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Original Research Paper

An Efficient Hybrid Scheme for Solving Time-Space Fractional Schrödinger Equation with Error Analysis

F. Hashemi
Shiraz University of Technology

A. Taleei*
Shiraz University of Technology

E. Hesameddini
Shiraz University of Technology

Abstract. A numerical approximation combining the fast finite difference method in time and the finite element method in space is studied to solve the distributed-order time and Riesz space fractional Schrödinger equation. In this work, a fast evaluation of the distributed-order time fractional derivative based on graded time mesh is applied to the time approximation of this equation. Also, the finite element method is used for space approximation. Moreover, the stability and convergence of the resulting discrete scheme are discussed. Finally, some numerical examples are presented to confirm the theoretical results.

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*Corresponding Author

1 Introduction

Consider the distributed-order time and Riesz space fractional Schrödinger equation (DOT-RSFSE) as follows:

$$iD_t^{\omega(\alpha)}\psi(x,t)+\delta\frac{\partial^{2\beta}}{\partial|x|^{2\beta}}\psi(x,t)-v(x,t)\psi(x,t)-\lambda|\psi(x,t)|^2\psi(x,t)-z(x,t)=0, \tag{1}$$

with the initial and Dirichlet boundary conditions

$$\psi(x,0)=f(x), \qquad x\in[a,b],$$

$$\psi(a,t)=\psi(b,t)=0, \qquad t\in[0,1],$$

where parameter δ is a real constant, z, f and unknown function ψ are complex-valued, v is general potential, $\frac{1}{2}<\beta\leq 1$, $\frac{\partial^{2\beta}}{\partial|x|^{2\beta}}$ is the Riesz fractional derivative, $D_t^{\omega(\alpha)}$ denotes the distributed-order fractional derivative and the function ω satisfies $0<\int_0^1\omega(\alpha)d\alpha<\infty$ [22]. Also, the Caputo fractional derivative and the Riesz fractional derivative are defined as follows, respectively

$${}_0^CD_t^{\alpha}\psi(x,t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t(t-s)^{-\alpha}\frac{\partial\psi}{\partial s}ds,$$

$$\frac{\partial^{2\beta}}{\partial|x|^{2\beta}}\psi(x,t)=\frac{-1}{2\cos(\beta\pi)}(D_{\mathcal{L}}^{2\beta}\psi+D_{\mathcal{R}}^{2\beta}\psi),$$

where the left-side and right-sided Riemann-Liouville derivatives on x are defined as

$$D_{\mathcal{L}}^{2\beta}\psi(x,t)=\frac{1}{\Gamma(2-2\beta)}\frac{\partial^2}{\partial x^2}\int_a^x(x-s)^{1-2\beta}\psi(x,s)ds,$$

$$D_{\mathcal{R}}^{2\beta}\psi(x,t)=\frac{1}{\Gamma(2-2\beta)}\frac{\partial^2}{\partial x^2}\int_x^b(s-x)^{1-2\beta}\psi(x,s)ds.$$

The fractional Schrödinger equations (FSEs) play an important role in describing physical phenomena, such as quantum mechanics, optics, seismology, and plasma physics [6, 11, 12, 21, 25]. Extensive theoretical research has been carried out on the FSEs that interested readers can refer to [5, 23]. Several numerical methods and their analysis for the linear and nonlinear fractional Schrödinger equations have been investigated [7, 8]. Bhrawy [2] used the Jacobi spectral collocation method to solve FSEs and their coupled kind. In [1], a fully spectral collocation approximation was developed to solve multi-dimensional time FSEs. Li et al. [17] applied the Galerkin finite element method for nonlinear space fractional Schrödinger equations. In [19], Li and Zhao considered a fast energy-conserving finite element method for nonlinear space fractional Schrödinger equations with wave operator. In [28], Mustafa and Almushaira presented a fast implicit difference scheme for solving high-dimensional time-space fractional nonlinear Schrödinger equation. Qayyum et al. [29] studied new solutions of time-space fractional coupled Schrödinger systems. Yin et al. [27] developed the structure-preserving difference scheme with fast algorithms for high dimensional space fractional Schrödinger equations. Li et al. studied a relaxation-type Galerkin FEM for nonlinear fractional Schrödinger equations in [16]. An unconditionally convergent L1-Galerkin FEMs is studied for nonlinear time fractional Schrödinger equation [15]. A fast L2-l_σ Galerkin FEMs is presented for generalized nonlinear coupled time fractional Schrödinger equation with Caputo derivative and obtained optimal order error estimate [18]. Wang et al. [26] considered the second-order and linear numerical schemes for the multi-dimensional nonlinear time fractional Schrödinger equation.

Although numerous numerical methods have been proposed for time-space fractional Schrödinger equations, most of them have limited to time fractional Schrödinger equations on uniform meshes. Recently, Heydari [9] presented a computational approach for a system of coupled distributed-order time fractional Klein-Gordon Schrödinger equations. Bhrawy and Zaky [3] developed an efficient spectral solution for distributed-order time fractional Schrödinger equations. Effective numerical methods and supporting error analysis for the time-space fractional Schrödinger equations with distributed order are still limited.

The main contribution of this work is to establish an optimal error estimate of the proposed linearized numerical scheme for nonlinear DOT-RSFSE without any step size restriction. We use the time-space error splitting argument presented in [13] for unconditionally error estimates.

In this paper, we develop a fast finite difference scheme based on graded time meshed in time. An optimal error estimate of this fully discrete system is obtained using the Galerkin finite element approximation in space. To the best of our knowledge, in the previous works, we mainly paid attention to established unconditionally stable and convergence results for time-space fractional Schrödinger equations using L1-FEM. However, there are few papers focusing on convergence analysis of numerical methods DOT-RSFSE.
A brief outline of the paper is as follows: In Section 2 the definitions and properties of fractional derivatives, fast finite difference method, and fractional Sobolev space are recalled. A semi-discrete variational scheme for Eq (1) is given in Section 3. Also, an implicit $L1$- Galerkin finite element method for the fully discrete system based on the standard Galerkin finite element method in space and the fast $L1$ algorithm in time are given. Section 4 is devoted to the semi-discrete system's unconditional stability and optimal error estimate. Moreover, L^2-norm and H^β-norm error estimates of the fully discrete scheme are presented for the DOT-RFSEs. In Section 5 some numerical examples are given to confirm our theoretical results. Finally, a brief conclusion is given in Section 6.

2 Preliminaries

In this section, we recall some definitions and lemmas needed in the numerical analysis of the presented algorithm.

Definition 2.1. *Let $\mu > 0$ and $\Omega = [a, b]$. Define the left and right norms as*

$$\|\psi\|_{J^\mu_{\mathcal{L}}(\Omega)} = (\|\psi\|^2_{L^2(\Omega)} + |\psi|^2_{J^\mu_{\mathcal{L}}(\Omega)})^{\frac{1}{2}},$$

$$\|\psi\|_{J^\mu_{\mathcal{R}}(\Omega)} = (\|\psi\|^2_{L^2(\Omega)} + |\psi|^2_{J^\mu_{\mathcal{R}}(\Omega)})^{\frac{1}{2}},$$

in which, $|\psi|_{J^\mu_{\mathcal{L}}(\Omega)} = \|D^\mu_{\mathcal{L}}\psi\|_{L^2(\Omega)}$ and $|\psi|_{J^\mu_{\mathcal{R}}(\Omega)} = \|D^\mu_{\mathcal{R}}\psi\|_{L^2(\Omega)}$ are the semi-norms. Also, we denote $J^\mu_{\mathcal{L}}(\Omega)$ and $J^\mu_{\mathcal{R}}(\Omega)$ as the closure of $C^\infty(\Omega)$ with respect to $\|.\|_{J^\mu_{\mathcal{L}}(\Omega)}$ and $\|.\|_{J^\mu_{\mathcal{R}}(\Omega)}$, respectively.

Definition 2.2. *(Fractional Sobolev space). Let $\mu > 0$ and $\mathcal{F}$ be the Fourier transform of ψ defined on Ω, i.e. $\mathcal{F}(\psi) = \int_\Omega \psi(x)e^{-iwx}dx$, with* *the variable w. Define the semi-norm of ψ as*

$$|\psi|_{H^{\mu}(\Omega)} = \||w|^{\mu}\mathcal{F}(\psi)\|_{L^2(\Omega)},$$

and the norm

$$\|\psi\|_{H^{\mu}(\Omega)} = (\|\psi\|^2_{L^2(\Omega)} + |\psi|^2_{H^{\mu}(\Omega)})^{\frac{1}{2}},$$

where H^{μ} be the closure of $C^{\infty}(\Omega)$ with respect to $\|.\|_{H^{\mu}(\Omega)}$.

Lemma 2.3. *[14] For $\mu > 0$, the following results hold*
a) $J^{\mu}_{\mathcal{R}}(\Omega)$, $J^{\mu}_{\mathcal{L}}(\Omega)$ and $H^{\mu}(\Omega)$ are equal with equivalent semi-norm and norm, respectively.
b) the following property in L^2-sense holds

$$(D^{\mu}_{\mathcal{L}}\psi, D^{\mu}_{\mathcal{R}}\psi) = \cos(\mu\pi)\|D^{\mu}_{\mathcal{L}}\|^2_{L^2(\Omega)}.$$

c) For $\psi \in H^{\mu}_0(\Omega)$ and $0 \leq r \leq \mu$, we have

$$\|\psi\|_{L^2(\Omega)} \leq C|\psi|_{H^{\mu}(\Omega)}, \quad ||\psi||_{H^{r}(\Omega)} \leq C|\psi|_{H^{\mu}(\Omega)},$$

d) If $0 < \mu < 1$, $\psi \in H^{2\mu}_0(\Omega)$ and $\phi \in H^{\mu}_0(\Omega)$, then

$$(D^{2\mu}_{\mathcal{L}}\psi, \phi) = (D^{\mu}_{\mathcal{L}}\psi, D^{\mu}_{\mathcal{R}}\phi), \quad (D^{2\mu}_{\mathcal{R}}\psi, \phi) = (D^{\mu}_{\mathcal{R}}\psi, D^{\mu}_{\mathcal{L}}\phi).$$

2.1 Fast L1 method

This part introduces a fast evaluation method for the time fractional derivative. Let $t_n = T(\frac{n}{N_T})^r$ $(n = 0, 1, 2, 3, ..., N_T)$ and choose the mesh parameter $r \geq 1$ and $\tau_{n+1} = t_{n+1} - t_n$.
Splitting the convolution integral Caputo derivative into a sum of the local part and history part results in

$$\begin{aligned}
&{}_0^C D_t^{\alpha}\psi(t) \mid_{t=t_n}= \\
&\frac{1}{\Gamma(1-\alpha)}\int_{t_{n-1}}^{t_n}\frac{\psi'(s)}{(t_n-s)^{\alpha}}ds + \frac{1}{\Gamma(1-\alpha)}\int_0^{t_{n-1}}\frac{\psi'(s)}{(t_n-s)^{\alpha}}ds \\
&:= C_l(t_n) + C_h(t_n), \qquad (2)
\end{aligned}$$

where C_l and C_h are local and history parts, respectively. For the local part, the function $\psi'(s)$ will be approximated by linear interpolation as follows,

$$C_l(t_n) = \frac{\psi(t_n) - \psi(t_{n-1})}{\tau_n \Gamma(1-\alpha)} \int_{t_n}^{t_{n-1}} \frac{1}{(t_n - s)^{\alpha}} ds = \frac{\psi(t_n) - \psi(t_{n-1})}{\tau_n^{\alpha} \Gamma(1-\alpha)}. \quad (3)$$

Also, for the history part using the integration by parts to eliminate $\psi'(s)$ and Theorem 2.1 in [24], one obtains

$$\begin{aligned} C_h(t_n) &= \frac{1}{\Gamma(1-\alpha)} \left[\frac{\psi(t_{n-1})}{\tau_n^{\alpha}} - \frac{\psi(t_0)}{t_n^{\alpha}} - \alpha \int_0^{t_{n-1}} \frac{\psi(s)}{(t_n - s)^{1+\alpha}} ds \right] \\ &\approx \frac{1}{\Gamma(1-\alpha)} \left[\frac{\psi(t_{n-1})}{\tau_n^{\alpha}} - \frac{\psi(t_0)}{t_n^{\alpha}} - \alpha \sum_{i=1}^{N_{exp}} w_i \psi_{hist,i}(t_n) \right], \end{aligned} \quad (4)$$

where $\psi_{hist,i}(t_n) = \int_0^{t_{n-1}} e^{-(t_n - s)s_i} \psi(s) ds$ $(n = 1, 2, 3, ..., N_T)$. So, it is easy to get

$$\psi_{hist,i}(t_n) = e^{-s_i \tau_n} \psi_{hist,i}(t_{n-1}) + \int_{t_{n-2}}^{t_{n-1}} e^{-(t_n - s)s_i} \psi(s) ds,$$

where $\psi_{hist,i}(t_1) = 0$. Besides, interpolation ψ via a linear function on $[t_{n-2}, t_{n-1}]$ yields

$$\begin{aligned} &\int_{t_{n-2}}^{t_{n-1}} e^{-(t_n - s)s_i} \psi(s) ds \approx \frac{e^{-s_i \tau_n}}{s_i^2 \tau_n} \Big[(e^{-s_i \tau_{n-1}} - 1 + s_i \tau_{n-1}) \psi(t_{n-1}) \\ &+ (1 - e^{-s_i \tau_{n-1}} - e^{-s_i \tau_{n-1}} s_i \tau_{n-1}) \psi(t_{n-2}) \Big]. \end{aligned}$$

Substituting (3) and (4) in (2), implies that

$$\begin{aligned} {}_0^F\mathfrak{D}_t^{\alpha} \psi(t_n) &= \frac{1}{\tau_n^{\alpha} \Gamma(1-\alpha)} \Big[\frac{\psi(t_n)}{1-\alpha} \big(\frac{\alpha}{1-\alpha} + \c{C}_{0,n}^{\alpha} \big) \psi(t^{n-1}) \\ &- \sum_{l=1}^{n-2} \big(\c{C}_{n-l-1,n}^{\alpha} + \dot{\c{C}}_{n-l-2,n}^{\alpha} \big) \psi(t_l) - \big(\dot{\c{C}}_{n-2,n}^{\alpha} + \big(\frac{\tau_n}{t_n} \big)^{\alpha} \big) \psi(t_0) \Big] \\ &= \frac{1}{\tau_n^{\alpha} \Gamma(1-\alpha)} \sum_{j=0}^{n} {}^{\alpha}\ddot{\c{C}}_j^{n} \psi(t_j), \end{aligned} \quad (5)$$

where the coefficients ${}^{\alpha}\ddot{\c{C}}_j^n$ are given by the following formula

$$
{}^{\alpha}\ddot{\c{C}}_j^n = \begin{cases} \frac{1}{1-\alpha}, & j=n, \\ -(\frac{\alpha}{1-\alpha}+\c{C}_{0,n}^{\alpha}), & j=n-1, \\ -(\c{C}_{n-j-1,n}^{\alpha}+\dot{\c{C}}_{n-j-2,n}^{\alpha}), & j=1,2,...,n-2, \\ -(\dot{\c{C}}_{n-2,n}^{\alpha}+(\frac{\tau_n}{t_n})^{\alpha}), & j=0, \end{cases}
$$

in which

$$
\c{C}_{j,n}^{\alpha} = \alpha\tau_n^{\alpha}\sum_{i=1}^{Nexp} w_i e^{-s_i(t_n-t_{n-j})}\ell_{i,n-j}^1,
$$

$$
\dot{\c{C}}_{j,n}^{\alpha} = \alpha\tau_n^{\alpha}\sum_{i=1}^{Nexp} w_i e^{-s_i(t_n-t_{n-j})}\ell_{i,n-j}^2,
$$

$$
\ell_{i,n}^1 = \frac{e^{-s_i\tau_n}}{s_i^2\tau_{n-1}}(e^{-s_i\tau_{n-1}}-1+s_i\tau_{n-1}),
$$

$$
\ell_{i,n}^2 = \frac{e^{-s_i\tau_n}}{s_i^2\tau_{n-1}}(1-e^{-s_i\tau_{n-1}}-s_i\tau_{n-1}e^{-s_i\tau_{n-1}}).
$$

Then,

$$
{}_0^CD_t^{\alpha}\psi(t_n) \approx \frac{1}{\tau_n^{\alpha}\Gamma(1-\alpha)}\sum_{j=0}^{n}{}^{\alpha}\ddot{\c{C}}_j^n\psi(t_j) := {}_0^F\mathfrak{D}_t^{\alpha}\psi(t_n).
$$

Using Lemma 2.5 in [24], the following error estimate for the presented fast L1 method will be obtained.

Remark: Assume that $0<\alpha<d$, and let

$$
{}^F\Xi^{n,\alpha}\psi(t_n) := {}_0^CD_t^{\alpha}\psi(t_n) - {}_0^F\mathfrak{D}_t^{\alpha}\psi(t_n), \quad n=1,2,3,...,N_T,
$$

then,

$$
|{}^F\Xi^{n,\alpha}\psi(t_n)| \preceq n^{-min(2-\alpha,r(\alpha+1))}+\epsilon, \tag{6}
$$

where ϵ is the tolerance error.

3 The Proposed Numerical Scheme

At first, we decompose the complex-valued functions ψ and z by their real and imaginary parts as $\psi(x,t) = \psi_R(x,t) + i\psi_I(x,t)$ and $z(x,t) = z_R(x,t) + iz_I(x,t)$. Inserting these complex forms in Eq (1), one obtains a coupled system of equations as

$$\begin{cases} D_t^{\omega(\alpha)}\psi_I - \delta\frac{\partial^{2\beta}}{\partial|x|^{2\beta}}\psi_R + \lambda|\psi|^2\psi_R + v\psi_R + z_R = 0, \\ D_t^{\omega(\alpha)}\psi_R + \delta\frac{\partial^{2\beta}}{\partial|x|^{2\beta}}\psi_I - \lambda|\psi|^2\psi_I - v\psi_I(x,t) - z_I = 0, \end{cases} \tag{7}$$

with the initial and boundary conditions as follows

$$\psi_R(x,0) = f_R(x), \qquad \psi_I(x,0) = f_I(x),$$

$$\psi_R(a,t) = \psi_R(b,t) = 0, \qquad \psi_I(a,t) = \psi_I(b,t) = 0,$$

in which indices R and I point to the real and imaginary parts, respectively.
By the Lemma 2.3.(d), we can derive the weak form of system (7) for any $\varphi_R, \varphi_I \in H_0^\beta$, as

$$\begin{cases} (D_t^{\omega(\alpha)}\psi_I, \varphi_R) - \Theta(\psi_R, \varphi_R) + \lambda(|\psi|^2\psi_R, \varphi_R) + (v\psi_R + z_R, \varphi_R) = 0, \\ (D_t^{\omega(\alpha)}\psi_R, \varphi_I) + \Theta(\psi_I, \varphi_I) - \lambda(|\psi|^2\psi_I, \varphi_I) - (v\psi_I - z_I, \varphi_I) = 0, \end{cases} \tag{8}$$

with the initial and boundary conditions

$$\psi_R(x,0) = f_R(x), \qquad \psi_I(x,0) = f_I(x),$$

$$\psi_R(a,t) = \psi_R(b,t) = 0, \qquad \psi_I(a,t) = \psi_I(b,t) = 0,$$

where $\Theta(.,.)$ is defined by

$$\Theta(\psi,\varphi) := \delta(\frac{\partial^{2\beta}}{\partial|x|^{2\beta}}\psi, \varphi) = C_x^\beta\big((D_{\mathcal{L}}^\beta\psi, D_{\mathcal{R}}^\beta\varphi) + (D_{\mathcal{R}}^\beta\psi, D_{\mathcal{L}}^\beta\varphi)\big), \tag{9}$$

in which $C_x^\beta = -\frac{\delta}{2\cos(\beta\pi)}$.

3.1 Semi-discrete scheme

To obtain the numerical solution of Eq (1), let $\sigma_\alpha = \frac{1}{M_\alpha}$ and $\alpha_m = (m+\frac{1}{2})\sigma_\alpha$where $m = 0, 1, 2, ..., M_\alpha$-1, $\alpha_0 < \alpha_1 < ... < \alpha_{M_\alpha}$ and $\omega \in C^2[0, 1]$. At first, we discretize the distributed-order time fractional derivative by midpoint rule [10], as follows

$$D_t^{\omega(\alpha)}\psi_I(x,t) = \int_0^1 \omega(\alpha){}_0^C D_t^\alpha \psi_I(x,t)d\alpha \approx \sigma_\alpha \sum_{m=0}^{M_\alpha-1} \omega(\alpha_m){}_0^C D_t^{\alpha_m}\psi_I,$$

$$D_t^{\omega(\alpha)}\psi_R(x,t) = \int_0^1 \omega(\alpha){}_0^C D_t^\alpha \psi_R(x,t)d\alpha \approx \sigma_\alpha \sum_{m=0}^{M_\alpha-1} \omega(\alpha_m){}_0^C D_t^{\alpha_m}\psi_R.$$

Substituting these relations in (8), yields

$$\begin{cases} \left(\sigma_\alpha \sum_{m=0}^{M_\alpha-1} \omega(\alpha_m){}_0^C D_t^{\alpha_m}\psi_I, \varphi_R\right) + \beth(\psi_R, \varphi_R) + (z_R, \varphi_R) = \Xi_I^{\omega(\alpha)}, \\ \left(\sigma_\alpha \sum_{m=0}^{M_\alpha-1} \omega(\alpha_m){}_0^C D_t^{\alpha_m}\psi_R, \varphi_I\right) + \beth(\psi_I, \varphi_I) - (z_I, \varphi_I) = \Xi_R^{\omega(\alpha)}, \end{cases}$$

where $\beth(.,.)$ is defined by

$$\beth(\psi_R, \varphi_R) = -\Theta(\psi_R, \varphi_R) + \lambda(|\psi|^2\psi_R, \varphi_R) + (v\psi_R, \varphi_R),$$

$$\beth(\psi_I, \varphi_I) = \Theta(\psi_I, \varphi_I) - \lambda(|\psi|^2\psi_I, \varphi_I) - (v\psi_I, \varphi_I).$$

Choosing $\Upsilon_\alpha^\tau(\omega(\alpha){}_0^C D_t^{\alpha_m}\psi) := \sigma_\alpha \sum_{m=0}^{M_\alpha-1} \omega(\alpha_m){}_0^C D_t^{\alpha_m}\psi$, this system can be written as

$$\begin{cases} \left(\Upsilon_\alpha^\tau(\omega(\alpha){}_0^C D_t^{\alpha_m}\psi_I), \varphi_R\right) + \beth(\psi_R, \varphi_R) + (z_R, \varphi_R) = \Xi_I^{\omega(\alpha)}, \\ \left(\Upsilon_\alpha^\tau(\omega(\alpha){}_0^C D_t^{\alpha_m}\psi_R), \varphi_I\right) + \beth(\psi_I, \varphi_I) - (z_I, \varphi_I) = \Xi_R^{\omega(\alpha)}. \end{cases} \tag{10}$$

Now, we consider the discretization of time Caputo fractional derivative of order $0 < \alpha < 1$ by fast $L1$ method. Applying relation (5) in system (10), we get the following semi-discrete time of Eq (1) as follows

$$\begin{cases} \left(\Upsilon_\alpha^\tau(\omega(\alpha){}_0^F \mathfrak{D}_t^{\alpha_m}\psi_I^n), \varphi_R\right) + \beth(\psi_R^n, \varphi_R) + (z_R^n, \varphi_R) = \Xi_I^{\omega(\alpha)} + {}^F\Xi_I^\alpha, \\ \left(\Upsilon_\alpha^\tau(\omega(\alpha){}_0^F \mathfrak{D}_t^{\alpha_m}\psi_R^n), \varphi_I\right) + \beth(\psi_I^n, \varphi_I) - (z_I^n, \varphi_I) = \Xi_R^{\omega(\alpha)} + {}^F\Xi_R^\alpha, \end{cases} \tag{11}$$

where

$$\beth(\psi_R^n,\varphi_R) = -\Theta(\psi_R^n,\varphi_R) + \lambda(|\psi^{n-1}|^2\psi_R^n,\varphi_R) + (v^n\psi_R^n,\varphi_R),$$

$$\beth(\psi_I^n,\varphi_I) = \Theta(\psi_I^n,\varphi_I) - \lambda(|\psi^{n-1}|^2\psi_I^n,\varphi_I) - (v^n\psi_I^n,\varphi_I),$$

and

$$\Xi^{\omega(\alpha)} = D_t^{\omega(\alpha)}\psi - \sigma_\alpha\sum_{m=0}^{M_\alpha-1}\omega(\alpha_m)\,{}_0^C D_t^{\alpha_m}\psi = O(\sigma_\alpha^2),$$

$${}^F\Xi^\alpha = {}_0^C D_t^{\alpha_m}\psi(t_n) - {}_0^F\mathfrak{D}_t^{\alpha_m}\psi(t_n) = O(n^{-min(2-\alpha,r(\alpha+1))}).$$

3.2 Fully discrete scheme

Let T_h be a family of subdivisions of $\Omega=[a,b]$, $\Omega_h=\{e_h|\ \ e_h\in T_h\}$ and $X_h^\beta=\{\varphi_h\in H_0^\beta\cap C^0(\Omega);\ \ \varphi_h|_{e_h}\in P_k(e_h),\ \ e_h\in\Omega_h\}$ where $P_k(e_h)$ is a set of polynomials with degree at most q, and h be the step size in space.
Assume that $\psi_{R,h}^n$ and $\psi_{I,h}^n\in X_h^\beta$ are the approximation of ψ_R^n and ψ_I^n. So, we have

$$\begin{cases}\left(\Upsilon_\alpha^\tau(\omega(\alpha){}_0^F\mathfrak{D}_t^{\alpha_m}\psi_{I,h}^n),\varphi_{R,h}\right)+\beth(\psi_{R,h}^n,\varphi_{R,h})-(z_R^n,\varphi_{R,h})=0,\\ \forall\varphi_{R,h}\in X_h^\beta,\\ \left(\Upsilon_\alpha^\tau(\omega(\alpha){}_0^F\mathfrak{D}_t^{\alpha_m}\psi_{R,h}^n),\varphi_{I,h}\right)+\beth(\psi_{I,h}^n,\varphi_{I,h})+(z_I^n,\varphi_{I,h})=0,\\ \forall\varphi_{I,h}\in X_h^\beta.\end{cases} \tag{12}$$

4 Error Analysis

4.1 Stability and convergence of the semi-discrete scheme

First, we recall the following lemma to analyze the stability and convergence of the presented numerical scheme.

Lemma 4.1. *[4]. (Gronwall inequality) Assume that $\{k_j\}_{j=0}^{n-1}$ and $\{\varepsilon_j\}_{j=0}^n$ be the non-negative sequences such that $\rho_0\geq 0$, $\varepsilon_0\leq\rho_0$ and*

$$\varepsilon_n\leq\rho_0+\sum_{j=0}^{n-1}k_j\varepsilon_j,\qquad n\geq 1,$$

then

$$\varepsilon_n \leq \rho_0 . \exp\big(\sum_{j=0}^{n-1} k_j\big), \qquad n \geq 1.$$

In the following to simplify the notations, we denote the norms of L^2 and H^β by $\|.\|$ and $\|.\|_\beta$. Now, we discuss some of the properties of the bilinear form, stability, and convergence of the scheme (11).

Lemma 4.2. *The bilinear form $\Theta(\psi, \varphi)$ is continuous and coercive i.e.*

$$\Theta(\psi, \varphi) \leq C^{\beta}_{max} \|\psi\|_\beta . \|\varphi\|_\beta,$$
$$\Theta(\psi, \psi) \geq 2C^{\beta}_{min} \|\psi\|^2_\beta,$$

where C^{β}_{max} and C^{β}_{min} are positive constants.

Proof. According to the definition of bilinear form in (9), we have

$$\Theta(\psi, \varphi) = C^{\beta}_{x}\bigg((D^{\beta}_{\mathcal{L}}\psi, D^{\beta}_{\mathcal{R}}\varphi) + (D^{\beta}_{\mathcal{R}}\psi, D^{\beta}_{\mathcal{L}}\varphi)\bigg) \leq$$
$$|C^{\beta}_{max}|\bigg(\big|(D^{\beta}_{\mathcal{L}}\psi, D^{\beta}_{\mathcal{R}}\varphi)\big| + \big|(D^{\beta}_{\mathcal{R}}\psi, D^{\beta}_{\mathcal{L}}\varphi)\big|\bigg).$$

Using the Cauchy-Schwartz inequality and the norm of fractional Sobolev space, one obtains

$$\Theta(\psi, \varphi) \leq |C^{\beta}_{max}|\bigg(\|D^{\beta}_{\mathcal{L}}\psi\|\|D^{\beta}_{\mathcal{R}}\varphi\| + \|D^{\beta}_{\mathcal{R}}\psi\|\|D^{\beta}_{L}\varphi\|\bigg) \leq 2|C^{\beta}_{max}|\|\psi\|_\beta\|\varphi\|_\beta,$$

thus

$$\Theta(\psi, \varphi) \leq 2|C^{\beta}_{max}|\|\psi\|_\beta\|\varphi\|_\beta,$$

and

$$\Theta(\psi, \psi) = 2C^{\beta}_{x}(D^{\beta}_{\mathcal{L}}\psi, D^{\beta}_{\mathcal{R}}\psi) = 2|C^{\beta}_{x}|\big|(D^{\beta}_{\mathcal{L}}\psi, D^{\beta}_{\mathcal{R}}\psi)\big|.$$

Also, by the fractional Poincare-Friedrichs inequality, we get

$$\Theta(\psi, \psi) \geq 2C^{\beta}_{min}\|D^{\beta}_{\mathcal{R}}\psi\|.\|D^{\beta}_{\mathcal{L}}\psi\| \geq 2C^{\beta}_{min}\|\psi\|^2_\beta.$$

Therefore, bilinear form $\Theta(.,.)$ is continuous and coercive.

Theorem 4.3. *The proposed numerical scheme* (11) *is unconditionally stable, in other words, we have*

$$\|\xi^k\|^2 + \|\eta^k\|^2 \le \|\xi^0\|^2 + \|\eta^0\|^2.$$

Proof. Suppose that ϕ_I^n and ϕ_R^n are another solution of the semi-discrete formulation (11), which means that

$$\begin{cases} \big(\Upsilon_\alpha^\tau(\omega(\alpha)_0^F\mathfrak{D}_t^{\alpha_m}\phi_I^n), \varphi_R\big) + \beth(\phi_R^n, \varphi_R) - (z_R^n, \varphi_R) = \Xi_I^{\omega(\alpha)} + {}^F\Xi_I^\alpha, \\ \big(\Upsilon_\alpha^\tau(\omega(\alpha)_0^F\mathfrak{D}_t^{\alpha_m}\phi_R^n), \varphi_I\big) + \beth(\phi_I^n, \varphi_I) + (z_I^n, \varphi_I) = \Xi_R^{\omega(\alpha)} + {}^F\Xi_R^\alpha. \end{cases} \tag{13}$$

Subtracting (11) from (13), implies that

$$\begin{cases} \big(\Upsilon_\alpha^\tau(\omega(\alpha)_0^F\mathfrak{D}_t^{\alpha_m}(\psi_I^n - \phi_I^n)), \varphi_R\big) + \beth(\psi_R^n - \phi_R^n, \varphi_R) = 0, \\ \big(\Upsilon_\alpha^\tau(\omega(\alpha)_0^F\mathfrak{D}_t^{\alpha_m}(\psi_R^n - \phi_R^n)), \varphi_I\big) + \beth(\psi_I^n - \phi_I^n, \varphi_I) = 0. \end{cases} \tag{14}$$

Set $\eta^n = \psi_R^n - \phi_R^n$ and $\xi^n = \psi_I^n - \phi_I^n$ in (14). Also, let $\varphi_I = \eta^n$ and $\varphi_R = \xi^n$. So we get

$$\begin{cases} \big(\Upsilon_\alpha^\tau(\omega(\alpha)_0^F\mathfrak{D}_t^{\alpha_m}\xi^n), \xi^n\big) + \beth(\eta^n, \xi^n) = 0, \\ \big(\Upsilon_\alpha^\tau(\omega(\alpha)_0^F\mathfrak{D}_t^{\alpha_m}\eta^n), \eta^n\big) + \beth(\xi^n, \eta^n) = 0. \end{cases} \tag{15}$$

Due to $\beth(\xi^n, \eta^n) = -\beth(\eta^n, \xi^n)$, summing equations (15) implies that

$$\big(\Upsilon_\alpha^\tau(\omega(\alpha)_0^F\mathfrak{D}_t^{\alpha_m}\xi^n), \xi^n\big) + \big(\Upsilon_\alpha^\tau(\omega(\alpha)_0^F\mathfrak{D}_t^{\alpha_m}\eta^n), \eta^n\big) = 0.$$

Since ${}^\alpha\ddot{\c{C}}_j^n < 0$, for $j = (0, 1, 2, ..., n-1)$ and ${}^\alpha\ddot{\c{C}}_j^n > 0$, for $j = n$, we have

$$\begin{aligned} &\sigma_\alpha \sum_{m=0}^{M_\alpha - 1} \omega(\alpha_m) \frac{1}{\tau_n^{\alpha_{[m]}}\Gamma(1-\alpha_m)} {}^\alpha\ddot{\c{C}}_n^n(\xi^n, \xi^n) \\ &+ \sigma_\alpha \sum_{m=0}^{M_\alpha - 1} \omega(\alpha_m) \frac{1}{\tau_n^{\alpha_m}\Gamma(1-\alpha_m)} {}^\alpha\ddot{\c{C}}_n^n(\eta^n, \eta^n) = \\ &- \sigma_\alpha \sum_{j=0}^{n-1} \sum_{m=0}^{M_\alpha - 1} \omega(\alpha_m) \frac{1}{\tau_n^{\alpha_m}\Gamma(1-\alpha_m)} {}^\alpha\ddot{\c{C}}_j^n(\xi^j, \xi^n) \\ &- \sigma_\alpha \sum_{j=0}^{n-1} \sum_{m=0}^{M_\alpha - 1} \omega(\alpha_m) \frac{1}{\tau_n^{\alpha_m}\Gamma(1-\alpha_m)} {}^\alpha\ddot{\c{C}}_j^n(\eta^j, \eta^n). \end{aligned}$$

Set $\chi = \sigma_\alpha \sum_{m=0}^{M_\alpha -1} \omega(\alpha_m) \frac{1}{\tau_n^{\alpha_m}\Gamma(1-\alpha_m)} {}^{\alpha}\ddot{\text{Ç}}_{n}^{n}$, then

$$
\begin{aligned}
\chi\|\xi^n\|^2 + \chi\|\eta^n\|^2 =& - \sigma_\alpha \sum_{j=0}^{n-1}\sum_{m=0}^{M_\alpha -1} \omega(\alpha_m) \frac{1}{\tau_n^{\alpha_m}\Gamma(1-\alpha_m)} {}^{\alpha}\ddot{\text{Ç}}_{j}^{n} (\xi^j, \xi^n) \\
& - \sigma_\alpha \sum_{j=0}^{n-1}\sum_{m=0}^{M_\alpha -1} \omega(\alpha_m) \frac{1}{\tau_n^{\alpha_m}\Gamma(1-\alpha_m)} {}^{\alpha}\ddot{\text{Ç}}_{j}^{n} (\eta^j, \eta^n).
\end{aligned}
$$

Using the Cauchy-Schwarz inequality, results in

$$
\begin{aligned}
\chi\|\xi^n\|^2 + \chi\|\eta^n\|^2 \leq& - \sigma_\alpha \sum_{j=0}^{n-1}\sum_{m=0}^{M_\alpha -1} \omega(\alpha_m) \frac{1}{\tau_n^{\alpha_m}\Gamma(1-\alpha_m)} {}^{\alpha}\ddot{\text{Ç}}_{j}^{n} \|\xi^j\|\|\xi^n\| \\
& - \sigma_\alpha \sum_{j=0}^{n-1}\sum_{m=0}^{M_\alpha -1} \omega(\alpha_m) \frac{1}{\tau_n^{\alpha_m}\Gamma(1-\alpha_m)} {}^{\alpha}\ddot{\text{Ç}}_{j}^{n} \|\eta^j\|\|\xi^n\|.
\end{aligned}
$$

Also, one can easily see that

$$
\chi\|\xi^n\|^2 + \chi\|\eta^n\|^2 \leq \chi \sum_{j=0}^{n-1} \frac{-{}^{\alpha}\ddot{\text{Ç}}_{j}^{n}}{{}^{\alpha}\ddot{\text{Ç}}_{n}^{n}} \|\xi^j\|\|\xi^n\| + \chi \sum_{j=0}^{n-1} \frac{-{}^{\alpha}\ddot{\text{Ç}}_{j}^{n}}{{}^{\alpha}\ddot{\text{Ç}}_{n}^{n}} \|\eta^j\|\|\eta^n\|.
$$

Using the Young's inequality, one obtains

$$
\begin{aligned}
\|\xi^n\|^2 + \|\eta^n\|^2 \leq& \frac{1}{2}\Bigg(\sum_{j=0}^{n-1} \frac{-{}^{\alpha}\ddot{\text{Ç}}_{j}^{n}}{{}^{\alpha}\ddot{\text{Ç}}_{n}^{n}} \left(\|\xi^j\|^2 + \|\xi^n\|^2\right) \\
& + \sum_{j=0}^{n-1} \frac{-{}^{\alpha}\ddot{\text{Ç}}_{j}^{n}}{{}^{\alpha}\ddot{\text{Ç}}_{n}^{n}} \left(\|\eta^j\|^2 + \|\eta^n\|^2\right) \Bigg). \qquad (16)
\end{aligned}
$$

To prove this relation, we use mathematical induction. In the case of $n = 1$, this relation holds as follows

$$
\|\xi^1\|^2 + \|\eta^1\|^2 \leq \|\xi^0\|^2 + \|\eta^0\|^2.
$$

Assume that (16) is true for $n = 2, 3, ..., k-1$, to prove this for $n = k$, we have

$$\|\xi^k\|^2+\|\eta^k\|^2 \leq \frac{1}{2}\bigg(\sum_{j=0}^{n-1}\frac{-{}^{\alpha}\ddot{\c{C}}_j^n}{{}^{\alpha}\ddot{\c{C}}_n^n}\left(\|\xi^j\|^2+\|\xi^k\|^2\right) + \sum_{j=0}^{n-1}\frac{-{}^{\alpha}\ddot{\c{C}}_j^n}{{}^{\alpha}\ddot{\c{C}}_n^n}\left(\|\eta^j\|^2+\|\eta^k\|^2\right)\bigg),$$

thus

$$\|\xi^k\|^2+\|\eta^k\|^2 \leq \frac{1}{2}\bigg(\sum_{j=0}^{n-1}\frac{-{}^{\alpha}\ddot{\c{C}}_j^n}{{}^{\alpha}\ddot{\c{C}}_n^n}\left(\|\xi^0\|^2+\|\xi^k\|^2\right) + \sum_{j=0}^{n-1}\frac{-{}^{\alpha}\ddot{\c{C}}_j^n}{{}^{\alpha}\ddot{\c{C}}_n^n}\left(\|\eta^0\|^2+\|\eta^k\|^2\right)\bigg),$$

moreover, $-\sum_{j=0}^{n-1}{}^{\alpha}\ddot{\c{C}}_j^n={}^{\alpha}\ddot{\c{C}}_n^n$, then

$$\|\xi^k\|^2+\|\eta^k\|^2 \leq \frac{1}{2}\left(\|\xi^0\|^2+\|\xi^k\|^2\right)+\frac{1}{2}\left(\|\eta^0\|^2+\eta^k\|^2\right).$$

Therefore, $\|\xi^k\|^2+\|\eta^k\|^2\leq\|\xi^0\|^2+\|\eta^0\|^2$, and the numerical scheme is unconditionally stable.

Theorem 4.4. *Assume that $\psi(x,t_n)$ is the exact solution of Eq (1) at $t=t_n$, then the numerical solution $\psi^n=\psi_R^n+i\psi_I^n$ of (11) satisfies the following relation*

$$\|\varepsilon^n\|\leq\frac{\exp(1)}{\chi}\left(n^{-\min(2-\alpha_{M_\alpha},r(\alpha_1+1))}+\sigma_\alpha^2\right),$$

where $\varepsilon^n=\psi(x,t_n)-\psi^n$.

Proof. Since $\psi_I(x,t)$ and $\psi_R(x,t)$ are the exact solutions of system (7) at $t=t_n$, then we have

$$\begin{cases}\left(\Upsilon_\alpha^\tau(\omega(\alpha){}_0^F\mathfrak{D}_t^{\alpha_m}\psi_I(x,t_n),\varphi_R\right)+\beth(\psi_R(x,t_n),\varphi_R)+(z_R(x,t_n),\varphi_R)=0,\\ \left(\Upsilon_\alpha^\tau(\omega(\alpha){}_0^F\mathfrak{D}_t^{\alpha_m}\psi_R(x,t_n),\varphi_I\right)+\beth(\psi_I(x,t_n),\varphi_I)-(z_I(x,t_n),\varphi_I)=0.\end{cases} \tag{17}$$

Subtracting (11) from (17) and setting $\varepsilon_R^n = \psi_R(x, t_n) - \psi_R^n$ and $\varepsilon_I^n = \psi_I(x, t_n) - \psi_I^n$, yields

$$\begin{cases} \big(\Upsilon_\alpha^\tau(\omega(\alpha)_0^F\mathfrak{D}_t^{\alpha_m}\varepsilon_I^n, \varphi_R\big) + \beth(\varepsilon_R^n, \varphi_R = (\Xi_I^{\omega(\alpha)} + {}^F\Xi_I^\alpha, \varphi_R), \\ \big(\Upsilon_\alpha^\tau(\omega(\alpha)_0^F\mathfrak{D}_t^{\alpha_m}\varepsilon_R^n, \varphi_I\big) + \beth(\varepsilon_I^n, \varphi_I) = (\Xi_R^{\omega(\alpha)} + {}^F\Xi_R^\alpha, \varphi_I). \end{cases}$$

Taking $\varphi_R = \varepsilon_I^n$ and $\varphi_I = \varepsilon_R^n$, we get

$$\begin{cases} \big(\Upsilon_\alpha^\tau(\omega(\alpha)_0^F\mathfrak{D}_t^{\alpha_m}\varepsilon_I^n), \varepsilon_I^n\big) + \beth(\varepsilon_R^n, \varepsilon_I^n) = (\Xi_I^{\omega(\alpha)} + {}^F\Xi_I^\alpha, \varepsilon_I^n), \\ \big(\Upsilon_\alpha^\tau(\omega(\alpha)_0^F\mathfrak{D}_t^{\alpha_m}\varepsilon_R^n), \varepsilon_R^n\big) + \beth(\varepsilon_I^n, \varepsilon_R^n) = (\Xi_R^{\omega(\alpha)} + {}^F\Xi_R^\alpha, \varepsilon_R^n). \end{cases}$$

Since ${}^\alpha\ddot{\c{C}}_j^n < 0$, for $j = (0, 1, 2, ..., n-1)$ and ${}^\alpha\ddot{\c{C}}_j^n > 0$, for $j = n$, then

$$\begin{cases} \chi\|\varepsilon_I^n\|^2 + \beth(\varepsilon_R^n, \varepsilon_I^n) \leqslant \\ -\sigma_\alpha \sum_{j=0}^{n-1}\sum_{m=0}^{M_\alpha-1}\omega(\alpha_m)\frac{1}{\tau_n^{\alpha_m}\Gamma(1-\alpha_m)}{}^\alpha\ddot{\c{C}}_j^n(\varepsilon_I^j, \varepsilon_I^n) + (\Xi_I^{\omega(\alpha)} + {}^F\Xi_I^\alpha, \varepsilon_I^n), \\ \chi\|\varepsilon_R^n\|^2 + \beth(\varepsilon_I^n, \varepsilon_R^n) \leqslant \\ -\sigma_\alpha \sum_{j=0}^{n-1}\sum_{m=0}^{M_\alpha-1}\omega(\alpha_m)\frac{1}{\tau_n^{\alpha_m}\Gamma(1-\alpha_m)}{}^\alpha\ddot{\c{C}}_j^n(\varepsilon_R^j, \varepsilon_R^n) + (\Xi_R^{\omega(\alpha)} + {}^F\Xi_R^\alpha, \varepsilon_R^n). \end{cases}$$

Using the Cauchy-Schwarz inequality, we have

$$\begin{cases} \chi\|\varepsilon_I^n\|^2 + \beth(\varepsilon_R^n, \varepsilon_I^n) \leq \\ -\sigma_\alpha \sum_{j=0}^{n-1}\sum_{m=0}^{M_\alpha-1}\omega(\alpha_m)\frac{1}{\tau_n^{\alpha_m}\Gamma(1-\alpha_m)}{}^\alpha\ddot{\c{C}}_j^n\|\varepsilon_I^j\|\|\varepsilon_I^n\| \\ +\|\Xi_I^{\omega(\alpha)} + {}^F\Xi_I^\alpha\|\|\varepsilon_I^n\|, \\ \chi\|\varepsilon_R^n\|^2 + \beth(\varepsilon_I^n, \varepsilon_R^n) \leq \\ -\sigma_\alpha \sum_{j=0}^{n-1}\sum_{m=0}^{M_\alpha-1}\omega(\alpha_m)\frac{1}{\tau_n^{\alpha_m}\Gamma(1-\alpha_m)}{}^\alpha\ddot{\c{C}}_j^n\|\varepsilon_R^j\|\|\varepsilon_R^n\| \\ +\|\Xi_R^{\omega(\alpha)} + {}^F\Xi_R^\alpha\|\|\varepsilon_R^n\|. \end{cases} \tag{18}$$

Due to $\beth(\varepsilon_I^n, \varepsilon_R^n) = -\beth(\varepsilon_R^n, \varepsilon_I^n)$ and summing equations (18), implies that

$$\chi\big(\|\varepsilon_I^n\|^2 + \|\varepsilon_R^n\|^2\big) \leq \chi\Big(\sum_{j=0}^{n-1}\frac{-{}^\alpha\ddot{\c{C}}_j^n}{{}^\alpha\ddot{\c{C}}_n^n}\Big)\Big(\|\varepsilon_I^j\|\|\varepsilon_I^n\| + \|\varepsilon_R^j\|\|\varepsilon_R^n\|\Big)$$
$$+ \|\Xi_R^{\omega(\alpha)} + {}^F\Xi_R^\alpha\|\|\varepsilon_R^n\| + \|\Xi_I^{\omega(\alpha)} + {}^F\Xi_I^\alpha\|\|\varepsilon_I^n\|.$$

Using the Holder's inequality, one obtains

$$\chi\big(\|\varepsilon_I^n\|^2+\|\varepsilon_R^n\|^2\big)\le$$
$$\chi\Big(\sum_{j=0}^{n-1}\frac{-{}^{\alpha}\ddot{Ç}_j^n}{{}^{\alpha}\ddot{Ç}_n^n}\Big)\Bigg(\big(\|\varepsilon_I^n\|^2+\|\varepsilon_R^n\|_2\big)^{\frac{1}{2}}\ \big(\|\varepsilon_I^j\|^2+\|\varepsilon_R^j\|^2\big)^{\frac{1}{2}}\Bigg)$$
$$+\Bigg(\|\Xi_R^{\omega(\alpha)}+{}^F\Xi_R^{\alpha}\|^2+\|\Xi_I^{\omega(\alpha)}+{}^F\Xi_I^{\alpha}\|^2\Bigg)^{\frac{1}{2}}$$
$$\Bigg(\|\varepsilon_R^n\|^2+\|\varepsilon_I^n\|^2\Bigg)^{\frac{1}{2}}.$$

It leads to

$$\|\varepsilon^n\|\le\sum_{j=0}^{n-1}\frac{-{}^{\alpha}\ddot{Ç}_j^n}{{}^{\alpha}\ddot{Ç}_n^n}\|\varepsilon^j\|+\frac{1}{\chi}\big(\|\Xi^{\omega(\alpha)}+{}^F\Xi^{\alpha}\|,$$

where $\|\varepsilon^n\|^2=\|\varepsilon_I^n\|^2+\|\varepsilon_R^n\|^2$, $\|\Xi^{\omega(\alpha)}+{}^F\Xi^{\alpha}\|^2=\|\Xi_R^{\omega(\alpha)}+{}^F\Xi_R^{\alpha}\|^2+\|\Xi_I^{\omega(\alpha)}+{}^F\Xi_I^{\alpha}\|^2$ and $\|\varepsilon^j\|^2=\|\varepsilon_I^j\|^2+\|\varepsilon_R^j\|^2$ for $j=0,...,n-1$.
Using Lemma 4.1, we get

$$\begin{aligned}\|\varepsilon^n\|&\le\frac{1}{\chi}\big(\|\Xi^{\omega(\alpha)}+{}^F\Xi^{\alpha}\|\exp\Big(\sum_{j=0}^{n-1}\frac{-{}^{\alpha}\ddot{Ç}_j^n}{{}^{\alpha}\ddot{Ç}_n^n}\Big)\\&\le\exp(1)\frac{1}{\chi}\big(\|\Xi^{\omega(\alpha)}+{}^F\Xi^{\alpha}\|\big)\\&\le\exp(1)\frac{1}{\chi}\big(n^{-\min(2-\alpha_{M_\alpha},r(\alpha_1+1))}+\sigma_\alpha^2\big),\end{aligned}$$

and the proof is completed.

4.2 Convergence of the fully discrete scheme

At first, we define the Ritz projector as $\Pi_h^\beta: H^\beta(\Omega)\to V_{0h}$ by $(\nabla\Pi_h^\beta\psi,\nabla\varphi_h)$ $=(\nabla\psi,\nabla\varphi_h)$ for all $\varphi_h\in V_{0h}$.

Lemma 4.5. *[14] Let $\frac{1}{2}<\beta<1$ and $\beta<\mu\le\rho+1$. If $\psi\in H^\beta\cap H^\mu$, then there exists a positive constant C independent of h such that*

$$\|\Pi_h^\beta\psi-\psi\|_\beta\le Ch^{\mu-\beta}\|\psi\|_\mu.$$

Theorem 4.6. *Assume that* $\psi(x,t) = \psi_R(x,t) + i\psi_I(x,t)$ *is the exact complex-valued solution of the equation* (1). *Then, the numerical solution* $\psi_h^n = \psi_{R,h}^n + i\psi_{I,h}^n$ (12) *satisfies the following relation*

$$\|\varrho_h^n\| \leq \frac{C}{\chi}\left(h^{\mu-\beta} + n^{-\min(2-\alpha_{M_\alpha}, r(\alpha_1+1))} + \sigma_\alpha^2\right),$$

where $\varrho_h^n = \psi(x,t_n) - \psi_h^n$

Proof: Since $\psi(x,t_n) = \psi_R(x,t_n) + i\psi_I(x,t_n)$ is the exact solution of equation (1) at $t = t_n$, taking $\psi_R(x,t_n)$ and $\psi_I(x,t_n)$ instead of $\psi_{R,h}^n$ and $\psi_{I,h}^n$ in (12), we get

$$\begin{cases} \left(\Upsilon_\alpha^\tau(\omega(\alpha)_0^F\mathfrak{D}_t^{\alpha_m}\psi_I(x,t_n)), \varphi_{R,h}\right) + \beth(\psi_R(x,t_n), \varphi_{R,h}) - (z_R(x,t_n), \varphi_{R,h}) \\ = (\Xi_I^{\omega(\alpha)} + {}^F\Xi_I^\alpha, \varphi_{R,h}), \quad \forall \varphi_{R,h} \in X_h^\beta, \\ \left(\Upsilon_\alpha^\tau(\omega(\alpha)_0^F\mathfrak{D}_t^{\alpha_m}\psi_R(x,t_n)), \varphi_{I,h}\right) + \beth(\psi_I(x,t_n), \varphi_{I,h}) + (z_I(x,t_n), \varphi_{I,h}) \\ = (\Xi_R^{\omega(\alpha)} + {}^F\Xi_R^\alpha, \varphi_{I,h}), \quad \forall \varphi_{I,h} \in X_h^\beta, \end{cases} \tag{19}$$

in which

$$\beth(\psi_R(x,t_n), \varphi_{R,h}) =$$
$$-\Theta(\psi_R(x,t_n), \varphi_{R,h}) + (\lambda|\psi(x,t_{n-1})|^2\psi_R(x,t_n) - v(x,t_n)\psi_R(x,t_n), \varphi_{R,h}),$$

$$\beth(\psi_I(x,t_n), \varphi_{I,h}) =$$
$$\Theta(\psi_I(x,t_n), \varphi_{I,h}) - (\lambda|\psi(x,t_{n-1})|^2\psi_I(x,t_n) + v(x,t_n)\psi_I(x,t_n), \varphi_{I,h}).$$

Subtracting (12) from (19) results in

$$\begin{cases} \left(\Upsilon_\alpha^\tau(\omega(\alpha)_0^F\mathfrak{D}_t^{\alpha_m}\varrho_I^n), \varphi_{R,h}\right) + \beth(\varrho_R^n, \varphi_{R,h}) = (\Xi_I^{\omega(\alpha)} + {}^F\Xi_I^\alpha, \varphi_{R,h}) \\ \forall \varphi_{R,h} \in X_h^\beta, \\ \left(\Upsilon_\alpha^\tau(\omega(\alpha)_0^F\mathfrak{D}_t^{\alpha_m}\varrho_R^n), \varphi_{I,h}\right) + \beth(\varrho_I^n, \varphi_{I,h}) = (\Xi_R^{\omega(\alpha)} + {}^F\Xi_R^\alpha, \varphi_{I,h}), \\ \forall \varphi_{I,h} \in X_h^\beta, \end{cases} \tag{20}$$

where $\varrho_I^n = \psi_I(x,t_n) - \psi_{I,h}^n$ and $\varrho_R^n = \psi_R(x,t_n) - \psi_{R,h}^n$.
Putting $\varrho_I^n = \epsilon_{I,h}^n + \theta_{I,h}^n$ and $\varrho_R^n = \epsilon_{R,h}^n + \theta_{R,h}^n$ in equation (20), where

$\theta^n_{I,h} = \Pi^\beta_h \psi_I - \psi^n_{I,h}$ and $\epsilon_{I,h} = \psi_I(x, t_n) - \Pi^\beta_h \psi^n_I$, implies that

$$\begin{cases} \big(\Upsilon^\tau_\alpha(\omega(\alpha){}^F_0\mathfrak{D}^{\alpha_m}_t \epsilon^n_{I,h} + \theta^n_{I,h})), \varphi_{R,h}\big) + \daleth(\epsilon^n_{R,h} + \theta^n_{R,h}, \varphi_{R,h}) \\ = (\Xi^{\omega(\alpha)}_I + {}^F\Xi^\alpha_I, \varphi_{R,h}), \quad \forall \varphi_R \in X^\beta_h, \\ \big(\Upsilon^\tau_\alpha(\omega(\alpha){}^F_0\mathfrak{D}^{\alpha_m}_t \epsilon^n_{R,h} + \theta^n_{R,h})), \varphi_{I,h}\big) + \daleth(\epsilon^n_{I,h} + \theta^n_{I,h}, \varphi_{I,h}) \\ = (\Xi^{\omega(\alpha)}_R + {}^F\Xi^\alpha_R, \varphi_{I,h}), \quad \forall \varphi_I \in X^\beta_h. \end{cases} \tag{21}$$

Taking $\psi_{R,h} = \theta^n_{I,h}$ and $\psi_{I,h} = \theta^n_{R,h}$ in (21), we get

$$\begin{cases} \big(\Upsilon^\tau_\alpha(\omega(\alpha){}^F_0\mathfrak{D}^{\alpha_m}_t (\epsilon^n_{I,h} + \theta^n_{I,h})), \theta^n_{I,h}\big) + \daleth(\epsilon^n_{R,h} + \theta^n_{R,h}, \theta^n_{I,h}) \\ = (\Xi^{\omega(\alpha)}_I + {}^F\Xi^\alpha_I, \theta^n_{I,h}), \quad \forall \varphi_{R,h} \in X^\beta_h, \\ \big(\Upsilon^\tau_\alpha(\omega(\alpha){}^F_0\mathfrak{D}^{\alpha_m}_t (\epsilon^n_{R,h} + \theta^n_{R,h})), \theta^n_{R,h}\big) + \daleth(\epsilon^n_{I,h} + \theta^n_{I,h}, \theta^n_{R,h}) \\ = (\Xi^{\omega(\alpha)}_R + {}^F\Xi^\alpha_R, \theta^n_{R,h}), \quad \forall \varphi_{I,h} \in X^\beta_h. \end{cases} \tag{22}$$

Due to $\daleth(\theta^n_{I,h} + \epsilon^n_{I,h}, \theta^n_{R,h}) = -\daleth(\theta^n_{R,h} + \epsilon^n_{R,h}, \theta^n_{I,h})$, summing equations (22) and using the Cauchy-Schwarz inequality, one obtains

$$\chi\|\theta^n_{I,h}\|^2 + \chi\|\theta^n_{R,h}\|^2 \leq \chi\Big(\sum_{j=0}^{n-1} \frac{-{}^\alpha\ddot{\c{C}}^n_j}{{}^\alpha\ddot{\c{C}}^n_n}\Big)\Big(\|\theta^j_{I,h}\|\|\theta^n_{I,h}\| + \|\theta^j_{R,h}\|\|\theta^n_{R,h}\|\Big) + \\ \|\Upsilon^\tau_\alpha(\omega(\alpha){}^F_0\mathfrak{D}^{\alpha_m}_t \epsilon^n_{R,h})\|\|\theta^n_{R,h}\| + \|\Upsilon^\tau_\alpha(\omega(\alpha){}^F_0\mathfrak{D}^{\alpha_m}_t \epsilon^n_{I,h})\|\|\theta^n_{I,h}\| + \\ \|\Xi^{\omega(\alpha)}_R + {}^F\Xi^\alpha_R\|\|\theta^n_{R,h}\| + \|\Xi^{\omega(\alpha)}_I + {}^F\Xi^\alpha_I\|\|\theta^n_{I,h}\|.$$

Using the Holder's inequality implies that

$$\|\theta^n_{I,h}\|^2 + \|\theta^n_{R,h}\|^2 \leq \Big(\sum_{j=0}^{n-1} \frac{-{}^\alpha\ddot{\c{C}}^n_j}{{}^\alpha\ddot{\c{C}}^n_n}\Big)\big(\|\theta^j_{I,h}\|^2 + \|\theta^j_{R,h}\|^2\big)^{\frac{1}{2}}\big(\|\theta^n_{I,h}\|^2 + \\ \|\theta^n_{R,h}\|^2\big)^{\frac{1}{2}} + \frac{1}{\chi}\Big(\big(\|\Upsilon^\tau_\alpha(\omega(\alpha){}^F_0\mathfrak{D}^{\alpha_m}_t \epsilon^n_{R,h})\|^2 + \|\Upsilon^\tau_\alpha(\omega(\alpha){}^F_0\mathfrak{D}^{\alpha_m}_t \epsilon^n_{I,h})\|^2\big)^{\frac{1}{2}} \\ (\|\theta^n_{R,h}\|^2 + \|\theta^n_{I,h}\|^2)^{\frac{1}{2}} + (\|\Xi^{\omega(\alpha)}_R + {}^F\Xi^\alpha_R\|^2 + \|\Xi^{\omega(\alpha)}_I + {}^F\Xi^\alpha_I\|^2)^{\frac{1}{2}} \\ (\|\theta^n_{I,h}\|^2 + \|\theta^n_{R,h}\|^2)^{\frac{1}{2}}\Big).$$

It leads to

$$\|\theta_h^n\| \leq (\sum_{j=0}^{n-1} \frac{-{}^{\alpha}C_j^n}{{}^{\alpha}C_n^n})\|\theta_h^j\| + \frac{1}{\chi}\Bigg(\big(\|\Upsilon_\alpha^\tau(\omega(\alpha){}_0^F\mathfrak{D}_t^{\alpha_m}\epsilon_{R,h}^n)\|^2 +$$

$$\|\Upsilon_\alpha^\tau(\omega(\alpha){}_0^F\mathfrak{D}_t^{\alpha_m}\epsilon_{I,h}^n)\|^2\big)^{\frac{1}{2}} + \|\Xi^{\omega(\alpha)} + {}^F\Xi^\alpha\|\Bigg),$$

where $\|\theta_h^n\|^2 = \|\theta_{I,h}^n\|^2 + \|\theta_{R,h}^n\|^2$, $\|\theta_h^j\|^2 = \|\theta_{I,h}^j\|^2 + \|\theta_{R,h}^j\|^2$ for $j = 0, ..., n-1$ and $\|\Xi^{\omega(\alpha)} + {}^F\Xi^\alpha\|^2 = \|\Xi_R^{\omega(\alpha)} + {}^F\Xi_R^\alpha\|^2 + \|\Xi_I^{\omega(\alpha)} + {}^F\Xi_I^\alpha\|^2$.
Using the Granwall inequality, we get

$$\begin{aligned}\|\theta_h^n\| \leq& \frac{1}{\chi}\Bigg(\big(\|\Upsilon_\alpha^\tau(\omega(\alpha){}_0^F\mathfrak{D}_t^{\alpha_m}\epsilon_{R,h}^n)\|^2 + \|\Upsilon_\alpha^\tau(\omega(\alpha){}_0^F\mathfrak{D}_t^{\alpha_m}\epsilon_{I,h}^n)\|^2\big)^{\frac{1}{2}} \\ &+ \|\Xi^{\omega(\alpha)} + {}^F\Xi^\alpha\|\Bigg) \exp\left(\sum_{j=0}^{n-1} \frac{-{}^{\alpha}\ddot{\c{C}}_j^n}{{}^{\alpha}\ddot{\c{C}}_n^n}\right) \\ \leq& \frac{\exp(1)}{\chi}\Bigg(\big(\|\Upsilon_\alpha^\tau(\omega(\alpha){}_0^F\mathfrak{D}_t^{\alpha_m}\epsilon_{R,h}^n)\|^2 + \|\Upsilon_\alpha^\tau(\omega(\alpha){}_0^F\mathfrak{D}_t^{\alpha_m}\epsilon_{I,h}^n)\|^2\big)^{\frac{1}{2}} \\ &+ \|\Xi^{\omega(\alpha)} + {}^F\Xi^\alpha\|\Bigg).\end{aligned}$$

Applying Lemma 4.5, we get

$$\begin{aligned}\|\Upsilon_\alpha^\tau(\omega(\alpha){}_0^F\mathfrak{D}_t^{\alpha_m}\epsilon_{R,h}^n)\| &\leq \|\Upsilon_\alpha^\tau(\omega(\alpha){}_0^F\mathfrak{D}_t^{\alpha_m}\epsilon_{R,h}^n)\|_\beta \\ &\leq h^{\mu-\beta}\|\Upsilon_\alpha^\tau(\omega(\alpha){}_0^F\mathfrak{D}_t^{\alpha_m}\psi_{R,h}^n)\|_\mu.\end{aligned}$$

Moreover, we have ${}_0^F\mathfrak{D}_t^\alpha\psi(t_n) = {}_0^CD_t^\alpha\psi(t_n) - {}^F\Xi^\alpha$, then

$$\begin{aligned}\|\Upsilon_\alpha^\tau(\omega(\alpha){}_0^F\mathfrak{D}_t^{\alpha_m}\epsilon_{R,h}^n)\| \leq& h^{\mu-\beta}\|\Upsilon_\alpha^\tau(\omega(\alpha){}_0^CD_t^{\alpha_m}\psi_{R,h}^n)\|_\mu \\ &+ h^{\mu-\beta}\|\Upsilon_\alpha^\tau(\omega(\alpha){}^F\Xi^{\alpha_m})\|_\mu.\end{aligned}$$

Using (6), positive constants c_1 and c_2 exist, such that

$$\|\Upsilon_\alpha^\tau(\omega(\alpha){}_0^F\mathfrak{D}_t^{\alpha_m}\epsilon_{R,h}^n)\| \leq c_2h^{\mu-\beta} + c_1h^{\mu-\beta}\big(n^{-\min(2-\alpha_{M_\alpha}, r(\alpha_1+1))} + \epsilon\big).$$

Similarly, we have

$$\|\Upsilon_\alpha^\tau(\omega(\alpha){}_0^F\mathfrak{D}_t^{\alpha_m}\epsilon_{I,h}^n)\| \leq c_4h^{\mu-\beta} + c_3h^{\mu-\beta}\big(n^{-\min(2-\alpha_{M_\alpha}, r(\alpha_1+1))} + \epsilon\big).$$

Therefore, one obtains

$$\|\theta_h^n\| \leq \frac{\exp(1)}{\chi}\Big(c_2 h^{\mu-\beta} + c_1 h^{\mu-\beta}\big(n^{-\min(2-\alpha_{M_\alpha}, r(\alpha_1+1))} + \epsilon\big) +$$

$$c_4 h^{\mu-\beta} + c_3 h^{\mu-\beta}\big(n^{-\min(2-\alpha_{M_\alpha}, r(\alpha_1+1))} + \epsilon\big) + \|\Xi^{\omega(\alpha)} + {}^F\Xi^{\alpha}\|\Big).$$

Consequently, we have

$$\|\theta_h^n\| \leq \frac{\exp(1)}{\chi}\Big(C\big(h^{\mu-\beta} + \big(n^{-\min(2-\alpha_{M_\alpha}, r(\alpha_1+1))} + \epsilon\big)\big)$$

$$+ \|\Xi^{\omega(\alpha)} + {}^F\Xi^{\alpha}\|\Big),$$

where C is a constant dependent on c_1, c_2, c_3 and c_4. Note that $\|\epsilon_{R,h}^n\| \leq c_R h^{\mu-\beta}$ and $\|\epsilon_{I,h}^n\| \leq c_I h^{\mu-\beta}$, therefore, the following inequality is satisfied

$$\|\varrho_h^n\| \leq \frac{\exp(1)}{\chi} C\Big(h^{\mu-\beta} + \big(n^{-\min(2-\alpha_{M_\alpha}, r(\alpha_1+1))} + \epsilon\big) + \sigma_\alpha^2\Big),$$

where

$$\|\varrho_h^n\|^2 = \|\varrho_{I,h}^n\|^2 + \|\varrho_{R,h}^n\|^2,$$

and the proof is completed.

5 Numerical Results

In this section, two numerical examples are given to confirm the validation of our theoretical results. In the following examples, the computational domain is $(x,t) \in ([0,1] \times [0,1])$, the weight function $\omega(\alpha) = \Gamma(3-\alpha)$ and $\epsilon = 10^{-8}$. In numerical experiments, piecewise linear basis functions are considered. We perform our computations using **MATLAB**2021 software on a desktop computer with $16GB$ of RAM and Core i7 -6700CPU @4.00GHZ.

Example 1**:** We consider the following distributed-order time and Riesz space fractional Schrödinger equation

$$iD_t^{\omega(\alpha)}\psi(x,t) = -\frac{\partial^{2\beta}}{\partial |x|^{2\beta}}\psi(x,t) + \psi(x,t) + z(x,t),$$

Table 1: Numerical results in space for Example 1 with $\beta = 0.75$

h	$\|\varrho_h^n\|$	order	$\|\varrho_h^n\|_\beta$	order
1/8	4.9926E-2	-	1.2263E-1	-
1/16	1.3751E-2	1.8603	5.5766E-2	1.1369
1/32	3.6544E-3	1.9118	3.5619E-2	1.2220
1/64	9.1857E-4	1.9922	1.0798E-2	1.2465

Table 2: Numerical results in space for Example 1 with $\beta = 0.9$

h	$\|\varrho_h^n\|$	order	$\|\varrho_h^n\|_\beta$	order
1/8	4.6175E-2	-	1.2013E-1	-
1/16	1.2239E-2	1.9156	5.5243E-2	1.1207
1/32	3.1784E-3	1.9451	2.4970E-2	1.1454
1/64	9.0962E-4	1.9730	1.1719E-2	1.0915

with homogeneous initial and Dirichlet boundary conditions. The function $z(x,t)$ is computed from the exact solution $\psi(x,t) = (1+i)\sin(\pi x)t^2$. We solve this problem with several step sizes in space and time at the final time of $T = 1$. Tables 1 and 2 show the L^2 and H^β- norms errors and the convergence orders in space with $\beta = 0.75$, $\beta = 0.9$ and $N_T = (\frac{1}{h})^2$, $M_\alpha = \frac{1}{h}$. In Table 3, we report the L^2-norm error and convergence order in time with $\beta = 0.8$ and $h = \frac{1}{100}$ and compare CPU time between the $L1$-method and the fast-$L1$-method. The results of Table 3 make it easy to find that the fast-$L1$-method can obtain the same convergence results as the $L1$-method, but the fast-$L1$-method consumes less CPU time during the program. As seen in all three tables, numerical results show that the computed orders are close to the theoretical orders.

Example 2: Consider the distributed-order time and Riesz space fractional Schrödinger equation

$$iD_t^{\omega(\alpha)}\psi(x,t) = -\frac{\partial^{2\beta}}{\partial|x|^{2\beta}}\psi(x,t) + |\psi(x,t)|^2\psi(x,t) + z(x,t),$$

with homogeneous initial and Dirichlet boundary conditions. The function $z(x,t)$ is computed from the exact solution $\psi(x,t) = (1+i)x^2(1-x)^2t^2$.

Table 3: Numerical results in time for Example 1 with $\beta = 0.8$

			$L1$	fast-$L1$
N_T	$\|\varrho_h^n\|$	order	CPU	CPU
500	1.6861E-3	-	1.9257E+2	8.3541E+1
1000	8.5406E-4	0.9813	3.7542E+3	1.6719E+2
2000	4.5345E-4	0.1934	8.3241E+3	3.0672E+2
4000	2.2058E-6	1.0396	4.7245E+5	8.3245E+2

Table 4: Numerical results in space for Example 2 with $\beta = 0.75$

h	$\|\varrho_h^n\|$	order	$\|\varrho_h^n\|_\beta$	order
1/8	7.0449E-4	-	1.5741E-2	-
1/16	1.9407E-4	1.8598	6.9745E-3	1.1744
1/32	5.2360E-5	1.8900	3.6289E-3	1.2110
1/64	1.3742E-5	1.9299	1.4719E-3	1.2288

For this example, the L^2 and H^β- norms errors and the convergence orders in space are shown in Tables 4 and 5. Moreover, comparison CPU time, errors, and convergence orders of $L1$- method and fast-$L1$- method for Example 2 with $\beta = 0.8$, $M_\alpha = \frac{1}{h}$ and different temporal steps are given in Table 6. In this example, the results are similar to the previous example.

6 Conclusion

In this paper, a combination of the fast-$L1$ method and the finite element method was proposed for a distributed-order time and Riesz space frac-

Table 5: Numerical results in space for Example 2 with $\beta = 0.9$

h	$\|\varrho_h^n\|$	order	$\|\varrho_h^n\|_\beta$	order
1/8	6.1952E-4	-	9.3614E-3	-
1/16	1.6948E-4	1.8700	4.4315E-3	1.0789
1/32	4.5097E-5	1.9100	2.0598E-3	1.1058
1/64	1.1753E-5	1.9388	9.4642E-4	1.1215

Table 6: Numerical results in time for Example 2 with $\beta = 0.8$

			$L1$	fast-$L1$
N_T	$\|\varrho_h^n\|$	order	CPU	CPU
500	1.2864E-5	-	1.7524E+2	7.4751E+1
1000	5.9503E-6	1.1123	3.3341E+3	2.2741E+2
2000	2.8004E-6	1.0380	8.2701E+3	3.9025E+2
4000	1.4082E-6	0.9918	3.5049E+5	8.9104E+2

tional Schrödinger equation (DOT-RSFSE). Using the proposed method, the semi-discrete and fully discrete variational formulations were obtained. Then the stability and convergence properties of the presented method were studied. The numerical scheme of the fast-$L1$-FEM was unconditionally stable and convergent. Two examples were implemented and the numerical results validated the theoretical results. Besides, the Performance of the fast-$L1$ finite element method was compared to the $L1$- finite element method. The obtained numerical results kept almost the same convergence order but showed that the fast-$L1$ finite element method has a lower computational cost.

References

[1] Ali H. Bhrawy and Mahmoud Abdelkawy. A fully spectral collocation approximation for multi-dimensional fractional Schrödinger equations. *Journal of Computational Physics*, 294 (2015), 462–483.

[2] Ali H Bhrawy, Eid H Doha, Samer S Ezz-Eldien, and Robert A Van Gorder. A new Jacobi spectral collocation method for solving 1+ 1 fractional schrödinger equations and fractional coupled schrödinger systems. *The European Physical Journal Plus*, 129 (2014), 1–21.

[3] Ali H Bhrawy and Mahmoud A Zaky. Numerical simulation of multi-dimensional distributed-order generalized

schrödinger equations. *Nonlinear Dynamics*, 89 (2017), 1415–1432.

[4] Hermann Brunner. *Collocation methods for Volterra integral and related functional differential equations*, volume 15, Cambridge University Press, (2004).

[5] Boling Guo, Yongqian Han, and Jie Xin. Existence of the global smooth solution to the period boundary value problem of fractional nonlinear Schrödinger equation. *Applied Mathematics and Computation*, 204 (2008), 468–477.

[6] Xiaoyi Guo and Mingyu Xu. Some physical applications of fractional Schrödinger equation. *Journal of Mathematical Physics*, 47 (2006), 082104.

[7] Mohammad Hossein Heydari and Mohsen Razzaghir. Highly accurate solutions for space-time fractional Schrödinger equations with nonsmooth continuous solution using the hybrid clique functions. *Mathematical Sciences*, 17 (2023), 31-42.

[8] Mohammad Hossein Heydari, Mohsen Razzaghir, Dumitru Baleanu. A numerical method based on the piecewise Jacobi function for distributed-order fractional Schrödinger equation. *Communications in Nonlinear Science and Numerical Simulation*, 116 (2023), 106873.

[9] Mohammad Hossein Heydari. A computational approach for a system of coupled distributed-order fractional klein–gordon–Schrödinger equations. *Results in Physics*, 51 (2023), 106750.

[10] David Kincaid, David Ronald Kincaid, and Elliott Ward Cheney. *Numerical analysis: Mathematics of Scientific Computing*, volume 2. American Mathematical Soc, 2009.

[11] Nick Laskin. Fractional quantum mechanics. *Physical Review E*, 62 (2000), 31-35.

[12] Nick Laskin. Fractional Schrödinger equation. *Physical Review E*, 66 (2002), 056108.

[13] Buyang Li and Weiwei Sun. Error analysis of linearized semi-implicit Galerkin finite element methods for nonlinear parabolic equations. *International Journal of Numerical Analysis and Modeling*, 10 (2013), 622-633.

[14] Changpin Li and Fanhai Zeng. *Numerical Methods for Fractional Calculus*, volume 24 . CRC Press, (2015).

[15] Dongfang Li, Jilu Wang, and Jiwei Zhang. Unconditionally convergent l1-Galerkin fems for nonlinear time-fractional Schrodinger equations. *SIAM Journal on Scientific Computing*, 39 (2017), A3067–A3088.

[16] Meng Li, Chengming Huang, and Wanyuan Ming. A relaxation-type Galerkin fem for nonlinear fractional Schrödinger equations. *Numerical Algorithms*, 83 (2020), 99–124.

[17] Meng Li, Chengming Huang, and Pengde Wang. Galerkin finite element method for nonlinear fractional schrödinger equations. *Numerical Algorithms*, 74 (2017), 499–525.

[18] Meng Li, Yifan Wei, Binqian Niu, and Yong-Liang Zhao. Fast l2-1σ Galerkin fems for generalized nonlinear coupled Schrödinger equations with Caputo derivatives. *Applied Mathematics and Computation*, 416 (2022), 126734.

[19] Meng Li and Yong-Liang Zhao. A fast energy conserving finite element method for the nonlinear fractional Schrödinger equation with wave operator. *Applied Mathematics and Computation*, 338 (2018), 758–773.

[20] Mark Naber. Distributed order fractional sub-diffusion. *Fractals*, 12 (2004), 23–32.

[21] Mark Naber. Time fractional Schrödinger equation. *Journal of Mathematical Physics*, 45 (2004), 3339–3352.

[22] Rezvan Salehi. Two implicit meshless finite point schemes for the two-dimensional distributed-order fractional equation. *Computational Methods in Applied Mathematics*, 19 (2017), 813–831.

[23] Simone Secchi and Marco Squassina. Soliton dynamics for fractional Schrödinger equations. *Applicable Analysis*, 93 (2014), 1702–1729.

[24] Jin-ye Shen, Zhi-zhong Sun, and Rui Du. Fast finite difference schemes for time-fractional diffusion equations with a weak singularity at initial time. *East Asian Journal on Applied Mathematics*, 8 (2018), 834–858.

[25] Shaowei Wang and Mingyu Xu. Generalized fractional Schrödinger equation with space-time fractional derivatives. *Journal of Mathematical Physics*, 48 (2007), 043502.

[26] Ying Wang, Gang Wang, Linlin Bu, and Liquan Mei. Two second-order and linear numerical schemes for the multi-dimensional nonlinear time-fractional Schrödinger equation. *Numerical Algorithms* (2021), pages 1–33.

[27] Baoli Yin, Jinfeng Wang, Yang Liu, and Hong Li. A structure-preserving difference scheme with fast algorithms for high dimensional nonlinear space-fractional Schrödinger equations. *Journal of Computational Physics*, 425 (2021), 109869.

[28] Mustafa, Almushaira. A fast implicit difference scheme for solving high-dimensional time-space fractional nonlinear Schrödinger equation. *International Journal of Computer Mathematics*, 100 (2023), 1419–1438.

[29] Qayyum, Mubashir and Ahmad, Efaza and Ahmad, Hijaz and Almohsen, Bandar. New solutions of time-space fractional coupled Schrödinger systems. *AIMS Mathematics*, 8 (2023), 27033–27051.

Fatemeh Hashemi
PhD student of Mathematics
Department of Mathematics
Shiraz University of Technology
Shiraz, Iran
E-mail: fa.hashemi@sutech.ac.ir

Ameneh Taleei
Assistant Professor of Mathematics
Department of Mathematics
Shiraz University of Technology
Shiraz, Iran
E-mail: a.taleei@sutech.ac.ir

Esmaiel Hesameddini
Full Professor of Mathematics
Department of Mathematics
Shiraz University of Technology
Shiraz, Iran
E-mail: hesameddini@sutech.ac.ir