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The Roman Domination Number of Comaximal Ideal Graph

S. Khojasteh*
Lahijan Branch, Islamic Azad University

F. Heydari
Karaj Branch, Islamic Azad University

Abstract. Let $G = (V(G), E(G))$ be a graph. A Roman dominating function φ is a coloring of the vertices of G with the colors $\{0, 1, 2\}$ such that every vertex colored 0 is adjacent to at least one vertex colored 2. The weight of φ is defined as $\sum_{x \in V(G)} \varphi(x)$. The weight of a Roman dominating function on G whose weight is minimum, is called the Roman domination number of G. In this paper, we compute the Roman domination number of comaximal ideal graph for all Artinian commutative rings.

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1 Introduction

Let $G = (V(G), E(G))$ be a graph. By $N(x)$, we mean the *neighborhood* of a vertex x. For a positive integer n, the complete graph with n vertices and its complement are denoted by K_n and $\overline{K_n}$, respectively.

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*Corresponding Author

Let S be a subset of $V(G)$. If every vertex of $V(G) \setminus S$ is adjacent to at least one vertex in S, then S is called a *dominating set* of G. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set in G. The Roman domination of a graph is an interesting variant of the domination which is introduced implicitly in [11, 12]. Since then, there have been several articles on Roman domination [6, 13, 15]. It was Emperor Constantine's defence strategy to assign two armies at any region which is adjacent to a region that is defenceless. Suppose S_0, S_1 and S_2 are subsets of $V(G)$ which are mutually disjoint such that their union is $V(G)$. By a function $\varphi = (S_0, S_1, S_2)$ on G, we mean a function $\varphi : V(G) \to \{0, 1, 2\}$ such that $\varphi(x) = j$ for all $x \in S_j$ $(j = 0, 1, 2)$. A *Roman dominating function* on G is a function $\varphi = (S_0, S_1, S_2)$ on G satisfying the condition that every vertex x for which $\varphi(x) = 0$ is adjacent to at least one vertex y for which $\varphi(y) = 2$. In other words, a Roman dominating function is a coloring of the vertices of G with the colors $\{0, 1, 2\}$ such that every vertex colored 0 is adjacent to at least one vertex colored 2.

Clearly, $\gamma(G) \leq |S_1| + |S_2|$. The weight of a Roman dominating function $\varphi = (S_0, S_1, S_2)$ is defined as $\sum_{x \in V(G)} \varphi(x) = |S_1| + 2|S_2|$. A $\gamma_{\mathfrak{R}}$-function on G is a Roman dominating function on G whose weight is minimum. The weight of a $\gamma_{\mathfrak{R}}$-function on G, denoted by $\gamma_{\mathfrak{R}}(G)$, is called the *Roman domination number* of G. For terms in graph theory not given here, the reader is referred to [16].

Throughout this paper, all rings are commutative with identity. We denote the Jacobson radical of a ring A by $J(A)$. Also, the set of all ideals and the set of all maximal ideals of A are denoted by $\mathbb{I}(A)$ and $\mathrm{Max}(A)$, respectively. If $|\mathrm{Max}(A)| = 1$, then A is called a *local ring*.

When we associate a graph with an algebraic structure a great number of questions arise from the translation of graph-theoretic parameters. We recommend to the reader the references [1, 2, 3, 5, 8, 9]. The *co-maximal ideal graph* of a commutative ring A, denoted by $\mathcal{C}(A)$, was first introduced in [17] and then has been studied by several authors, for instance see [10, 14]. It is a simple graph whose vertices are the proper ideals I of A such that $I \nsubseteq J(A)$, and two vertices I and J are adjacent if and only if $I + J = A$.

Here, we present some auxiliary results which will be used several times in the sequel.

Proposition 1.1. ([6, Proposition 1]). *For any graph G, $\gamma(G) \leq \gamma_{\mathfrak{R}}(G) \leq 2\gamma(G)$.*

Proposition 1.2. ([6, Proposition 2]). *For any graph G of order n, $\gamma(G) = \gamma_{\mathfrak{R}}(G)$ if and only if $G \cong \overline{K_n}$.*

Theorem 1.3. ([7, Theorem 2.23]). *If A is a ring, $|\mathrm{Max}(A)| \geq 3$ and $\gamma(\mathcal{C}(A))$ is finite, then $|\mathrm{Max}(A)| = \gamma(\mathcal{C}(A))$.*

Corollary 1.4. ([7, Corollary 2.25]). *Let A be a ring. Then $\gamma(\mathcal{C}(A)) = 1$ if and only if $A \cong F_1 \times A_1$, where F_1 is a field and A_1 is a local ring.*

Corollary 1.5. ([7, Corollary 2.26]). *Let A be a ring. Then $\gamma(\mathcal{C}(A)) = 2$ if and only if $A \cong A_1 \times A_2$, where $(A_i, \mathfrak{m}_i)$ is a local ring and $\mathfrak{m}_1, \mathfrak{m}_2 \neq 0$.*

In this paper, the Roman domination number of comaximal ideal graphs is studied. We prove that if A is a ring, then there exists a $\gamma_{\mathfrak{R}}$-function on $\mathcal{C}(A)$ say $\varphi = (S_0, S_1, S_2)$ such that $S_2 \subseteq \mathrm{Max}(A)$. Let $A \cong A_1 \times \cdots \times A_n$ be a ring such that $n \geq 4$ and $(A_i, \mathfrak{m}_i)$ is a local ring, for $i = 1, \ldots, n$. We show that if at most one of A_i's is not a field, then $\gamma_{\mathfrak{R}}(\mathcal{C}(A)) = 2n - 1$; otherwise, $\gamma_{\mathfrak{R}}(\mathcal{C}(A)) = 2n$. Also, the Roman domination number of $\mathcal{C}(A)$ is computed for all Artinian commutative rings A.

1.1 Main Results

In this article, we assume that the domination number of $\mathcal{C}(A)$ is finite. We start with the following key theorem.

Theorem 1.6. *Let A be a ring. Then the following statements hold:*

(i) If $\varphi = (S_0, S_1, S_2)$ is a $\gamma_{\mathfrak{R}}$-function on $\mathcal{C}(A)$, then the elements of S_2 are pairwise incomparable.

(ii) If $\varphi = (S_0, S_1, S_2)$ is a $\gamma_{\mathfrak{R}}$-function on $\mathcal{C}(A)$, then each maximal ideal of A contains at most one ideal in S_2.

(iii) There exists a $\gamma_{\mathfrak{R}}$-function $\varphi = (S_0, S_1, S_2)$ on $\mathcal{C}(A)$ such that $S_2 \subseteq \mathrm{Max}(A)$.

Proof. (i) By contradiction, assume that $I_1, I_2 \in S_2$ and $I_1 \subseteq I_2$. We know that φ is a Roman dominating function whose weight is minimum. Then $\phi = (S_0, S_1 \cup \{I_1\}, S_2 \setminus \{I_1\})$ is a Roman dominating function on $\mathcal{C}(A)$ with a weight less than φ, a contradiction. Hence, the elements of S_2 are pairwise incomparable.

(ii) Let M be a maximal ideal of A. By contrary, first suppose that M contains at least three distinct ideals I_1, I_2 and I_3 in S_2, that is, $I_1, I_2, I_3 \subseteq M$. By the previous part, $M \notin S_2$, so if $M \in S_0$, then $\phi = (S_0 \setminus \{M\}, S_1 \cup \{I_1, I_2, I_3\}, S_2 \cup \{M\} \setminus \{I_1, I_2, I_3\})$ and if $M \in S_1$, then $\phi = (S_0, S_1 \cup \{I_1, I_2, I_3\} \setminus \{M\}, S_2 \cup \{M\} \setminus \{I_1, I_2, I_3\})$ is a Roman dominating function on $\mathcal{C}(A)$ with a weight less than φ, a contradiction. Next, assume that M contains two distinct ideals I_1 and I_2 in S_2. Therefore, $M \notin S_2$. In what follows, we consider two cases.
Case 1. $|S_2| = 2$. Since M is not adjacent to I_1 and I_2, so $M \in S_1$. Hence, the Roman dominating function $\phi = (S_0, S_1 \cup \{I_1, I_2\} \setminus \{M\}, \{M\})$ with a weight less than φ leads to a contradiction.
Case 2. $|S_2| > 2$. Let $I_3 \in S_2 \setminus \{I_1, I_2\}$. Then there exists $M' \in \mathrm{Max}(A)$ such that $I_3 \subseteq M'$. Since $I_3 \nsubseteq M$, so $M \neq M'$. Also, M is adjacent to I_3 and hence, $M \in S_0$. On the other hand, $I_1 \nsubseteq M'$ or $I_2 \nsubseteq M'$. Without loss of generality, assume that $I_2 \nsubseteq M'$ which implies that M' is adjacent to I_2. Now, if $I_3 \neq M'$ (resp. $I_3 = M'$), then the Roman dominating function $\phi = (S_0 \cup \{I_2, I_3\} \setminus \{M, M'\}, S_1 \cup \{I_1\}, S_2 \cup \{M, M'\} \setminus \{I_1, I_2, I_3\})$ (resp. $\phi = (S_0 \cup \{I_2\} \setminus \{M\}, S_1 \cup \{I_1\}, S_2 \cup \{M\} \setminus \{I_1, I_2\})$) with a weight less than φ concludes a contradiction.

(iii) Let $\varphi = (S_0, S_1, S_2)$ be a $\gamma_{\mathfrak{R}}$-function on $\mathcal{C}(A)$. Suppose that $S_2 = \{I_1, \ldots, I_t\}$, M_i is a maximal ideal of A and $I_i \subseteq M_i$, for $i = 1, \ldots, t$. By Part (ii), the maximal ideals $M_1, \ldots, M_t$ are pairwise distinct and $I_i \nsubseteq M_j$, for each $j \neq i$. There are two following cases:
Case 1. $|S_2| = 1$. Thus, $S_2 = \{I_1\}$. We may assume that $I_1 = M_1$; otherwise, M_1 is not adjacent to I_1 and hence, $M_1 \in S_1$ which yields that $\phi = (S_0, S_1 \cup \{I_1\} \setminus \{M_1\}, \{M_1\})$ is a $\gamma_{\mathfrak{R}}$-function on $\mathcal{C}(A)$, so we consider ϕ instead of φ.
Case 2. $|S_2| \geq 2$. So, we deduce that $M_i \notin S_1$, for $i = 1, \ldots, t$. We may assume that $S_2 = \{M_1, \ldots, M_t\}$; otherwise, consider $\phi = (S_0 \cup \{I_1, \ldots, I_t\} \setminus \{M_1, \ldots, M_t\}, S_1, \{M_1, \ldots, M_t\})$ instead of φ. □

Lemma 1.7. *Let $n > 1$ be a positive integer number, $F_1, \ldots, F_n$ be* *fields, $(A_1,\mathfrak{m}_1)$ be a local ring and let $A \cong F_1 \times \cdots \times F_n \times A_1$. Then $\gamma_{\mathfrak{R}}(\mathcal{C}(A)) \leq 2\gamma(\mathcal{C}(A)) - 1$.*

Proof. Let $S_1 = \{0\times\cdots\times 0\times A_1\}, S_2 = \{0\times F_2\times\cdots\times F_n\times A_1, F_1\times 0\times F_3\times\cdots\times F_n\times A_1,\ldots,F_1\times\cdots\times F_{n-1}\times 0\times A_1\}$ and $S_0 = V(\mathcal{C}(A))\setminus(S_1\cup S_2)$. Then $\varphi = (S_0,S_1,S_2)$ is a Roman dominating function on $\mathcal{C}(A)$ of weight $2n+1$. Clearly, $\gamma(\mathcal{C}(A)) = n+1$ and so $\gamma_{\mathfrak{R}}(\mathcal{C}(A)) \leq 2\gamma(\mathcal{C}(A))-1$. $\square$

Now, we are in a position to prove one of the main results.

Theorem 1.8. *Let $A \cong A_1 \times \cdots \times A_n$ be a ring such that $n \geq 4$ and $(A_i,\mathfrak{m}_i)$ is a local ring, for $i = 1,\ldots,n$. If at most one of A_i's is not a field, then $\gamma_{\mathfrak{R}}(\mathcal{C}(A)) = 2n-1$; otherwise, $\gamma_{\mathfrak{R}}(\mathcal{C}(A)) = 2n$.*

Proof. According to Theorem 1.6, suppose that $\varphi = (S_0,S_1,S_2)$ is a $\gamma_{\mathfrak{R}}$-function on $\mathcal{C}(A)$ such that $S_2 \subseteq \mathrm{Max}(A)$. Let M_i be the maximal ideal of A whose ith component is $\mathfrak{m}_i$, for $i = 1,\ldots,n$. With no loss of generality, assume that $S_2 = \{M_1,\ldots,M_t\}$, with $t \leq n$. We have the following cases:

Case 1. $|S_2| = 1$, that is $S_2 = \{M_1\}$. Clearly, $\{I \in V(\mathcal{C}(A))|I \subset M_1\} \subseteq S_1$. Then $\{\mathfrak{m}_1\times J_2\times\cdots\times J_n|J_i = 0, A_i, \text{ for } i = 2,\ldots,n\}\setminus\{M_1,\mathfrak{m}_1\times 0\times\cdots\times 0\} \subseteq S_1$. Therefore, $\gamma_{\mathfrak{R}}(\mathcal{C}(A)) = |S_1|+2|S_2| \geq 2^{n-1}-2+2 > 2n-1$, for each $n \geq 4$. If at most one of A_i's is not a field, then by Lemma 1.7, $\gamma_{\mathfrak{R}}(\mathcal{C}(A)) \leq 2n-1$, a contradiction. Otherwise, by Proposition 1.1, $\gamma_{\mathfrak{R}}(\mathcal{C}(A)) = 2n$.

Case 2. $|S_2| \geq 2$. Then $\{I \in V(\mathcal{C}(A))|I = J_1\times\cdots\times J_n,\ J_i \trianglelefteq A_i, \text{for } i = 1,\ldots,n, \text{ and } J_j \subseteq \mathfrak{m}_j \text{ for } j = 1,\ldots,t\} \subseteq S_1$.

Now, if at most one of A_i's is not a field, then there are two following subcases:

Subcase 1. A_i is a field, for every i, $1 \leq i \leq n$. Then we conclude that $\gamma_{\mathfrak{R}}(\mathcal{C}(A)) = |S_1|+2|S_2| \geq 2^{n-t}-1+2t \geq 2n-1$, for each $n \geq 4$. Thus, by Lemma 1.7, $\gamma_{\mathfrak{R}}(\mathcal{C}(A)) = 2n-1$.

Subcase 2. A_k is not a field, for some k, $1 \leq k \leq n$. If $1 \leq k \leq t$, then $|S_1|+2|S_2| \geq 2^{n-t+1}-2+2t > 2n-1$. This contradicts Lemma 1.7 which is impossible. If $t+1 \leq k \leq n$, then $|S_1|+2|S_2| \geq 3\times 2^{n-t-1}-2+2t \geq 2n-1$. According to Lemma 1.7, $\gamma_{\mathfrak{R}}(\mathcal{C}(A)) = 2n-1$.

Otherwise, a simple calculation gives that $\gamma_{\mathfrak{R}}(\mathcal{C}(A)) = |S_1|+2|S_2| > 2n-1$, for each $n \geq 4$. Hence, by Proposition 1.1, $\gamma_{\mathfrak{R}}(\mathcal{C}(A)) = 2n$. This completes the proof. $\square$

Clearly, if $A \cong A_1 \times \cdots \times A_n$ is a ring such that $n \geq 4$ and $(A_i, \mathfrak{m}_i)$ is a local ring, for $i = 1, \ldots, n$, then $\gamma_{\mathfrak{R}}(\mathcal{C}(A)) \geq 7$. Next, we determine all Artinian rings A with $\gamma_{\mathfrak{R}}(\mathcal{C}(A)) \leq 6$. Throughout the rest of the paper, A is an Artinian ring with $|\mathrm{Max}(A)| > 1$. By the structure theorem of Artinian rings [4, Theorem 8.7], there exists an integer $n > 1$ such that $A \cong A_1 \times A_2 \times \cdots \times A_n$ and $(A_i, \mathfrak{m}_i)$ is a local ring for all $1 \leq i \leq n$. Clearly, $\mathcal{C}(A)$ is not a null graph. Note that if $\varphi = (S_0, S_1, S_2)$ is a $\gamma_{\mathfrak{R}}$-function on $\mathcal{C}(A)$, we may assume that $S_2 \neq \emptyset$. First, we show that $\gamma_{\mathfrak{R}}(\mathcal{C}(A)) \neq 1$.

Lemma 1.9. *If A is a ring, then $\gamma_{\mathfrak{R}}(\mathcal{C}(A)) \neq 1$.*

Proof. By contradiction, suppose that $\gamma_{\mathfrak{R}}(\mathcal{C}(A)) = 1$. Then $\gamma(\mathcal{C}(A)) = 1$ and so $\gamma_{\mathfrak{R}}(\mathcal{C}(A)) = \gamma(\mathcal{C}(A)) = 1$. According to Proposition 1.2, $\mathcal{C}(A)$ is a null graph, which is impossible. $\square$

Now, we determine all rings A with $\gamma_{\mathfrak{R}}(\mathcal{C}(A)) = 2$.

Theorem 1.10. *Let A be a ring. Then $\gamma_{\mathfrak{R}}(\mathcal{C}(A)) = 2$ if and only if $A \cong F_1 \times A_1$ such that F_1 is a field and A_1 is a local ring.*

Proof. First, suppose that $A \cong F_1 \times A_1$ such that F_1 is a field and A_1 is a local ring. We may assume that $S_2 = \{0 \times A_1\}, S_0 = V(\mathcal{C}(A)) \setminus S_2$ and $S_1 = \emptyset$. Clearly, every vertex of S_0 is adjacent to $0 \times A_1$. Thus, $\varphi = (S_0, S_1, S_2)$ is a Roman domination function on $\mathcal{C}(A)$ of weight 2. Hence, $\gamma_{\mathfrak{R}}(\mathcal{C}(A)) = 2$.

Conversely, assume that $\gamma_{\mathfrak{R}}(\mathcal{C}(A)) = 2$. Then $\gamma(\mathcal{C}(A)) \leq 2$. Since $\mathcal{C}(A)$ is not a null graph, $\gamma(\mathcal{C}(A)) = 1$. By Corollary 1.4, we find that $A \cong F_1 \times A_1$, where F_1 is a field and A_1 is a local ring. The proof is complete. $\square$

In the next theorem, we study the case that $\gamma_{\mathfrak{R}}(\mathcal{C}(A)) = 3$.

Theorem 1.11. *Let A be a ring. Then $\gamma_{\mathfrak{R}}(\mathcal{C}(A)) = 3$ if and only if $A \cong A_1 \times A_2$ such that $(A_1, \mathfrak{m}_1)$ and $(A_2, \mathfrak{m}_2)$ are local rings, $\mathfrak{m}_1, \mathfrak{m}_2 \neq 0$, and $|\mathbb{I}(A_i)| = 3$, for some $i = 1, 2$.*

Proof. First, consider $\gamma_{\mathfrak{R}}(\mathcal{C}(A)) = 3$. According to Theorem 1.6, suppose that $\varphi = (S_0, S_1, S_2)$ is a $\gamma_{\mathfrak{R}}$-function on $\mathcal{C}(A)$ such that $S_2 \subseteq \mathrm{Max}(A)$. It is clear that $|S_1| = |S_2| = 1$. Since $\mathcal{C}(A)$ is not a null graph,

$\gamma(\mathcal{C}(A)) \leq 2$. If $\gamma(\mathcal{C}(A)) = 1$, then by Corollary 1.4, $A \cong F_1 \times A_1$ such that F_1 is a field and A_1 is a local ring. Theorem 1.10 shows that $\gamma_R(\mathcal{C}(A)) = 2$, a contradiction. Therefore, $\gamma(\mathcal{C}(A)) = 2$. By Corollary 1.5, $A \cong A_1 \times A_2$ such that $(A_i, \mathfrak{m}_i)$ is a local ring and $\mathfrak{m}_i \neq 0$, for every $i = 1, 2$. Next, we prove that $|\mathbb{I}(A_i)| = 3$, for some $i = 1, 2$. By contradiction, suppose that $|\mathbb{I}(A_1)|, |\mathbb{I}(A_2)| \geq 4$. We know that $V(\mathcal{C}(A)) = \{I_1 \times A_2 | I_1 \lhd A_1\} \cup \{A_1 \times I_2 | I_2 \lhd A_2\}$. Let $J \in S_2$. Since $S_2 \subseteq \mathrm{Max}(A)$, we may assume that $J = \mathfrak{m}_1 \times A_2$. Let $I \in \mathbb{I}(A_1) \backslash \{0, \mathfrak{m}_1, A_1\}$. Then J and the vertices of the set $\{I \times A_2, 0 \times A_2\}$ are non-adjacent. This yields that J and at least 2 vertices are non-adjacent. So $|S_1|$ should be at least 2, a contradiction. Therefore, $|\mathbb{I}(A_i)| = 3$, for some $i = 1, 2$.

Conversely, suppose that $A \cong A_1 \times A_2$ such that A_1, A_2 are local rings, $\mathfrak{m}_1, \mathfrak{m}_2 \neq 0$, and $|\mathbb{I}(A_1)| = 3$. Then $V(\mathcal{C}(A)) = \{0 \times A_2, \mathfrak{m}_1 \times A_2\} \cup \{A_1 \times I_2 | I_2 \lhd A_2\}$. Let $S_0 = \{A_1 \times I_2 | I_2 \lhd A_2\}, S_1 = \{0 \times A_2\}$ and let $S_2 = \{\mathfrak{m}_1 \times A_2\}$. Clearly, $\varphi = (S_0, S_1, S_2)$ is a Roman dominating function on $\mathcal{C}(A)$ of weight 3. Therefore, $\gamma_{\mathfrak{R}}(\mathcal{C}(A)) = 3$. □

The following theorem shows that for every ring A with $\gamma_{\mathfrak{R}}(\mathcal{C}(A)) = 4$, there are two possibilities.

Theorem 1.12. *Let A be a ring. Then $\gamma_{\mathfrak{R}}(\mathcal{C}(A)) = 4$ if and only if one of the following holds:*

(i) $A \cong A_1 \times A_2$ such that A_1, A_2 are local rings and $|\mathbb{I}(A_1)|, |\mathbb{I}(A_2)| \geq 4$.

(ii) $A \cong F_1 \times F_2 \times F_3$ such that F_i is a field, for each $i = 1, 2, 3$.

Proof. If (i) holds, then by Proposition 1.1 and Corollary 1.5, $\gamma_R(\mathcal{C}(A)) \in \{3, 4\}$. Since $|\mathbb{I}(A_1)|, |\mathbb{I}(A_2)| \geq 4$, Theorem 1.11 shows that $\gamma_{\mathfrak{R}}(\mathcal{C}(A)) \neq 3$. Therefore, $\gamma_{\mathfrak{R}}(\mathcal{C}(A)) = 4$. Now, consider (ii) holds. By Theorem 1.11 and Proposition 1.1, we find that $\gamma_{\mathfrak{R}}(\mathcal{C}(A)) \geq 4$. Let $S_0 = \{0 \times F_2 \times F_3, F_1 \times F_2 \times 0, 0 \times F_2 \times 0\}, S_1 = \{F_1 \times 0 \times 0, 0 \times 0 \times F_3\}$ and $S_2 = \{F_1 \times 0 \times F_3\}$. It is not hard to see that S_2 dominates S_0. So, $\varphi = (S_0, S_1, S_2)$ is a Roman dominating function on $\mathcal{C}(A)$ of weight 4. Therefore, $\gamma_{\mathfrak{R}}(\mathcal{C}(A)) = 4$.

Conversely, suppose that $\gamma_{\mathfrak{R}}(\mathcal{C}(A)) = 4$. From Theorem 1.6, suppose that $\varphi = (S_0, S_1, S_2)$ is a $\gamma_{\mathfrak{R}}$-function on $\mathcal{C}(A)$ such that $S_2 \subseteq \mathrm{Max}(A)$. Hence, $|S_1| = 0, |S_2| = 2$ or $|S_1| = 2, |S_2| = 1$. If $|S_1| = 0, |S_2| = 2$,

then we find that $\gamma(\mathcal{C}(A)) = 2$. By Corollary 1.5 and Theorem 1.11, we conclude that (i) holds. Now, consider $|S_1| = 2, |S_2| = 1$. We have $\gamma(\mathcal{C}(A)) \in \{2, 3\}$. There are two following cases:
Case 1. $\gamma(\mathcal{C}(A)) = 2$. Then by Corollary 1.5, we conclude that $A \cong A_1 \times A_2$ such that $(A_i, \mathfrak{m}_i)$ is a local ring, for $i = 1, 2$ and $\mathfrak{m}_1, \mathfrak{m}_2 \neq 0$. If $|\mathbb{I}(A_i)| = 3$, for some $i = 1, 2$, then by Theorem 1.11, we have $\gamma_R(\mathcal{C}(A)) = 3$, a contradiction. Therefore, (i) holds.
Case 2. $\gamma(\mathcal{C}(A)) = 3$. Then by Theorem 1.3, we find that $A \cong A_1 \times A_2 \times A_3$ such that A_1, A_2, A_3 are local rings. We prove that (ii) holds. Assume to the contrary that A_1 is not a field. Suppose that $I = I_1 \times I_2 \times I_3 \in S_2$. Three following subcases hold:
Subcase 1. $I_1 \neq A_1$. Since $I \in \mathrm{Max}(A)$, $I = \mathfrak{m}_1 \times A_2 \times A_3$. Hence, I and all the vertices of the set $\{0 \times A_2 \times 0, 0 \times 0 \times A_3, \mathfrak{m}_1 \times A_2 \times 0\}$ are non-adjacent.
Subcase 2. $I_2 \neq A_2$. Hence, $I_1 = A_1, I_3 = A_3$. Then I and all the vertices of the set $\{0 \times 0 \times A_3, A_1 \times 0 \times 0, \mathfrak{m}_1 \times 0 \times A_3\}$ are non-adjacent.
Subcase 3. $I_3 \neq A_3$. We know that $I_1 = A_1, I_2 = A_2$. Then I and all the vertices of the set $\{0 \times A_2 \times 0, A_1 \times 0 \times 0, \mathfrak{m}_1 \times A_2 \times 0\}$ are non-adjacent.

By the above subcases, we find that $|S_1| \geq 3$, a contradiction. Therefore, A_1 is a field. Similarly, A_2 and A_3 are fields and (ii) holds. □

Now, we determine all rings A whose comaximal ideal graphs have a Roman domination number equal to 5.

Theorem 1.13. *Let A be a ring. Then $\gamma_{\mathfrak{R}}(\mathcal{C}(A)) = 5$ if and only if $A \cong F_1 \times F_2 \times A_1$, where F_i is a field, for $i = 1, 2$ and $(A_1, \mathfrak{m}_1)$ is a local ring with $\mathfrak{m}_1 \neq 0$.*

Proof. First, suppose that $A \cong F_1 \times F_2 \times A_1$, F_1 and F_2 are fields, and $(A_1, \mathfrak{m}_1)$ is a local ring with $\mathfrak{m}_1 \neq 0$. By Lemma 1.7, $\gamma_{\mathfrak{R}}(\mathcal{C}(A)) \leq 5$. On the other hand, Proposition 1.2 and Theorem 1.12 show that $\gamma_{\mathfrak{R}}(\mathcal{C}(A)) = 5$ and the proof is complete.

Conversely, assume that $A \cong A_1 \times \cdots \times A_n$, for some positive integer n, $(A_i, \mathfrak{m}_i)$ is a local ring, for $1 \leq i \leq n$, and $\gamma_{\mathfrak{R}}(\mathcal{C}(A)) = 5$. By Theorem 1.6, suppose that $\varphi = (S_0, S_1, S_2)$ is a $\gamma_{\mathfrak{R}}$-function on $\mathcal{C}(A)$ such that $S_2 \subseteq \mathrm{Max}(A)$. Then we have $|S_1| = 1, |S_2| = 2$ or $|S_1| = 3, |S_2| = 1$. It is clear that $\gamma(\mathcal{C}(A)) \in \{3, 4\}$.

First, consider $\gamma(\mathcal{C}(A)) = 4$. By Theorem 1.3, we have $A \cong A_1 \times A_2 \times A_3 \times A_4$, where $(A_i, \mathfrak{m}_i)$ is a local ring, for $1 \leq i \leq 4$. Clearly, $|S_1| =$

$3, |S_2| = 1$. Let $I = I_1 \times I_2 \times I_3 \times I_4 \in S_2$. Without loss of generality, we may assume that $I_1 \neq A_1$. Set $\mathfrak{A} = \{0 \times J_2 \times J_3 \times J_4 | J_i = 0, A_i$, for $i = 2, 3, 4\} \setminus \{0\}$. Clearly, I and at least 6 vertices of $\mathfrak{A}$ are non-adjacent. This yields that $|S_1| \geq 6$, a contradiction. Therefore, $\gamma(\mathcal{C}(A)) \neq 4$, and so $\gamma(\mathcal{C}(A)) = 3$. Then Theorem 1.3 shows that $A \cong A_1 \times A_2 \times A_3$, $(A_i, \mathfrak{m}_i)$ is a local ring, for all $1 \leq i \leq 3$. We claim that two rings of A_i's are fields. By contradiction, we may assume that A_1, A_2 are not fields. Two following cases hold:

Case 1. $|S_1| = 3, |S_2| = 1$. Let $I = I_1 \times I_2 \times I_3 \in S_2$. There are the following subcases:

Subcase 1. $I_3 = A_3$. Then $I \in \{\mathfrak{m}_1 \times A_2 \times A_3, A_1 \times \mathfrak{m}_2 \times A_3\}$. Therefore, I and all vertices of the set $\{0 \times 0 \times A_3, 0 \times \mathfrak{m}_2 \times A_3, \mathfrak{m}_1 \times \mathfrak{m}_2 \times A_3, \mathfrak{m}_1 \times 0 \times A_3\}$ are non-adjacent. This contradicts $|S_1| = 3$.

Subcase 2. $I_3 \neq A_3$. Then $I_1 = A_1, I_2 = A_2$. Hence, I and all vertices of the set $\{0 \times A_2 \times 0, \mathfrak{m}_1 \times A_2 \times 0, A_1 \times 0 \times 0, A_1 \times \mathfrak{m}_2 \times 0\}$ are non-adjacent. This contradicts $|S_1| = 3$.

Case 2. $|S_1| = 1, |S_2| = 2$. Let $I = I_1 \times I_2 \times I_3, J = J_1 \times J_2 \times J_3 \in S_2$. Since $S_2 \subseteq \mathrm{Max}(A)$, $I_i = A_i$ or $J_i = A_i$, for every i, $1 \leq i \leq 3$. With no loss of generality, assume the following subcases:

Subcase 1. $I_1 = A_1, J_1 \neq A_1, I_2 \neq A_2, J_2 = A_2$. Then $\{0 \times 0 \times A_3, \mathfrak{m}_1 \times 0 \times A_3\} \subseteq S_1$ and so $|S_1| \neq 1$. This is a contradiction.

Subcase 2. $I_1 = A_1, J_1 \neq A_1, I_3 \neq A_3, J_3 = A_3$. Then $\{0 \times A_2 \times 0, \mathfrak{m}_1 \times A_2 \times 0\} \subseteq S_1$. This yields that $|S_1| \neq 1$, a contradiction.

The above cases show that two rings of A_i's are fields and the claim is proved. Now, by Part (ii) of Theorem 1.12, we find that $A \cong F_1 \times F_2 \times A_1$ such that F_1 and F_2 are fields and A_1 is a local ring with $|\mathbb{I}(A_1)| \geq 3$ and the proof is complete. $\square$

Finally, we answer the question of when the Roman domination number is equal to 6.

Theorem 1.14. *Let A be a ring. Then $\gamma_{\mathfrak{R}}(\mathcal{C}(A)) = 6$ if and only if $A \cong A_1 \times A_2 \times A_3$, where A_i is a local ring for every $i = 1, 2, 3$, and at most one of A_i's is a field.*

Proof. Let $A \cong A_1 \times A_2 \times A_3$ and at most one of A_i's is a field. Thus, $\gamma(\mathcal{C}(A)) = 3$. Hence, by Proposition 1.1, we have $3 \leq \gamma_{\mathfrak{R}}(\mathcal{C}(A)) \leq 6$. Therefore, by Theorems 1.11, 1.12 and 1.13, we are done.

Now, suppose that $\gamma_{\mathfrak{R}}(\mathcal{C}(A)) = 6$. By Theorem 1.6, assume that $\varphi = (S_0, S_1, S_2)$ is a $\gamma_{\mathfrak{R}}$-function on $\mathcal{C}(A)$ such that $S_2 \subseteq \mathrm{Max}(A)$. By Propositions 1.1 and 1.2, $3 \leq \gamma(\mathcal{C}(A)) \leq 5$. First, assume that $\gamma(\mathcal{C}(A)) = 5$, and $A \cong A_1 \times \cdots \times A_5$ such that $(A_i, \mathfrak{m}_i)$ is a local ring, for all $1 \leq i \leq 5$. Thus, we conclude that $|S_1| = 4$ and $|S_2| = 1$. Let $I = I_1 \times \cdots \times I_5 \in S_2$. We may assume that $I_1 \neq A_1$. Clearly, I is not adjacent to at least 14 vertices of $\{0 \times J_2 \times \cdots \times J_5 | J_i = 0, A_i$, for $i = 2, \ldots, 5\} \setminus \{0\}$, a contradiction.

Next, suppose that $\gamma(\mathcal{C}(A)) = 4$, and $A \cong A_1 \times \cdots \times A_4$, where $(A_i, \mathfrak{m}_i)$ is a local ring, for all $1 \leq i \leq 4$. Hence, $|S_1| = 4, |S_2| = 1$ or $|S_1| = 2, |S_2| = 2$. If $|S_1| = 4, |S_2| = 1$, then we assume that $I = I_1 \times \cdots \times I_4 \in S_2$ and $I_1 \neq A_1$. Similarly, I is not adjacent to at least 6 vertices of $\{0 \times J_2 \times J_3 \times J_4 | J_i = 0, A_i$, for $i = 2, 3, 4\} \setminus \{0\}$, a contradiction. If $|S_1| = 2, |S_2| = 2$, then suppose that $I = I_1 \times \cdots \times I_4, J = J_1 \times \cdots \times J_4 \in S_2$. Since $S_2 \subseteq \mathrm{Max}(A)$, we may assume that $I_1 = S_1, J_1 \neq A_1, I_2 \neq A_2, J_2 = A_2$. Then $\{0 \times 0 \times A_3 \times 0, 0 \times 0 \times 0 \times A_4, 0 \times 0 \times A_3 \times A_4\} \subseteq S_1$ and hence, $|S_1| \geq 3$, a contradiction.

Finally, suppose that $\gamma(\mathcal{C}(A)) = 3$. Thus, $A \cong A_1 \times A_2 \times A_3$ such that $(A_i, \mathfrak{m}_i)$ is a local ring, for all $1 \leq i \leq 3$. By Theorems 1.12 and 1.13, we find that at most one of A_i's is a field. The proof is complete. □

As an immediate consequence of Theorems 1.8 and 1.10–1.14, we have the following results.

Corollary 1.15. *Let $A \cong A_1 \times \cdots \times A_n$ be a ring such that $n \geq 2$ and $(A_i, \mathfrak{m}_i)$ is a local ring for $i = 1, \ldots, n$. Then*

$$\gamma_{\mathfrak{R}}(\mathcal{C}(A)) = \begin{cases} 2, & \text{if } n = 2, \text{ and } \mathfrak{m}_i = 0, \text{ for some } i = 1, 2; \\ 3, & \text{if } n = 2, \ \mathfrak{m}_1, \mathfrak{m}_2 \neq 0, \text{ and } |\mathbb{I}(A_i)| = 3, \text{ for some } \\ & \ i = 1, 2; \\ 4, & \text{if } n = 2 \text{ with } |\mathbb{I}(A_1)|, |\mathbb{I}(A_2)| \geq 4 \text{ or } n = 3 \text{ with} \\ & \mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3 = 0; \\ 5, & \text{if } n = 3 \text{ and two of } A_i\text{'s are fields and another} \\ & \ \text{is not a field;} \\ 6, & \text{if } n = 3 \text{ and at most one of } A_i\text{'s is a field;} \\ 2n-1, & \text{if } n \geq 4 \text{ and at most one of } A_i\text{'s is not a field;} \\ 2n, & \text{otherwise.} \end{cases}$$

A graph G is called a *Roman graph* if $\gamma_{\mathfrak{R}}(G) = 2\gamma(G)$. From the above corollary, we conclude the next result.

Corollary 1.16. *Let $A \cong A_1 \times \cdots \times A_n$ be a ring such that $n \geq 2$ and $(A_i, \mathfrak{m}_i)$ is a local ring for $i = 1, \ldots, n$. Then $\mathcal{C}(A)$ is a Roman graph if and only if one of the following holds:*
(i) $n = 2$ and $\mathfrak{m}_i = 0$, for some $i = 1, 2$.
(ii) $n = 2$ with $|\mathbb{I}(A_1)|, |\mathbb{I}(A_2)| \geq 4$.
(iii) $n \geq 3$ and at least two of A_i's are not fields.

We close this paper by the following example.

Example 1.17. Let $m = p_1^{\alpha_1} \cdots p_s^{\alpha_s} > 1$ be an integer, where p_i's are distinct primes and α_i's are positive integers and let $\mathbb{Z}_m$ be the integers modulo m. Then the following statements hold:
(i) $\gamma_{\mathfrak{R}}(\mathcal{C}(\mathbb{Z}_m)) = 2$ if and only if $m = p_1 p_2^{\alpha_2}$.
(ii) $\gamma_{\mathfrak{R}}(\mathcal{C}(\mathbb{Z}_m)) = 3$ if and only if $m = p_1^2 p_2^{\alpha_2}$, $\alpha_2 \geq 2$.
(iii) $\gamma_{\mathfrak{R}}(\mathcal{C}(\mathbb{Z}_m)) = 4$ if and only if $m = p_1^{\alpha_1} p_2^{\alpha_2}$, $\alpha_1, \alpha_2 \geq 3$ or $m = p_1 p_2 p_3$.
(iv) $\gamma_{\mathfrak{R}}(\mathcal{C}(\mathbb{Z}_m)) = 5$ if and only if $m = p_1 p_2 p_3^{\alpha_3}$, $\alpha_3 \geq 2$.
(v) $\gamma_{\mathfrak{R}}(\mathcal{C}(\mathbb{Z}_m)) = 6$ if and only if $m = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3}$, $\alpha_2, \alpha_3 \geq 2$.
(vi) $\gamma_{\mathfrak{R}}(\mathcal{C}(\mathbb{Z}_m)) = 2s - 1$ if and only if $s \geq 4$ and $m = p_1 \cdots p_{s-1} p_s^{\alpha_s}$.
(vii) $\gamma_{\mathfrak{R}}(\mathcal{C}(\mathbb{Z}_m)) = 2s$ if and only if $s \geq 4$ and $m = p_1^{\alpha_1} \cdots p_s^{\alpha_s}$, $\alpha_{s-1}, \alpha_s \geq 2$.

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Soheila Khojasteh
Assistant Professor of Mathematics
Department of Mathematics
Lahijan Branch, Islamic Azad University
Lahijan, Iran
E-mail: soheila.khojasteh@iau.ac.ir; khojaste.soheila@gmail.com

Farideh Heydari
Associate Professor of Mathematics
Department of Mathematics
Karaj Branch, Islamic Azad University
Karaj, Iran
E-mail: f-heydari@kiau.ac.ir; faride.heydari@gmail.com