aleph_\star\left(\zeta_n - \zeta_m, 1\right) = \xi_\star^{(-1)}\left(\frac{\|\zeta_n - \zeta_m\|}{1}\right) > 1 - \alpha_1 > 0.$$

So $\xi_\star^{(-1)}\left(\|\zeta_n - \zeta_m\|\right) = \xi_\star^{-1}\left(\|\zeta_n - \zeta_m\|\right)$ and consequently

$$\xi_\star^{-1}\left(\|\zeta_n - \zeta_m\|\right) > 1 - \alpha_1.$$

Because $\xi_\star$ is strictly decreasing, we obtain

$$\|\zeta_n - \zeta_m\| < \xi_\star\left(1 - \alpha_1\right) < \epsilon$$

for all $n > m \geq K$. This shows that $\{\zeta_n\}_{n\geq 1}$ is a Cauchy sequence. The completeness of Z implies that there exists $\zeta \in Z$ such that $\zeta_n \xrightarrow{\|\cdot\|} \zeta$.

For $t > 0$, since $\lim_{n\longrightarrow\infty} \frac{\|\zeta_n - \zeta\|}{t} = 0$, there exists $n_0 \in \mathbb{N}$ such that $\frac{\|\zeta_n - \zeta\|}{t} < \xi_\star(0)$ for all $n \geq n_0$. Hence

$$\begin{aligned}
\lim_{n\longrightarrow\infty} \aleph_\star(\zeta_n - \zeta, t) &= \lim_{n\longrightarrow\infty} \xi_\star^{(-1)}\left(\frac{\|\zeta_n - \zeta\|}{t}\right) \\
&= \lim_{n\longrightarrow\infty} \xi_\star^{-1}\left(\frac{\|\zeta_n - \zeta\|}{t}\right) \\
&= \xi_\star^{-1}\left(\lim_{n\longrightarrow\infty} \frac{\|\zeta_n - \zeta\|}{t}\right) \\
&= \xi_\star^{-1}(0) \\
&= 1.
\end{aligned}$$

So $\{\zeta_n\}_{n\geq1}$ is fuzzy convergent to ζ. This shows that Z is a fuzzy Banach space. □

Definition 3.10. *[2] Let Z be an algebra, $\star_1, \star_2$ be t-norms and $(Z, \aleph, \star_1)$ be a fuzzy normed space. If*

$$\aleph(\zeta\gamma, st) \geq \aleph(\zeta, s) \star_2 \aleph(\gamma, t) \tag{2}$$

for all $\zeta, \gamma \in Z$ and $s, t \in \mathbb{R}$, then $(Z, \aleph, \star_1, \star_2)$ is called a FNA (Fuzzy normed algebra).

Corollary 3.11. *Let $(Z, \|\cdot\|)$ be a normed algebra and $\star_H$ and $\star_p$ be defined as follows :*

$$\alpha \star_H \beta = \begin{cases} 0, & \alpha = \beta = 0 \\ \dfrac{\alpha\beta}{\alpha + \beta - \alpha\beta}, & o.w. \end{cases}$$

and $\alpha \star_p \beta = \alpha\beta$ for all $(\alpha, \beta) \in [0,1] \times [0,1]$. Then $(Z, \aleph_{\star_H}, \star_H, \star_p)$ is a FNA.

Proof. To prove, we only check inequality (2) for $\aleph_{\star_H}$. Since $\xi_{\star_H}(\alpha) = \frac{1-\alpha}{\alpha}$ which $\xi_{\star_H}(0) = \infty$ and $\xi_{\star_H}^{(-1)}(\alpha) = \xi_{\star_H}^{-1}(\alpha) = \frac{1}{1+\alpha}$, by Theorem

3.1 we have

$$\aleph_{\star_H}(\zeta,t)=\begin{cases}\dfrac{t}{t+\|\zeta\|}, & t>0\\ 0, & t\leq 0\end{cases}.$$

If $s\leq 0$ or $t\leq 0$, then inequality

$$\aleph_{\star_H}(\zeta\gamma,st)\geq\aleph_{\star_H}(\zeta,s)\star_P\aleph_{\star_H}(\gamma,t)$$

is obvious. If $s>0$ and $t>0$, we show that

$$\frac{st}{st+\|\zeta\gamma\|}\geq\left(\frac{s}{s+\|\zeta\|}\right)\left(\frac{t}{t+\|\gamma\|}\right). \tag{3}$$

But

$$\begin{aligned}\|\zeta\gamma\|&\leq\|\zeta\|\,\|\gamma\|\\&\leq\|\zeta\|\,\|\gamma\|+s\,\|\gamma\|+t\,\|\zeta\|.\end{aligned}$$

So

$$\begin{aligned}st+\|\zeta\gamma\|&\leq st+\|\zeta\|\,\|\gamma\|+s\,\|\gamma\|+t\,\|\zeta\|\\&=(s+\|\zeta\|)(t+\|\gamma\|)\end{aligned}$$

and consequently

$$\frac{st}{st+\|\zeta\gamma\|}\geq\frac{s}{s+\|\zeta\|}\cdot\frac{t}{t+\|\gamma\|}.$$

Therefore inequality (3) holds. □

Proposition 3.12. *Let Z be a normed algebra, $\star$ be a CATN and $\xi_\star$ be a CAG of $\star$ such that $\xi_\star(0)\leq 1$. Then $(Z,\aleph_\star,\star,\star)$ is a FNA.*

Proof. We only prove inequality (2) for $\aleph_\star$. Let $\zeta,\gamma\in Z$ and $s,t>0$. If $\xi_\star(0)\leq\dfrac{\|\zeta\|}{s}$ or $\xi_\star(0)\leq\dfrac{\|\gamma\|}{t}$, then $\aleph_\star(\zeta,s)=0$ or $\aleph_\star(\gamma,t)=0$. Hence

$$\aleph_\star(\zeta\gamma,st)\geq\aleph_\star(\zeta,s)\star\aleph_\star(\gamma,t)=0.$$

If $0 \leq \dfrac{\|\zeta\|}{s} < \xi_\star(0) \leq 1$ and $0 \leq \dfrac{\|\gamma\|}{t} < \xi_\star(0) \leq 1$, then

$$\frac{\|\zeta\gamma\|}{st} \leq \frac{\|\zeta\|}{s}\frac{\|\gamma\|}{t} \leq \frac{\|\zeta\|}{s} + \frac{\|\gamma\|}{t}.$$

Hence,

$$\begin{aligned}
\aleph_\star(\zeta\gamma, st) &= \xi_\star^{(-1)}\left(\frac{\|\zeta\gamma\|}{st}\right) \\
&\geq \xi_\star^{(-1)}\left(\frac{\|\zeta\|}{s} + \frac{\|\gamma\|}{t}\right) \\
&= \xi_\star^{(-1)}\left(\xi_\star\left(\xi_\star^{-1}\left(\frac{\|\zeta\|}{s}\right)\right) + \xi_\star\left(\xi_\star^{-1}\left(\frac{\|\gamma\|}{t}\right)\right)\right) \\
&= \xi_\star^{-1}\left(\frac{\|\zeta\|}{s}\right) \star \xi_\star^{-1}\left(\frac{\|\gamma\|}{t}\right) \\
&= \xi_\star^{(-1)}\left(\frac{\|\zeta\|}{s}\right) \star \xi_\star^{(-1)}\left(\frac{\|\gamma\|}{t}\right) \\
&= \aleph_\star(\zeta, s) \star \aleph_\star(\gamma, t).
\end{aligned}$$

□

We will show further that for a CAG of $\star$ where $\xi_\star(0) > 1$, it is possible to reach an algebra fuzzy norm.

Lemma 3.13. *Suppose that $\star$ is a CATN and $\xi_\star$ is a CAG of $\star$. Then*

1) $(k\xi_\star)^{(-1)}(\lambda) = \xi_\star^{(-1)}\left(\dfrac{\lambda}{k}\right), \quad 0 < k < \infty.$

2) $\alpha \star \beta = \xi_\star^{(-1)}(\xi_\star(\alpha) + \xi_\star(\beta)) = (k\xi_\star)^{(-1)}(k\xi_\star(\alpha) + k\xi_\star(\beta))$

for each $0 \leq \alpha \leq 1, 0 \leq \beta \leq 1$ *and* $0 < k < \infty$.

Proof. 1) Suppose that $\lambda \in [0, \infty]$ and $0 < k < \infty$. Therefore

$$\begin{aligned}
(k\xi_\star)^{(-1)}(\lambda) &= \sup\{0 \leq \alpha \leq 1,\ k\xi_\star(\alpha) > \lambda\} \\
&= \sup\left\{0 \leq \alpha \leq 1,\ \xi_\star(\alpha) > \frac{\lambda}{k}\right\} \\
&= \xi_\star^{(-1)}\left(\frac{\lambda}{k}\right).
\end{aligned}$$

2) Let $(\alpha,\beta)\in[0,1]\times[0,1]$ and $0<k<\infty$. Therefore by part 1 we have,

$$\begin{aligned}(k\xi_\star)^{(-1)}\left(k\xi_\star(\alpha)+k\xi_\star(\beta)\right)&=\xi_\star^{(-1)}\left(\frac{k\xi_\star(\alpha)+k\xi_\star(\beta)}{k}\right)\\&=(\xi_\star)^{(-1)}\left(\xi_\star(\alpha)+\xi_\star(\beta)\right)\\&=\alpha\star\beta.\end{aligned}$$

□

Lemma (3.13) states that t-norms produced by $\xi_\star$ and $k\xi_\star$ are the same.

Remark 3.14. Let $\star$ be a CATN and $0<k<\infty$. Then the fuzzy norm $\aleph_\star$ generated by $k\xi_\star$ as a CAG on a norm space $(Z,\|\cdot\|)$ is as follows :

$$\aleph_\star(\zeta,t)=\begin{cases}(k\xi_\star)^{(-1)}\left(\frac{\|\zeta\|}{t}\right), & t>0\\ 0, & t\le 0\end{cases}$$
$$=\begin{cases}(k\xi_\star)^{-1}\left(\frac{\|\zeta\|}{t}\right), & 0\le\frac{\|\zeta\|}{t}<k\xi_\star(0),\ t>0\\ 0, & k\xi_\star(0)\le\frac{\|\zeta\|}{t},\ t>0\\ 0, & t\le 0\end{cases}$$

for all $\zeta\in Z$ and $t\in\mathbb{R}$.

Proposition 3.15. *Let $(Z,\|\cdot\|)$ be a normed algebra, $\star$ be a CATN and $\xi_\star$ be a CAG of $\star$ such that $1<\xi_\star(0)<\infty$. If $\aleph_\star$ is the fuzzy norm generated by $k\xi_\star$ where $k=\dfrac{1}{\xi_\star(0)}$, then $(Z,\aleph_\star,\star,\star)$ is a FNA.*

Proof. By Lemma 3.13, $k\xi_\star$ is a CAG of $\star$. Since $k\xi_\star(0)=1$, by applying Proposition 3.12 the proof is trivial. □

Proposition 3.16. *Let $\star$ be a CATN and $\xi_\star$ be a CAG of $\star$ such that $\xi_\star(0)=\infty$. If $h:[0,1]\longrightarrow[0,\infty]$ is defined by $h(a)=\xi_\star\left(\mathrm{e}^{(a-1)}\right)$ for each $0\le a\le 1$, then*

$$h^{(-1)}(b)=\begin{cases}h^{-1}(b), & 0\le b<\xi_\star\left(\frac{1}{\mathrm{e}}\right)\\ 0, & b\ge\xi_\star\left(\frac{1}{\mathrm{e}}\right)\end{cases}$$

where $h^{-1}(b) = \mathrm{Ln}\left(\xi_\star^{-1}(b)\right) + 1$ *for all* $b \in \left[0, \xi_\star\left(\frac{1}{\mathrm{e}}\right)\right]$. *Moreover* $\star' : [0,1]^2 \longrightarrow [0,1]$ *defined by*

$$\alpha \star' \beta = h^{(-1)}\left(h(\alpha) + h(\beta)\right)$$

is a CATN.

Proof. Since h is a CAG of $\star'$, by Theorem 2.11, $\star'$ is a CATN. $\square$

Proposition 3.17. *Let* $(Z, \|\cdot\|)$ *be a normed algebra,* $\star$ *be a CATN and* $\xi_\star$ *be a CAG of* $\star$ *such that* $\xi_\star(0) = \infty$. *If* $h(a) = \xi_\star\left(\mathrm{e}^{(a-1)}\right)$ *for all* $0 \le a \le 1$, $\alpha \star' \beta = h^{(-1)}\left(h(\alpha) + h(\beta)\right)$ *for all* $(\alpha, \beta) \in [0,1] \times [0,1]$ *and* $\aleph_{\star'}$ *is the fuzzy norm generated by* kh *where* $k = \dfrac{1}{\xi_\star\left(\frac{1}{\mathrm{e}}\right)}$, *then* $(Z, \aleph_{\star'}, \star', \star')$ *is a FNA.*

Proof. Since $kh(0) = 1$, by Proposition 3.12, the proof is obvious. $\square$

By Propositions 3.12, 3.15 and 3.17 and using a CATN $\star$, a FNA can be produced. Within the next proposition, we are going appear that with each CATN, a new CATN can be made.

Proposition 3.18. *Let* $\star$ *be a CATN and* $\xi_\star$ *and* $\varrho_\star$ *be the corresponding CAG and CMG of* $\star$ *respectively. If* $h : [0,1] \longrightarrow [0,\infty]$ *is defined by*

$$h(a) = (\xi_\star \circ \varrho_\star)(a)$$

for all $0 \le a \le 1$, *then* h *is a CAG of* $\star'$ *where*

$$\alpha \star' \beta = h^{(-1)}\left(h(\alpha) + h(\beta)\right)$$

for all $(\alpha, \beta) \in [0,1] \times [0,1]$.

Proof. Since h is a CAG of $\star'$, by Theorem 2.11, $\star'$ is a CATN. $\square$

Corollary 3.19. *Let* $\star$ *be a CATN and* $\xi_\star^{(1)}, \xi_\star^{(2)}$ *be two CAGs of* $\star$. *If* $\aleph_\star^{(1)}$ *and* $\aleph_\star^{(2)}$ *are fuzzy norms generated by* $\xi_\star^{(1)}$ *and* $\xi_\star^{(2)}$ *on a normed space* $(Z, \|\cdot\|)$ *respectively, then there exists* $\alpha \in (0, \infty)$ *such that*

$$\aleph_\star^{(2)}(\zeta, t) = \aleph_\star^{(1)}(\zeta, \alpha t)$$

for all $\zeta \in Z$ *and* $t \in \mathbb{R}$. *In particular if* $\alpha \leq 1$, *then*

$$\aleph_{\star}^{(2)}(\zeta, t) \leq \aleph_{\star}^{(1)}(\zeta, t)$$

for all $\zeta \in Z$ *and* $t \in \mathbb{R}$. *If* $\alpha > 1$, *then*

$$\aleph_{\star}^{(2)}(\zeta, t) \geq \aleph_{\star}^{(1)}(\zeta, t)$$

for all $\zeta \in Z$ *and* $t \in \mathbb{R}$.

Proof. Since for each CATN $\star$, a CAG of $\star$ is uniquely determined up to a positive multiplicative constant [5], then we assume that

$$\xi_{\star}^{(2)} = \alpha \xi_{\star}^{(1)}$$

for some $\alpha \in (0, \infty)$. Hence

$$\begin{aligned}
\aleph_{\star}^{(2)}(\zeta, t) &= \begin{cases} \left(\xi_{\star}^{(2)}\right)^{(-1)} \left(\dfrac{\|\zeta\|}{t}\right), & t > 0 \\ 0, & t \leq 0 \end{cases} \\
&= \begin{cases} \left(\alpha \xi_{\star}^{(1)}\right)^{(-1)} \left(\dfrac{\|\zeta\|}{t}\right), & t > 0 \\ 0, & t \leq 0 \end{cases} \\
&= \begin{cases} \left(\xi_{\star}^{(1)}\right)^{(-1)} \left(\dfrac{\|\zeta\|}{\alpha t}\right), & t > 0 \\ 0, & t \leq 0 \end{cases} \\
&= \begin{cases} \left(\xi_{\star}^{(1)}\right)^{(-1)} \left(\dfrac{\|\zeta\|}{\alpha t}\right), & \alpha t > 0 \\ 0, & \alpha t \leq 0 \end{cases} \\
&= \aleph_{\star}^{(1)}(\zeta, \alpha t)
\end{aligned}$$

for all $\zeta \in Z$ and $t \in \mathbb{R}$. Since $\aleph_{\star}(\zeta, \cdot)$ is increasing on $\mathbb{R}$, the remain of the proof is obvious. $\square$

Conclusion

In this paper, we conclude that it is possible to reach a fuzzy norm and an algebra fuzzy norm on a normed space $(Z, \|\cdot\|)$ by applying a CATN $\star$ using the mentioned methods.

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