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Original Research Paper

On the Fuzzy Solution of Time-Fractional Cauchy Reaction-Diffusion Equation

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Abstract. In the current article, fuzzy Sumudu transform iterative method is defined and used to obtain an analytical fuzzy triangular solution of the time-fractional Cauchy reaction-diffusion equation under generalized Hukuhara partial differentiability. On this basis, we prove some properties of fuzzy Sumudu transform. The merits and applicability of the proposed theory are validated through numerical simulation.

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1 Introduction

For the last two decades, classical calculus has been extended to modern fuzzy and fuzzy fractional calculus, like differential equations to fractional order and fuzzy fractional order. The fuzzy fractional calculus has been given much attentions by the researchers due to its significant applications and realistic description of many physical and biological phenomenon [15, 18, 12, 13, 14, 21]. Some real-life problems have been model by partial differential equations, because during the study of natural phenomena, we often faced several variables simultaneously[9, 10, 17]. Even some time partial differential equations is not the best option to study real life problems due to fuzziness in the problems.

Over the past decades, a significant development in fractional calculus has been used widely by the researchers. Fractional operators are much better at explaining the physical phenomena (Biological population models, predator-prey models, infectious diseases models, etc) more accurately compared to ordinary operators. The time-fractional cauchy reaction-diffusion equation is a fractional partial differential equation that deals with the study of fluid velocity and convection temperature dynamics [1, 3, 16]. The concepts of fuzzy fractional integral, Caputo partial differentiability based on generalized Hukurara differentiability for the fuzzy multivariable functions, and fuzzy fractional partial differential equations are examined by H. Viet Long et al. [19]. The fuzzy Caputo-Katugampola fractional differential equations in fuzzy space are considered in [11], and under generalized Lipschitz condition, the existence and uniqueness of the solution are proved.

In the following, We consider generalized Hukuhara partial differentiability of the solution and an analytical fuzzy triangular solution of time-fractional Cauchy Reaction-Diffusion equation with fuzzy triangular initial conditions, is achieved using the fuzzy Sumudu transform.

The rest of this paper is organized as follows. In Section 2, some notations and preliminaries used throughout the paper is introduced. In Section 3, we define the definition of fuzzy Sumudu transform and prove some properties. We obtain an analytical fuzzy triangular solution for the time-fractional Cauchy Reaction-Diffusion equation by the fuzzy Sumudu transform based on the type of Caputo gH-differentiability in Section 4. Some examples are given in the final section to illustrate our theory.

2 Basic Preliminaries

In this section, we introduce notations, definitions, and preliminary facts used throughout this paper. We use $\mathbf{E}$ to show the fuzzy numbers space, and $\mathbf{T}$ is the set of all triangular fuzzy numbers, which is defined by an ordered triple $\mathcal{A} = (\mathrm{a}_1, \mathrm{a}_2, \mathrm{a}_3)$, where $\mathrm{a}_1 \le \mathrm{a}_2 \le \mathrm{a}_3$.

For $0 < r \le 1$ denote $[\mathcal{A}]^r = \left\{ \delta \in \mathbb{R}^n \middle| \mathcal{A}(\delta) \ge r \right\} = [\mathcal{A}^-(r),\ \mathcal{A}^+(r)]$. The r-level set $[A]^r$ is a closed interval for all $r \in [0,\ 1]$, and for every triangular fuzzy number $\mathcal{A} = (\mathrm{a}_1, \mathrm{a}_2, \mathrm{a}_3)$, we have $\mathcal{A}^-(r) = \mathrm{a}_1 + (\mathrm{a}_2 - a_1)r$ and $\mathcal{A}^+(r) = \mathrm{a}_3 - (\mathrm{a}_3 - \mathrm{a}_2)r$.

The Hausdorff distance between fuzzy numbers is given by $D : \mathbf{E} \times \mathbf{E} \longrightarrow \mathbb{R}^+ \cup \{0\}$ as in [4]

$$
\begin{aligned}
D(\mathcal{A},\ \mathcal{B}) &= \sup_{r\in[0,\ 1]} d\Big([\mathcal{A}]^r,\ [\mathcal{B}]^r\Big) \\
&= \sup_{r\in[0,\ 1]} \max\Big\{ |\mathcal{A}^-(r) - \mathcal{B}^-(r)|,\ |\mathcal{A}^+(r) - \mathcal{B}^+(r)| \Big\}
\end{aligned}
$$

where d is the Hausdorff metric [4].

Now consider $\mathcal{A} = (\mathrm{a}_1, \mathrm{a}_2, \mathrm{a}_3)$ And $\mathcal{B} = (\mathrm{b}_1, \mathrm{b}_2, \mathrm{b}_3)$ are two triangular fuzzy numbers. The generalized Hukuhara difference, $A \ominus_{gH} B$, is defined as follows

$$
\mathcal{A} \ominus_{gH} \mathcal{B} = \Big(\min\{\mathrm{a}_1 - \mathrm{b}_1, \mathrm{a}_3 - \mathrm{b}_3\}, \mathrm{a}_2 - \mathrm{b}_2, \max\{\mathrm{a}_1 - \mathrm{b}_1, \mathrm{a}_3 - \mathrm{b}_3\} \Big) \quad (1)
$$

Let $\mathbb{J} \subseteq \mathbb{R}^2$ and $\psi : \mathbb{J} \to \mathbf{E}$ is a fuzzy function. A fuzzy function $\psi(\delta, \mathrm{t}) = \Big(\psi_1(\delta, \mathrm{t}), \psi_2(\delta, \mathrm{t}), \psi_3(\delta, \mathrm{t})\Big)$ is called a triangular fuzzy function provided that $\psi_1(\delta, \mathrm{t}), \psi_2(\delta, \mathrm{t})$ and $\psi_3(\delta, \mathrm{t})$ are real-valued functions such that $\psi_1(\delta, \mathrm{t}) \le \psi_2(\delta, \mathrm{t}) \le \psi_3(\delta, \mathrm{t})$ for all $(\delta, \mathrm{t}) \in \mathbb{J}$.

2.1 Fuzzy Differentiation

Definition 2.1. ([8]) *The first generalized Hukuhara partial derivative ([gH-p]-derivative for short) of a fuzzy value function* $\psi(\delta, \mathrm{t}) : \mathbb{J} \to \boldsymbol{E}$

at $(\delta_0, \mathrm{t}_0) \in \mathbb{J}$ *concerning variable* t *is a function* $\frac{\partial\psi(\delta_0,\mathrm{t}_0)}{\partial\mathrm{t}}$ *such that*

$$\frac{\partial\psi(\delta_0, \mathrm{t}_0)}{\partial\mathrm{t}} = \lim_{k\to 0} \frac{\psi(\delta_0, t_0 + k) \ominus_{gH} \psi(\delta_0, \mathrm{t}_0)}{k},$$

provided that $\frac{\partial\psi(\delta_0,\mathrm{t}_0)}{\partial\mathrm{t}} \in \boldsymbol{E}$.

Definition 2.2. (See [8]) *Let* $\psi(\delta, \mathrm{t})$ *and* $\frac{\partial\psi(\delta,\mathrm{t})}{\partial\mathrm{t}}$ *are triangular fuzzy functions and [gH-p]-differentiable at* $(\delta, \mathrm{t}) \in \mathbb{J}$*. Moreover. there aren't any switching points on* $\mathbb{J}$ *and* $\psi_1(\delta, \mathrm{t}), \psi_2(\delta, \mathrm{t})$ *and* $\psi_3(\delta, \mathrm{t})$ *are differentiable at* (δ_0, t_0)*. Then is called*

(i). $\psi(\delta, \mathrm{t})$ *is* $[(i) - p]$*-differentiable with respect to* t *at* $(\delta, \mathrm{t}) \in \mathbb{J}$ *if* $\frac{\partial\psi(\delta,\mathrm{t})}{\partial\mathrm{t}} = \Big(\frac{\partial\psi_1(\delta,\mathrm{t})}{\partial\mathrm{t}}, \frac{\partial\psi_2(\delta,\mathrm{t})}{\partial\mathrm{t}}, \frac{\partial\psi_3(\delta,\mathrm{t})}{\partial\mathrm{t}}\Big)$.

(ii). $\psi(\delta, \mathrm{t})$ *is* $[(ii) - p]$*-differentiable with respect to* t *at* $(\delta, \mathrm{t}) \in \mathbb{J}$ *if* $\frac{\partial\psi(\delta,\mathrm{t})}{\partial\mathrm{t}} = \Big(\frac{\partial\psi_3(\delta,\mathrm{t})}{\partial\mathrm{t}}, \frac{\partial\psi_2(\delta,\mathrm{t})}{\partial\mathrm{t}}, \frac{\partial\psi_1(\delta,\mathrm{t})}{\partial\mathrm{t}}\Big)$.

Definition 2.3. (See [2], [20]) *Let* $\psi(\delta, \mathrm{t})$ *be a fuzzy function which [gH-p]-differentiable for* t *up to order one. The generalized Hukuhara fractional Caputo derivative* $\psi(\delta, \mathrm{t})$ *of order* α *is defined as follows*

$${}^{C}_{gH}\mathfrak{D}^{\alpha}_{t}\psi(\delta, \mathrm{t}) = \frac{1}{\Gamma(1-\alpha)} \int_0^{\mathrm{t}} \frac{\partial\psi(\delta, \xi)}{\partial\xi} \frac{1}{(\mathrm{t} - \xi)^{\alpha}} d\xi,$$

where $0 < \alpha \leq 1$.

Let $\psi(\delta, \mathrm{t})$ *is a triangular fuzzy function and [gH-p]-differentiable, then*

- $\psi(\delta, \mathrm{t})$ *is* $(1)-$*Caputo gH-differentiable w.r.t* t*, if* ${}^{C}_{gH}\mathfrak{D}^{\alpha}_{t}\psi(\delta, \mathrm{t}) \in \boldsymbol{T}$ *for all* $(\delta, \mathrm{t}) \in \mathbb{J}$ *and*

$${}^{C}_{gH}\mathfrak{D}^{\alpha}_{1}\psi(\delta, \mathrm{t}) = \Big(\, {}^{C}D^{\alpha}\psi_1(\delta, \mathrm{t}),\ {}^{C}D^{\alpha}\psi_2(\delta, \mathrm{t}),\ {}^{C}D^{\alpha}\psi_3(\delta, \mathrm{t})\Big).$$

- $\psi(\delta, \mathrm{t})$ *is* $(2)-$*Caputo gH-differentiable w.r.t* t*, if* ${}^{C}_{gH}\mathfrak{D}^{\alpha}_{t}\psi(\delta, \mathrm{t}) \in \boldsymbol{T}$ *for all* $(\delta, \mathrm{t}) \in \mathbb{J}$ *and*

$${}^{C}_{gH}\mathfrak{D}^{\alpha}_{2}\psi(\delta, \mathrm{t}) = \Big(\, {}^{C}D^{\alpha}\psi_3(\delta, \mathrm{t}),\ {}^{C}D^{\alpha}\psi_2(\delta, \mathrm{t}),\ {}^{C}D^{\alpha}\psi_1(\delta, \mathrm{t})\Big).$$

2.2 Fuzzy Integration

Let $\psi : \mathbb{J} \to \mathbf{T}$ be a triangular continuous fuzzy function. Based on the results in [5] and [8], we have

$$\int_{\mathrm{a}}^{\mathrm{b}} \psi(\delta, \mathrm{t})\mathrm{dt} = \Big(\int_{\mathrm{a}}^{\mathrm{b}} \psi_1(\delta, \mathrm{t})\mathrm{dt}, \int_{\mathrm{a}}^{\mathrm{b}} \psi_2(\delta, \mathrm{t})\mathrm{dt}, \int_{\mathrm{a}}^{\mathrm{b}} \psi_3(\delta, \mathrm{t})\mathrm{dt} \Big). \tag{2}$$

Moreover,

$$\int_{\mathrm{a}}^{\infty} \psi(\delta, \mathrm{t})\mathrm{dt} = \lim_{\mathcal{P} \to \infty} \int_{\mathrm{a}}^{\mathcal{P}} \psi(\delta, \mathrm{t})\mathrm{dt}.$$

In fact

$$\int_{\mathrm{a}}^{\infty} \psi(\delta, \mathrm{t})\mathrm{dt} = \lim_{\mathcal{P} \to \infty} \Big(\int_{\mathrm{a}}^{\mathcal{P}} \psi_1(\delta, \mathrm{t})\mathrm{dt}, \ \int_{\mathrm{a}}^{\mathcal{P}} \psi_2(\delta, \mathrm{t})\mathrm{dt}, \int_{\mathrm{a}}^{\mathcal{P}} \psi_3(\delta, \mathrm{t})\mathrm{dt} \ \Big),$$

provided that the limits exist as a finite numbers.

Lemma 2.4. *Consider* a *and* b, a.b > 0 *are real constants. If* $\psi(\delta, \mathrm{t})$ *and* $\varphi(\delta, \mathrm{t})$ *are triangular fuzzy functions, then,*

i). $\int_0^{\infty} \Big(\mathrm{a} \ \psi(\delta, \mathrm{t}) \oplus \mathrm{b} \ \varphi(\delta, \mathrm{t}) \Big)\mathrm{dt} = \mathrm{a} \int_0^{\infty} \psi(\delta, \mathrm{t})\mathrm{dt} \oplus \mathrm{b} \int_0^{\infty} \varphi(\delta, \mathrm{t})\mathrm{dt}.$

ii). $\int_0^{\infty} \Big(\mathrm{a} \ \psi(\delta, \mathrm{t}) \ominus_{gH} \mathrm{b} \ \varphi(\delta, \mathrm{t}) \Big)\mathrm{dt} = \mathrm{a} \int_0^{\infty} \psi(\delta, \mathrm{t})\mathrm{dt} \ominus_{gH} \mathrm{b} \int_0^{\infty} \varphi(\delta, \mathrm{t})\mathrm{dt}.$

Proof. Since the proofs of case (i) and (ii) are similar, here we will prove case (ii). Let us consider a and b are positive real constants. By equation (1) and equation (2) we have

$$
\begin{aligned}
&\int_0^\infty \Big(\mathrm{a}\ \psi(\delta,\mathrm{t}) \ominus_{gH} \mathrm{b}\ \varphi(\delta,\mathrm{t})\Big)\mathrm{dt} = \\
&\int_0^\infty \Bigg(\min\Big\{a\psi_1(\delta,\mathrm{t}) - \mathrm{b}\varphi_1(\delta,\mathrm{t}),\ \mathrm{a}\psi_3(\delta,\mathrm{t}) - \mathrm{b}\varphi_3(\delta,\mathrm{t})\Big\}, \\
&\ \mathrm{a}\psi_2(\delta,\mathrm{t}) - \mathrm{b}\varphi_2(\delta,\mathrm{t}), \\
&\ \max\Big\{\mathrm{a}\psi_1(\delta,\mathrm{t}) - \mathrm{b}\varphi_1(\delta,\mathrm{t}),\ \mathrm{a}\psi_3(\delta,\mathrm{t}) - \mathrm{b}\varphi_3(\delta,\mathrm{t})\Big\}\Bigg)\mathrm{dt}, \\
&\ = \Bigg(\min\Big\{\mathrm{a}\int_0^\infty \psi_1(\delta,\mathrm{t})\mathrm{dt} - \mathrm{b}\int_0^\infty \varphi_1(\delta,\mathrm{t})\mathrm{dt}, \\
&\ \mathrm{a}\int_0^\infty \psi_3(\delta,\mathrm{t})\mathrm{dt} - \mathrm{b}\int_0^\infty \varphi_3(\delta,\mathrm{t})\mathrm{dt}\Big\}, \qquad (3) \\
&\ \mathrm{a}\int_0^\infty \psi_2(\delta,\mathrm{t})\mathrm{dt} - \mathrm{b}\int_0^\infty \varphi_2(\delta,\mathrm{t})\mathrm{dt} \\
&\ \ ,\ \max\Big\{\mathrm{a}\int_0^\infty \psi_1(\delta,\mathrm{t})\mathrm{dt} - \mathrm{b}\int_0^\infty \varphi_1(\delta,\mathrm{t})\mathrm{dt} \\
&\qquad ,\ \mathrm{a}\int_0^\infty \psi_3(\delta,\mathrm{t})\mathrm{dt} - \mathrm{b}\int_0^\infty \varphi_3(\delta,\mathrm{t})\mathrm{dt}\Big\}\Bigg) \\
&\ = \mathrm{a}\int_0^\infty \psi(\delta,\mathrm{t})\mathrm{dt} \ominus_{gH} \mathrm{b}\int_0^\infty \varphi(\delta,\mathrm{t})\mathrm{dt} \qquad (4)
\end{aligned}
$$

Now, consider a and b are negative real constants. Therefore

$$
\begin{aligned}
\int_0^\infty \mathrm{a}\psi(\delta,\mathrm{t})\mathrm{dt} &= \int_0^\infty \Big(\mathrm{a}\psi_3(\delta,\mathrm{t}), \mathrm{a}\psi_2(\delta,\mathrm{t}), \mathrm{a}\psi_1(\delta,\mathrm{t})\Big)\mathrm{dt} \\
&= \Big(\mathrm{a}\int_0^\infty \psi_3(\delta,\mathrm{t})\mathrm{dt}, \mathrm{a}\int_0^\infty \psi_2(\delta,\mathrm{t})\mathrm{dt}, \mathrm{a}\int_0^\infty \psi_1(\delta,\mathrm{t})\mathrm{dt}\Big) \\
&= \mathrm{a}\Big(\int_0^\infty \psi_1(\delta,\mathrm{t})\mathrm{dt}, \int_0^\infty \psi_2(\delta,\mathrm{t})\mathrm{dt}, \int_0^\infty \psi_3(\delta,\mathrm{t})\mathrm{dt}\Big) \\
&= \mathrm{a}\int_0^\infty \psi(\delta,\mathrm{t})\mathrm{dt} \qquad (5)
\end{aligned}
$$

Similar to Eq.(3) and using Eq.(5), for negative constants a and b, we observe that

$$
\int_0^\infty \Big(\mathrm{a}\psi(\delta,\mathrm{t})\mathrm{dt} \ominus_{gH} \mathrm{b}\varphi(\delta,\mathrm{t})\Big)\mathrm{dt} = \mathrm{a}\int_0^\infty \psi(\delta,\mathrm{t})\mathrm{dt} \ominus_{gH} \mathrm{b}\int_0^\infty \varphi(\delta,\mathrm{t})\mathrm{dt}
$$

That is, proving the claim. $\square$

Proposition 2.5. *([19]) Let $\psi(\delta, \mathrm{t})$ be a continuous fuzzy function.*

i). *If ψ is [i-p]-differentiable w.r.t t, with no switching point on $\mathbb{R}\times[a, \mathrm{b}]$ then $\frac{\partial\psi}{\partial \mathrm{t}}$ is integrable on $[a, \mathrm{b}]$ and*

$$\int_{\mathrm{a}}^{\mathrm{b}} \frac{\partial\psi(\delta, \mathrm{t})}{\partial \mathrm{t}} \mathrm{dt} = \psi(\delta, \mathrm{b}) \ominus \psi(\delta, a).$$

ii). *If ψ is [ii-p]-differentiable w.r.t t, with no switching point on $\mathbb{R}\times[a, \mathrm{b}]$ then $\frac{\partial\psi}{\partial \mathrm{t}}$ is integrable on $[a, \mathrm{b}]$ and*

$$\int_{\mathrm{a}}^{\mathrm{b}} \frac{\partial\psi(\delta, \mathrm{t})}{\partial \mathrm{t}} \mathrm{dt} = (-1)\psi(\delta, a) \ominus (-1)\psi(\delta, \mathrm{b}).$$

Proposition 2.6. *Let $\mathrm{h}(\mathrm{t})$ be a positive and decreasing continuous real valued function and ψ is a [gH-p]-differentiable respect to t at every $(\delta, \mathrm{t}) \in \mathbb{J}$ such that there are not any switching points on $\mathbb{J}$.*

i). *Suppose that $k(\delta, \mathrm{t}) = \mathrm{h}(\mathrm{t})\psi(\delta, \mathrm{t})$ and $\psi(\delta, \mathrm{t})$ is [i-p]-differentiable w.r.t t, then $k(\delta, \mathrm{t})$ is [i-p]-differentiable w.r.t t and*

$$\int_{\mathrm{a}}^{\mathrm{b}} \mathrm{h}(\mathrm{t})\frac{\partial\psi(\delta, \mathrm{t})}{\partial \mathrm{t}} \mathrm{dt} = k(\delta, \mathrm{b}) \ominus k(\delta, a) \oplus \int_{\mathrm{a}}^{\mathrm{b}} (-1)h'(\mathrm{t})\psi(\delta, \mathrm{t})\mathrm{dt}$$

ii). *If $k(\delta, \mathrm{t}) = \mathrm{h}(\mathrm{t})\psi(\delta, \mathrm{t})$ and $\psi(\delta, \mathrm{t})$ is [ii-p]-differentiable w.r.t t, then $k(\delta, \mathrm{t})$ is [ii-p]-differentiable w.r.t t and*

$$\int_{\mathrm{a}}^{\mathrm{b}} \mathrm{h}(\mathrm{t})\frac{\partial\psi(\delta, \mathrm{t})}{\partial \mathrm{t}} \mathrm{dt} = (-1)k(\delta, a) \ominus (-1)k(\delta, \mathrm{b}) \ominus_{gH} \int_{\mathrm{a}}^{\mathrm{b}} h'(\mathrm{t})\psi(\delta, \mathrm{t})\mathrm{dt}.$$

Proof. Let $\psi(\delta,\mathrm{t})$ is [i-p]-differentiable w.r.t t. h(t) be a positive and decreasing continuous real valued function, then $\mathrm{h}(\mathrm{t})>0$ and $h'(\mathrm{t})<0$

$$\begin{aligned}
&\mathrm{h}(\mathrm{t})\frac{\partial\psi(\delta,\mathrm{t})}{\partial\mathrm{t}}\ominus((-1)h'(\mathrm{t}))\psi(\delta,\mathrm{t})\\
&=\mathrm{h}(\mathrm{t})\Big(\frac{\partial\psi_1(\delta,\mathrm{t})}{\partial\mathrm{t}},\frac{\psi_2(\delta,\mathrm{t})}{\partial\mathrm{t}},\frac{\psi_3(\delta,\mathrm{t})}{\partial\mathrm{t}}\Big)\\
&\ominus(-h'(\mathrm{t}))\Big(\psi_1(\delta,\mathrm{t}),\psi_2(\delta,\mathrm{t}),\psi_3(\delta,\mathrm{t})\Big)\\
&=\Big(\mathrm{h}(\mathrm{t})\frac{\partial\psi_1(\delta,\mathrm{t})}{\partial\mathrm{t}},\mathrm{h}(\mathrm{t})\frac{\psi_2(\delta,\mathrm{t})}{\partial\mathrm{t}},\mathrm{h}(\mathrm{t})\frac{\psi_3(\delta,\mathrm{t})}{\partial\mathrm{t}}\Big)\\
&\ominus\Big(-h'(\mathrm{t})\psi_1(\delta,\mathrm{t}),-h'(\mathrm{t})\psi_2(\delta,\mathrm{t}),-h'(\mathrm{t})\psi_3(\delta,\mathrm{t})\Big)\\
&=\Big(\mathrm{h}(\mathrm{t})\frac{\partial\psi_1(\delta,\mathrm{t})}{\partial\mathrm{t}}+h'(\mathrm{t})\psi_1(\delta,\mathrm{t}),\mathrm{h}(\mathrm{t})\frac{\psi_2(\delta,\mathrm{t})}{\partial\mathrm{t}}\\
&+h'(\mathrm{t})\psi_2(\delta,\mathrm{t}),\mathrm{h}(\mathrm{t})\frac{\psi_3(\delta,\mathrm{t})}{\partial\mathrm{t}}+h'(\mathrm{t})\psi_3(\delta,\mathrm{t})\Big)\\
&=\frac{\partial k(\delta,\mathrm{t})}{\partial\mathrm{t}}.
\end{aligned}$$

Therefore $k(\delta,\mathrm{t})$ be [i-p]-differentiable w.r.t t and

$$\frac{\partial k(\delta,\mathrm{t})}{\partial\mathrm{t}}=\mathrm{h}(\mathrm{t})\frac{\partial\psi(\delta,\mathrm{t})}{\partial\mathrm{t}}\ominus((-1)h'(\mathrm{t}))\psi(\delta,\mathrm{t}). \tag{6}$$

Take the integral both side of equation (6). Using Proposition 2.5 and taking into account the fact that $k(\delta,\mathrm{t})$ is [i-p]-differentiable w.r.t t

$$\int_{\mathrm{a}}^{\mathrm{b}}\mathrm{h}(\mathrm{t})\frac{\partial\psi(\delta,\mathrm{t})}{\partial\mathrm{t}}\mathrm{dt}=k(\delta,\mathrm{b})\ominus k(\delta,a)\oplus\int_{\mathrm{a}}^{\mathrm{b}}-h'(\mathrm{t})\psi(\delta,\mathrm{t})\mathrm{dt}.$$

The other case can be proved in the same way. □

3 The Fuzzy Sumudu Transform

In this section, the fuzzy Sumudu transform is defined for a fuzzy function, and some properties for this fuzzy transform will be proved. Let $\mathfrak{A}$ be a set defined as

$$\mathfrak{A}=\Big\{\psi(\delta,\mathrm{t})|\ \exists M,\tau_1\ and/or\ \tau_2>0 \text{ such that}$$
$$D\Big(\psi(\delta,\mathrm{t}),0\Big)<Me^{\frac{|\mathrm{t}|}{\tau_j}}\text{ for t}\in(-1)^j\times[0,\infty)\Big\},$$

where D is Hausdorff distance, the constant M should be a finite, while τ_1 and τ_2 donot have to exist simultaneously and each can be infinite.

For every $\psi(\delta,\mathrm{t}) \in \mathfrak{A}$, the Sumudu transform with respect to t, $\mathbf{S}_{\mathrm{t}}[\psi(\delta,\mathrm{t})]$ is defined by

$$\mathbf{S}_{\mathrm{t}}[\psi(\delta,\mathrm{t})] = \begin{cases} \int_0^\infty \psi(\delta,\xi\mathrm{t})\exp[-\mathrm{t}]\mathrm{dt}, & 0 \leq \xi < \tau_2, \\ \int_0^\infty \psi(\delta,\xi\mathrm{t})\exp[-\mathrm{t}]\mathrm{dt}, & -\tau_1 < \xi \leq 0. \end{cases} \tag{7}$$

If a fuzzy function is defined for non-negative t, the fuzzy Sumudu transform of this function is just defined for non-negative ξ [6]. All equations in this paper will be defined for $\mathrm{t} \geq 0$, and henceforward ξ belongs to $[0,\tau_2)$.

Let $\omega = \xi\mathrm{t}$, then for every fuzzy function $\psi(\delta,\mathrm{t})$ defined for $\mathrm{t} \geq 0$, equation (7) can be written as

$$\mathbf{S}_{\mathrm{t}}[\psi(\delta,\mathrm{t})] = \frac{1}{\xi}\int_0^\infty \exp\left(\frac{-\omega}{\xi}\right)\psi(\delta,\omega)\mathrm{d}\omega = U(\delta,\xi), \qquad \xi \in [0,\tau_2). \tag{8}$$

Provided the integral exists for some ξ.

Definition 3.1. *Let* $\mathrm{h}(\mathrm{t})$ *be a real-valued piecewise continuous function, and* $\psi(\delta,\mathrm{t})$ *is a triangular fuzzy continuous function. The convolution of two functions* $\mathrm{h}(\mathrm{t})$ *and* $\psi(\delta,\mathrm{t})$ *for* $\mathrm{t} > 0$ *is given by*

$$(\psi * \mathrm{h})(\delta,\mathrm{t}) = \int_0^{\mathrm{t}} \psi(\delta,\zeta)\mathrm{h}(\mathrm{t}-\zeta)\mathrm{d}\zeta.$$

Lemma 3.2. *Consider* $\psi(\delta,\mathrm{t})$ *and* $\phi(\delta,\mathrm{t})$ *are fuzzy functions whose the fuzzy Sumudu transform exist. Let* a *and* b *are two real constants such that* $\mathrm{a},\mathrm{b} \geq 0$ *(or* $\mathrm{a},\mathrm{b} \leq 0$*) . Hence*

i). $\mathrm{a}\ \mathbf{S}_{\mathrm{t}}[\psi(\delta,\mathrm{t})] \ominus_{gH} \mathrm{b}\ \mathbf{S}_{\mathrm{t}}[\phi(\delta,\mathrm{t})] = \mathbf{S}_{\mathrm{t}}[\mathrm{a}\ \psi(\delta,\mathrm{t}) \ominus_{gH} \mathrm{b}\ \phi(\delta,\mathrm{t})]$.

ii). $\mathrm{a}\ \mathbf{S}_{\mathrm{t}}[\psi(\delta,\mathrm{t})] \oplus \mathrm{b}\ \mathbf{S}_{\mathrm{t}}[\phi(\delta,\mathrm{t})] = \mathbf{S}_{\mathrm{t}}[\mathrm{a}\ \psi(\delta,\mathrm{t}) \oplus \mathrm{b}\ \phi(\delta,\mathrm{t})]$.

Proof. Using equation (8) and Lemma 2.4 , we conclude that

$$
\begin{aligned}
&\mathrm{a}\ \mathbf{S}_{\mathrm{t}}[\mathrm{a}\ \psi(\delta,\mathrm{t})] \ominus_{gH} \mathrm{b}\ \mathbf{S}_{\mathrm{t}}[\phi(\delta,\mathrm{t})] \\
&= \frac{1}{\xi}\int_0^\infty \mathrm{a}\ \exp\left(\frac{-\mathrm{t}}{\xi}\right)\psi(\delta,\mathrm{t})\mathrm{dt} \ominus_{gH} \frac{1}{\xi}\int_0^\infty \mathrm{b}\ \exp\left(\frac{-\mathrm{t}}{\xi}\right)\phi(\delta,\mathrm{t})\mathrm{dt} \\
&= \frac{1}{\xi}\int_0^\infty \Big(\mathrm{a}\ \exp\left(\frac{-\mathrm{t}}{\xi}\right)\psi(\delta,\mathrm{t}) \ominus_{gH} \mathrm{b}\ \exp\left(\frac{-\mathrm{t}}{\xi}\right)\phi(\delta,\mathrm{t})\Big)\mathrm{dt} \\
&= \frac{1}{\xi}\int_0^\infty \exp\left(\frac{-\mathrm{t}}{\xi}\right)\Big(\mathrm{a}\ \psi(\delta,\mathrm{t}) \ominus_{gH} \mathrm{b}\ \phi(\delta,\mathrm{t})\Big)\mathrm{dt} \\
&= \mathbf{S}_{\mathrm{t}}[\mathrm{a}\ \psi(\delta,\mathrm{t}) \ominus_{gH} \mathrm{b}\ \phi(\delta,\mathrm{t})].
\end{aligned}
$$

The proof for part (ii) will be obtained similarly. □

Theorem 3.3. *Let $\psi(\delta,\mathrm{t})$ be [gH-p]-differentiable in $\mathbb{J}$ with respect to t provided that the type of [gH-p]-differentiability does not change in $\mathbb{J}$.*

i). *If $\psi(\delta,\mathrm{t})$ is [i-p]-differentiable with respect to t then*

$$\mathbf{S}_{\mathrm{t}}[\frac{\partial\psi(\delta,\mathrm{t})}{\partial\mathrm{t}}] = \frac{1}{\xi}\Big(\mathbf{S}_{\mathrm{t}}[\psi(\delta,\mathrm{t})] \ominus \psi(\delta,0)\Big).$$

ii). *If $\psi(\delta,\mathrm{t})$ is [ii-p]-differentiable with respect to t then*

$$\mathbf{S}_{\mathrm{t}}[\frac{\partial\psi(\delta,\mathrm{t})}{\partial\mathrm{t}}] = \frac{1}{\xi}\Big((-1)\psi(\delta,0) \ominus_{gH} (-1)\mathbf{S}_{\mathrm{t}}[\psi(\delta,\mathrm{t})]\Big).$$

Proof. Let $\psi(\delta,\mathrm{t})$ be [i-p]-differentiable with respect to t. According to the definition of the fuzzy Sumudu transform for a fuzzy function and Proposition 2.6, we have

$$
\begin{aligned}
\mathbf{S}_{\mathrm{t}}[\frac{\partial\psi(\delta,\mathrm{t})}{\partial\mathrm{t}}] &= \int_0^\infty \frac{1}{\xi}\exp\left(\frac{-\mathrm{t}}{\xi}\right)\frac{\partial\psi(\delta,\mathrm{t})}{\partial\mathrm{t}}\mathrm{dt} \\
&= \lim_{p\to\infty}\int_0^p \frac{1}{\xi}\exp\left(\frac{-\mathrm{t}}{\xi}\right)\frac{\partial\psi(\delta,\mathrm{t})}{\partial\mathrm{t}}\mathrm{dt} \\
&= \lim_{p\to\infty}\Bigg[\frac{1}{\xi}\exp\left(\frac{-p}{\xi}\right)\psi(\delta,p) \ominus \frac{1}{\xi}\psi(\delta,0) \\
&\qquad \oplus \frac{1}{\xi^2}\int_0^p \exp\left(\frac{-\mathrm{t}}{\xi}\right)\psi(\delta,\mathrm{t})\mathrm{dt}\Bigg] \\
&= \frac{1}{\xi}\Big[\mathbf{S}_{\mathrm{t}}[\psi(\delta,\mathrm{t})] \ominus \psi(\delta,0)\Big].
\end{aligned}
$$

Now, let $\psi(\delta, \mathrm{t})$ be [ii-p]-differentiable with respect to t. So, as in the procedure outlined above

$$\begin{aligned}
\mathbf{S}_{\mathrm{t}}[\frac{\partial \psi(\delta, \mathrm{t})}{\partial \mathrm{t}}] &= \int_0^\infty \frac{1}{\xi} \exp\Big(\frac{-\mathrm{t}}{\xi}\Big) \frac{\partial \psi(\delta, \mathrm{t})}{\partial \mathrm{t}} \mathrm{dt} \\
&= \lim_{p \to \infty} \Big[-\frac{1}{\xi}\psi(\delta, 0) \ominus \frac{-1}{\xi} \exp\Big(\frac{-p}{\xi}\Big)\psi(\delta, p) \\
&\qquad \ominus_{gH} \frac{-1}{\xi^2} \int_0^p \exp\Big(\frac{-\mathrm{t}}{\xi}\Big)\psi(\delta, \mathrm{t})\mathrm{dt}\Big] \\
&= \frac{1}{\xi}\Big(-\psi(\delta, 0) \ominus_{gH} (-1)\mathbf{S}_{\mathrm{t}}[\psi(\delta, \mathrm{t}))\Big].
\end{aligned}$$

So, the desired result was obtained. $\square$

Theorem 3.4. *(Convolution Theorem) Assume that $\psi(\delta, \mathrm{t})$ is a triangular fuzzy continuous function on $[0, \infty)$ and $\mathrm{h}(\mathrm{t})$ is a real-valued piecewise continuous function on $[0, \infty)$. Then*

$$\mathbf{S}_{\mathrm{t}}[\psi * h] = \xi\Big(\mathbf{S}_{\mathrm{t}}[\psi(\delta, \mathrm{t})] \odot \mathbf{S}_{\mathrm{t}}[\mathrm{h}(\mathrm{t})]\Big).$$

Proof. By using the definition of the fuzzy Sumudu transform, we have

$$\begin{aligned}
\mathbf{S}_{\mathrm{t}}[\psi(\delta, \mathrm{t})] \odot \mathbf{S}_{\mathrm{t}}[\mathrm{h}(\mathrm{t})] &= \Big(\frac{1}{\xi}\int_0^\infty \exp\Big(\frac{-\zeta}{\xi}\Big)\psi(\delta, \zeta) d\zeta\Big) \\
&\qquad \Big(\frac{1}{\xi}\int_0^\infty \exp\Big(\frac{-\sigma}{\xi}\Big) h(\sigma) d\sigma\Big) \\
&= \frac{1}{\xi^2}\int_0^\infty \Big(\int_0^\infty \exp\Big(\frac{-1}{\xi}(\zeta + \sigma)\Big)\psi(\delta, \zeta) d\zeta\Big) h(\sigma) d\sigma.
\end{aligned}$$

Let σ is fix in the interior integral and substitute $t = \zeta + \sigma$ and $d\zeta = \mathrm{dt}$, then

$$\begin{aligned}
\mathbf{S}_{\mathrm{t}}[\psi(\delta, \mathrm{t})] \odot \mathbf{S}_{\mathrm{t}}[\mathrm{h}(\mathrm{t})] &= \frac{1}{\xi^2}\int_0^\infty \Big[\int_\sigma^\infty \exp\Big(\frac{-\mathrm{t}}{\xi}\Big)\psi(\delta, \mathrm{t} - \sigma)\mathrm{dt}\Big] h(\sigma) d\sigma \\
&= \frac{1}{\xi^2}\int_0^\infty \Big[\int_\sigma^\infty \exp\Big(\frac{-\mathrm{t}}{\xi}\Big)\psi(\delta, \mathrm{t} - \sigma) h(\sigma)\mathrm{dt}\Big] d\sigma.
\end{aligned}$$

Using this fact that $\psi(\delta,\mathrm{t}) = \Big(\psi_1(\delta,\mathrm{t}), \psi_2(\delta,\mathrm{t}), \psi_3(\delta,\mathrm{t})\Big)$ and equation (2) yield to

$$\begin{aligned}
&\mathbf{S}_{\mathrm{t}}[\psi(\delta,\mathrm{t})] \odot \mathbf{S}_{\mathrm{t}}[\mathrm{h(t)}] \\
&= \frac{1}{\xi^2}\Bigg(\int_0^\infty \left[\int_\sigma^\infty \exp\left(\frac{-\mathrm{t}}{\xi}\right) \psi_1(\delta,\mathrm{t}-\sigma) h(\sigma) \mathrm{dt} \right] d\sigma, \\
&\int_0^\infty \left[\int_\sigma^\infty \exp\left(\frac{-\mathrm{t}}{\xi}\right) \psi_2(\delta,\mathrm{t}-\sigma) h(\sigma) \mathrm{dt} \right] d\sigma, \\
&\qquad \int_0^\infty \left[\int_\sigma^\infty \exp\left(\frac{-\mathrm{t}}{\xi}\right) \psi_3(\delta,\mathrm{t}-\sigma) h(\sigma) \mathrm{dt} \right] d\sigma \Bigg).
\end{aligned}$$

Now, using Theorem 3.2 in [8], we can reverse the order of integration

$$\begin{aligned}
&\mathbf{S}_{\mathrm{t}}[\psi(\delta,\mathrm{t})] \odot \mathbf{S}_{\mathrm{t}}[\mathrm{h(t)}] = \\
&= \frac{1}{\xi^2}\Bigg(\int_0^\infty \left[\int_0^{\mathrm{t}} \exp\left(\frac{-\mathrm{t}}{\xi}\right) \psi_1(\delta,\mathrm{t}-\sigma) h(\sigma) d\sigma \right] \mathrm{dt}, \\
&\int_0^\infty \left[\int_0^{\mathrm{t}} \exp\left(\frac{-\mathrm{t}}{\xi}\right) \psi_2(\delta,\mathrm{t}-\sigma) h(\sigma) d\sigma \right] \mathrm{dt}, \\
&\qquad \int_0^\infty \left[\int_0^{\mathrm{t}} \exp\left(\frac{-\mathrm{t}}{\xi}\right) \psi_3(\delta,\mathrm{t}-\sigma) h(\sigma) d\sigma \right] \mathrm{dt} \Bigg) \\
&= \frac{1}{\xi}\Bigg(\frac{1}{\xi} \int_0^\infty \exp\left(\frac{-\mathrm{t}}{\xi}\right) \left[\int_0^{\mathrm{t}} \psi_1(\delta,\mathrm{t}-\sigma) h(\sigma) d\sigma \right] \mathrm{dt}, \\
&\frac{1}{\xi} \int_0^\infty \exp\left(\frac{-\mathrm{t}}{\xi}\right) \left[\int_0^{\mathrm{t}} \psi_2(\delta,\mathrm{t}-\sigma) h(\sigma) d\sigma \right] \mathrm{dt}, \\
&\qquad \frac{1}{\xi} \int_0^\infty \exp\left(\frac{-\mathrm{t}}{\xi}\right) \left[\int_0^{\mathrm{t}} \psi_3(\delta,\mathrm{t}-\sigma) h(\sigma) d\sigma \right] \mathrm{dt} \Bigg).
\end{aligned}$$

Then we obtain

$$\begin{aligned}
\mathbf{S}_{\mathrm{t}}[\psi(\delta,\mathrm{t})] \odot \mathbf{S}_{\mathrm{t}}[\mathrm{h(t)}] &= \frac{1}{\xi}\Bigg(\frac{1}{\xi} \int_0^\infty \exp\left(\frac{-\mathrm{t}}{\xi}\right) \left[\int_0^{\mathrm{t}} \psi(\delta,\mathrm{t}-\sigma) h(\sigma) d\sigma \right] \mathrm{dt} \Bigg) \\
&= \frac{1}{\xi} \mathbf{S}_{\mathrm{t}}[\psi * h].
\end{aligned}$$

□

Theorem 3.5. *If $0 < \alpha \leq 1$ and $\psi(\delta, \mathrm{t})$, $\frac{\partial \psi(\delta,\mathrm{t})}{\partial \mathrm{t}}$ are fuzzy continuous on $[0, \infty)$. Moreover ${}^{C}_{gH}\mathfrak{D}^{\alpha}_{t}\psi(\delta, \mathrm{t})$ is fuzzy continuous on $[0, \infty)$. Then*

i). *If $\psi(\delta, \mathrm{t})$ is (1)-Caputo gH-differentiable, then*

$$\mathbf{S}_{\mathrm{t}}[{}^{C}_{gH}\mathfrak{D}^{\alpha}_{1}\psi(\delta, \mathrm{t})] = \xi^{-\alpha}\mathbf{S}_{\mathrm{t}}[\psi(\delta, \mathrm{t})] \ominus \xi^{-\alpha}\psi(\delta, 0).$$

ii). *If $\psi(\delta, \mathrm{t})$ is (2)-Caputo gH-differentiable, then*

$$\mathbf{S}_{\mathrm{t}}[\, {}^{C}_{gH}\mathfrak{D}^{\alpha}_{2}\psi(\delta, \mathrm{t})] = (-1)\xi^{-\alpha}\psi(\delta, 0) \ominus (-1)\xi^{-\alpha}\mathbf{S}_{\mathrm{t}}[\psi(\delta, \mathrm{t})].$$

Proof. By Definition 2.3 we have

$$\begin{aligned} {}^{C}_{gH}\mathfrak{D}^{\alpha}_{1}\psi(\delta, \mathrm{t}) &= \frac{1}{\Gamma(1-\alpha)} \int_0^{\mathrm{t}} (\mathrm{t}-\tau)^{-\alpha} \frac{\partial \phi(\delta, \tau)}{\partial \tau} d\tau \\ &= \frac{1}{\Gamma(1-\alpha)} \Big(t^{-\alpha} * \frac{\partial \phi(\delta, \mathrm{t})}{\partial \mathrm{t}} \Big). \end{aligned}$$

According to the assumptions of the theorem in case (i), $\psi(\delta, \mathrm{t})$ is (1)-Caputo gH-differentiable. Applying the fuzzy Sumudu transform for both sides of the above equation and by Theorem 3.4 conclude that

$$\mathbf{S}_{\mathrm{t}}[\, {}^{C}_{gH}\mathfrak{D}^{\alpha}_{1}\psi(\delta, \mathrm{t})] = \frac{1}{\Gamma(1-\alpha)} \Big(\mathbf{S}_{\mathrm{t}}[t^{-\alpha}] \odot \mathbf{S}_{\mathrm{t}}[\frac{\partial \phi(\delta, \mathrm{t})}{\partial \mathrm{t}}] \Big).$$

Given that $\mathbf{S}_{\mathrm{t}}[t^{-\alpha}] = \xi^{-\alpha}\Gamma(1-\alpha)$ [6] and Theorem 3.3 can be conclude that

$$\mathbf{S}_{\mathrm{t}}[\, {}^{C}_{gH}\mathfrak{D}^{\alpha}_{1}\psi(\delta, \mathrm{t})] \;=\; \xi^{-\alpha}\mathbf{S}_{\mathrm{t}}[\psi(\delta, \mathrm{t})] \ominus \xi^{-\alpha}\psi(\delta, 0).$$

Furthermore, a similar approach can be applied to (2)-Caputo gH-differentiable function. □

Example 3.6. Let $\psi(\delta, \mathrm{t}) = kf(\delta, \mathrm{t})$ such that k is a fuzzy triangular number and $f(\delta, \mathrm{t})$ is a real-valued continuous function, then

$$\begin{aligned} \mathbf{S}_{\mathrm{t}}\Big[\psi(\delta, \mathrm{t})\Big] &= \frac{1}{\xi} \int_0^{\infty} \exp\Big(\frac{-\mathrm{t}}{\xi}\Big) kf(\delta, \mathrm{t}) \mathrm{dt} \\ &= k\frac{1}{\xi} \int_0^{\infty} \exp\Big(\frac{-\mathrm{t}}{\xi}\Big) f(\delta, \mathrm{t}) \mathrm{dt} \\ &= k\mathbf{S}_{\mathrm{t}}\Big[f(\delta, \mathrm{t})\Big] \end{aligned}$$

Table A.1 in the article [6] can be used to get $\mathbf{S}_{\mathrm{t}}\Big[f(\delta, \mathrm{t})\Big]$.

4 The Fuzzy Sumudu Transform Iterative Method (FSTIM)

In this section, we present the derivation of a fuzzy solution for the fuzzy time-fractional partial differential with fuzzy initial condition as

$$\begin{cases} {}^{C}_{gH}\mathfrak{D}^{\alpha}_{t}\psi(\delta,\mathrm{t}) = L\psi(\delta,\mathrm{t}) \oplus g(\delta,\mathrm{t}), \quad (t>0, \delta\in\mathbb{R}), \quad 0<\alpha\leq 1 \\ \psi(\delta,0) = \phi(\delta), \end{cases} \tag{9}$$

where ${}^{C}_{gH}\mathfrak{D}^{\alpha}_{t}$ be the fuzzy Caputo fractional partial derivative with respect to t, L is a linear operator, $g(\delta,\mathrm{t}) = \Big(g_1(\delta,\mathrm{t}), g_2(\delta,\mathrm{t}), g_3(\delta,\mathrm{t})\Big)$ is a given fuzzy continuous function and $\phi(\delta) = \Big(\phi_1(\delta), \phi_2(\delta), \phi_3(\delta)\Big)$ is the fuzzy initial condition.

Let us apply the fuzzy Sumudu transform on both sides of equation (9)

$$\mathbf{S}_{\mathrm{t}}\Big[{}^{C}_{gH}\mathfrak{D}^{\alpha}_{t}\psi(\delta,\mathrm{t})\Big] = \mathbf{S}_{\mathrm{t}}\Big[L\psi(\delta,\mathrm{t})\Big] \oplus \mathbf{S}_{\mathrm{t}}\Big[g(\delta,\mathrm{t})\Big]. \tag{10}$$

- Let $\psi(\delta,\mathrm{t})$ is (1)-Caputo gH-differentiable. Theorem 3.5 concludes that

  $$\mathbf{S}_{\mathrm{t}}[\psi(\delta,\mathrm{t})] = \psi(\delta,0) \oplus \xi^{\alpha}\mathbf{S}_{\mathrm{t}}[L\psi(\delta,\mathrm{t})] \oplus \xi^{\alpha}\mathbf{S}_{\mathrm{t}}[g(\delta,\mathrm{t})]. \tag{11}$$

  We apply the inverse Sumudu transform on both sides of the equation (11)

  $$\psi(\delta,\mathrm{t}) = \mathbf{S}^{-1}_{\mathrm{t}}\Big[\psi(\delta,0)\Big] \oplus \mathbf{S}^{-1}_{\mathrm{t}}\Big[\xi^{\alpha}\mathbf{S}_{\mathrm{t}}\Big[L\psi(\delta,\mathrm{t})\Big]\Big] \oplus \mathbf{S}^{-1}_{\mathrm{t}}\Big[\xi^{\alpha}\mathbf{S}_{\mathrm{t}}\Big[g(\delta,\mathrm{t})\Big]\Big].$$

  Assume that

  $$\begin{cases} f(\delta,\mathrm{t}) = \phi(\delta) \oplus \mathbf{S}^{-1}_{\mathrm{t}}\Big[\xi^{\alpha}\mathbf{S}_{\mathrm{t}}\Big[g(\delta,\mathrm{t})\Big]\Big], \\ K(\psi(\delta,\mathrm{t})) = \mathbf{S}^{-1}_{\mathrm{t}}\Big[\xi^{\alpha}\mathbf{S}_{\mathrm{t}}\Big[L\psi(\delta,\mathrm{t})\Big]\Big]. \end{cases}$$

So, equation (11) can be rewritten as follows,

$$\psi(\delta,\mathrm{t}) = f(\delta,\mathrm{t}) \oplus K(\psi(\delta,\mathrm{t})), \tag{12}$$

where $f(\delta,\mathrm{t})$ is a known fuzzy function, and K is a linear operator of ψ. Now, suppose that the solution of equation (9) is as follows

$$\psi(\delta,\mathrm{t}) = \sum_{i=0}^{\infty} \psi_i(\delta,\mathrm{t}).$$

K is a linear operator, then

$$K\Big(\sum_{i=0}^{\infty} \psi_i(\delta,\mathrm{t})\Big) = \sum_{i=0}^{\infty} K(\psi_i(\delta,\mathrm{t})).$$

Consequently, equation (12) can be rewritten as

$$\sum_{i=0}^{\infty} \psi_i(\delta,\mathrm{t}) = \psi_0(\delta,\mathrm{t}) \oplus \sum_{i=0}^{\infty} K(\psi_i(\delta,\mathrm{t})),$$

and the following recursive equation is obtained

$$\begin{cases} \psi_0(\delta,\mathrm{t}) = f(\delta,\mathrm{t}) \\ \\ \psi_{m+1}(\delta,\mathrm{t}) = K(\psi_m(\delta,\mathrm{t})), \qquad m = 0, 1, ... \end{cases} \tag{13}$$

Finally, the (1)-Caputo gH-differentiable solution of equation (9) is given as

$$\psi(\delta,\mathrm{t}) = f(\delta,\mathrm{t}) \oplus \sum_{m=1}^{\infty} \psi_m(\delta,\mathrm{t}). \tag{14}$$

- Let $\psi(\delta,\mathrm{t})$ be (2)-Caputo gH-differentiable. By the process discussed in detail in the previous part, we obtain the following recursive equation

$$\begin{cases} \psi_0(\delta,\mathrm{t}) = \phi(\delta) \ominus (-1)\mathbf{S}_{\mathrm{t}}^{-1}\Big[\xi^{\alpha}\mathbf{S}_{\mathrm{t}}\Big[g(\delta,\mathrm{t})\Big]\Big], \\ \\ \psi_{m+1}(\delta,\mathrm{t}) = \ominus(-1)K(\psi_m(\delta,\mathrm{t})), \qquad m = 0, 1, ... \end{cases} \tag{15}$$

Finally, the (2)-Caputo gH-differentiable solution of equation (9) is given as

$$\psi(\delta,\mathrm{t}) = \psi_0(\delta,\mathrm{t}) \oplus \sum_{m=1}^{\infty} \psi_m(\delta,\mathrm{t}). \tag{16}$$

5 Examples

In this section, the proposed method is utilized to study some examples of the fuzzy time-fractional Cauchy equations. The computations associated with the examples are performed using Mathematica software.

Example 5.1. Consider the following time-fractional Cauchy equation

$$\begin{cases} {}^{C}_{gH}\mathfrak{D}^{\alpha}_{t}\psi(\delta,\mathrm{t}) = \frac{\partial^2\psi(\delta,\mathrm{t})}{\partial\delta^2} \oplus \frac{\partial}{\partial\delta}(\delta\psi(\delta,\mathrm{t})) \\ \psi(\delta,0) = \Big(0.1, 4.5, 7.8\Big) \end{cases} \tag{17}$$

Let $\psi(\delta,\mathrm{t})$ be (1)-Caputo gH-differentiable. Applying the fuzzy Sumudu transform with respect to t on both sides of of equation (17)

$$\begin{aligned}
\psi_0(\delta,\mathrm{t}) &= \Big(0.1, 4.5, 7.8\Big), \\
\psi_1(\delta,\mathrm{t}) &= \mathbf{S}_{\mathrm{t}}^{-1}\Big[\xi^{\alpha}\mathbf{S}_{\mathrm{t}}\Big[\frac{\partial^2\psi_0(\delta,\mathrm{t})}{\partial\delta^2}\Big]\Big] \oplus \mathbf{S}_{\mathrm{t}}^{-1}\Big[\xi^{\alpha}\mathbf{S}_{\mathrm{t}}\Big[\frac{\partial}{\partial\delta}(\delta\Big(0.1, 4.5, 7.8\Big))\Big]\Big] \\
&= \mathbf{S}_{\mathrm{t}}^{-1}\Big[\xi^{\alpha}\Big(0.1, 4.5, 7.8\Big)\Big] \\
&= \frac{\Big(0.1, 4.5, 7.8\Big)t^{\alpha}}{\Gamma(\alpha+1)}, \\
\psi_2(\delta,\mathrm{t}) &= \mathbf{S}_{\mathrm{t}}^{-1}\Big[\xi^{\alpha}\mathbf{S}_{\mathrm{t}}\Big[\frac{\partial^2}{\partial\delta^2}\Big(\frac{\Big(0.1, 4.5, 7.8\Big)t^{\alpha}}{\Gamma(\alpha+1)}\Big)\Big]\Big] \\
&\oplus \mathbf{S}_{\mathrm{t}}^{-1}\Big[\xi^{\alpha}\mathbf{S}_{\mathrm{t}}\Big[\frac{\partial}{\partial\delta}(\delta\frac{\Big(0.1, 4.5, 7.8\Big)t^{\alpha}}{\Gamma(\alpha+1)})\Big]\Big] \\
&= \frac{\Big(0.1, 4.5, 7.8\Big)t^{2\alpha}}{\Gamma(2\alpha+1)}
\end{aligned}$$

$$
\begin{aligned}
\psi_3(\delta,\mathrm{t}) &= \mathbf{S}_{\mathrm{t}}^{-1}\Big[\xi^{\alpha}\mathbf{S}_{\mathrm{t}}\Big[\frac{\partial}{\partial\delta}\big(\delta\frac{\Big(0.1,4.5,7.8\Big)t^{2\alpha}}{\Gamma(2\alpha+1)}\big)\Big]\Big]\\
&= \frac{\Big(0.1,4.5,7.8\Big)t^{3\alpha}}{\Gamma(3\alpha+1)}\\
&\vdots\\
\psi_m(\delta,\mathrm{t}) &= \frac{\Big(0.1,4.5,7.8\Big)t^{m\alpha}}{\Gamma(m\alpha+1)}
\end{aligned}
$$

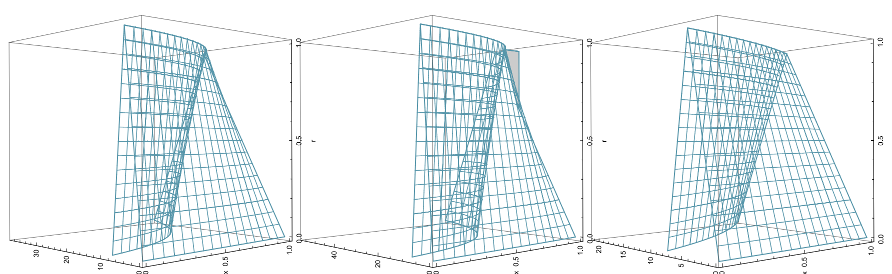
Therefore, the (1)-Caputo gH-differentiable solution of the problem (17) is

$$
\begin{aligned}
\psi(\delta,\mathrm{t}) &= \Big(0.1,4.5,7.8\Big)\oplus\frac{\Big(0.1,4.5,7.8\Big)t^{\alpha}}{\Gamma(\alpha+1)}\oplus\cdots\oplus\frac{\Big(0.1,4.5,7.8\Big)t^{m\alpha}}{\Gamma(m\alpha+1)}\\
&= \Big(0.1,4.5,7.8\Big)E_{\alpha}(t^{\alpha}) \qquad (18)
\end{aligned}
$$

where is $E_{\alpha}(z)$ the Mittag-Leffler function [7]

$$E_{\alpha}(z)=\sum_{i=0}^{\infty}\frac{z^{j}}{\Gamma(\alpha j+1)}.$$

The $r-$cut of this solution, $\psi(\delta,t;r)=[0.1+4.4r,7.8-3.3r]E_{\alpha}(t^{\alpha})$, for different values of α and $0\le r\le 1$, are showed in Figures 1.



(a). $\alpha=\frac{1}{2}$. (b). $\alpha=\frac{1}{3}$. (c). $\alpha=1$.

Figure 1: Plots of $\psi(\delta,t;r)=[0.1+4.4r,7.8-3.3r]E_{\alpha}(t^{\alpha})$ for different values of α and $r\in[0,1]$.

Example 5.2. Consider the following time-fractional Cauchy reaction-diffusion equation

$$\begin{cases} {}^{C}_{gH}\mathfrak{D}_t^{\frac{1}{2}}\psi(\delta,\mathrm{t}) = \frac{\partial^2\psi(\delta,\mathrm{t})}{\partial\delta^2} \ominus_{gH} \Big(\frac{0.4}{\sqrt{\pi}},\ \frac{10.6}{\sqrt{\pi}},\ \frac{19}{\sqrt{\pi}}\Big)\delta\mathrm{DawsonF}(\sqrt{t}) \\ \psi(\delta,0) = \Big(0.4\delta,\ 10.6\delta,\ 19\delta\Big) \end{cases} \tag{19}$$

where Dawson function is defined as $\mathrm{DawsonF}(z) = \exp(-z^2)\int_0^z \exp(t^2)\mathrm{d}t$.

Applying the fuzzy Sumudu transform with respect to t on both sides of equation (19). We want to find a (2)-Caputo differentiable solution, then, we will use equations (15) and (16)

$$\begin{aligned} \mathbf{S}_{\mathrm{t}}[\psi(\delta,\mathrm{t})] &= \psi(\delta,0) \\ &\ominus (-1)\xi^{\frac{1}{2}}\Bigg(\mathbf{S}_{\mathrm{t}}\Big[\frac{\partial^2\psi(\delta,\mathrm{t})}{\partial\delta^2}\Big] \\ &\ominus_{gH} \mathbf{S}_{\mathrm{t}}\Big[\Big(\frac{0.4}{\sqrt{\pi}},\ \frac{10.6}{\sqrt{\pi}},\ \frac{19}{\sqrt{\pi}}\Big)\delta\mathrm{DawsonF}(\sqrt{t})\Big]\Bigg) \\ &= \psi(\delta,0) \oplus (-1)\xi^{\frac{1}{2}}\mathbf{S}_{\mathrm{t}}\Big[\Big(\frac{0.4}{\sqrt{\pi}},\ \frac{10.6}{\sqrt{\pi}},\ \frac{19}{\sqrt{\pi}}\Big)\delta\mathrm{DawsonF}(\sqrt{t})\Big]\Bigg) \\ &\ominus (-1)\xi^{\frac{1}{2}}\mathbf{S}_{\mathrm{t}}\Big[\frac{\partial^2\psi(\delta,\mathrm{t})}{\partial\delta^2}\Big] \end{aligned}$$

we have

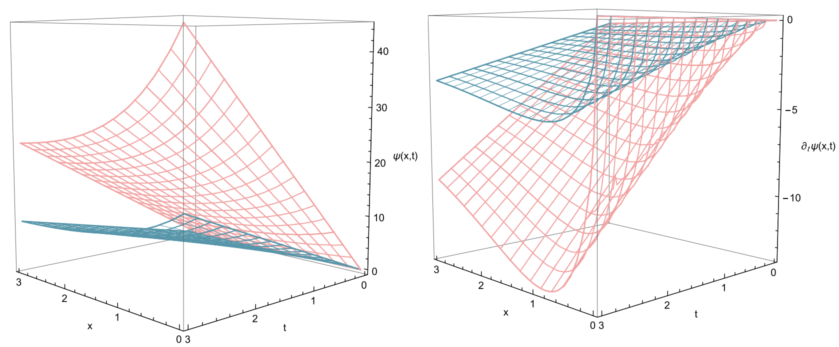
$$\mathbf{S}_{\mathrm{t}}\Big[\Big(\frac{0.4}{\sqrt{\pi}},\ \frac{10.6}{\sqrt{\pi}},\ \frac{19}{\sqrt{\pi}}\Big)\delta\mathrm{DawsonF}(\sqrt{t})\Big] = \Big(\frac{0.2\delta\sqrt{\xi}}{1+\xi}, \frac{5.3\delta\sqrt{\xi}}{1+\xi}, \frac{9.5\delta\sqrt{\xi}}{1+\xi}\Big)$$

Under process discussed in detail in Section 4, we obtain the following recursive equation

$$\begin{aligned} \psi_0(\delta,\mathrm{t}) &= \Big(0.4\delta,\ 10.6\delta,\ 19\delta\Big) \oplus (-1)\mathbf{S}_{\mathrm{t}}^{-1}\Bigg[\Bigg(\Big(\frac{0.2\delta\xi}{1+\xi}, \frac{5.3\delta\xi}{1+\xi}, \frac{9.5\delta\xi}{1+\xi}\Big)\Bigg]\Bigg] \\ &= \Big(0.4\delta,\ 10.6\delta,\ 19\delta\Big) \\ &\oplus \Big(0.2\delta(\exp(-\mathrm{t})-1),\ 5.3\delta(\exp(-\mathrm{t})-1),\ 9.5\delta(\exp(-\mathrm{t})-1)\Big), \\ \psi_{m+1}(\delta,\mathrm{t}) &= \ominus(-1)\mathbf{S}_{\mathrm{t}}^{-1}\Big[\xi^{\alpha}\mathbf{S}_{\mathrm{t}}\Big[\frac{\partial^2\psi_m(\delta,\mathrm{t})}{\partial\delta^2}\Big]\Big], \qquad m = 0,1,... \end{aligned}$$

We have $\psi_{m+1}(\delta,\mathrm{t})=0$ and, by iteration, the following exact (2)-Caputo gH-differentiable solution is obtained

$$\psi(\delta,\mathrm{t})=\Big(0.2\delta(\exp(-\mathrm{t})+1),5.3\delta(\exp(-\mathrm{t})+1),9.5\delta(\exp(-\mathrm{t})+1)\Big).$$



(a). $\psi(\delta,\mathrm{t})$. (b). ${}^{C}_{gH}\mathfrak{D}^{\frac{1}{2}}_{t}\psi(\delta,\mathrm{t})$.

Figure 2: Plots of $\psi(\delta,\mathrm{t})$ and ${}^{C}_{gH}\mathfrak{D}^{\frac{1}{2}}_{t}\psi(\delta,\mathrm{t})$ for $r=\frac{1}{2}$.

To illustrate the behavior of the (2)-Caputo gH-differentiable solution , $\psi(\delta,t;r)=[0.2+5.1r,9.5-4.2r]\delta(\exp(-\mathrm{t})+1)$, $\psi(\delta,t;r)$ and ${}^{C}_{gH}\mathfrak{D}^{\frac{1}{2}}_{t}\psi(\delta,t;r)$ are presented in Figure 2 (a) and (b) for $r=\frac{1}{2}$, respectively. As can be see in Figure 2(b), the position of lower cut(blue) and upper cut(red) for ${}^{C}_{gH}\mathfrak{D}^{\frac{1}{2}}_{t}\psi(\delta,\mathrm{t})$ is changed. It shows that, $\psi(\delta,\mathrm{t})$ is (2)-Caputo gH-differentiable with respect to t.

Example 5.3. Consider the following time-fractional Cauchy reaction-diffusion equation

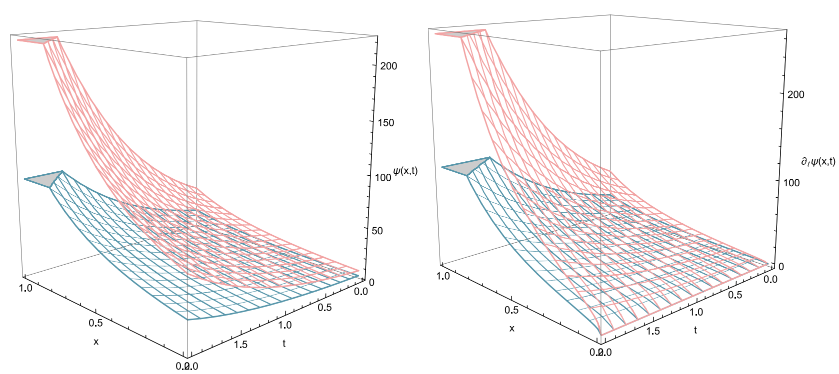
$$\begin{cases} {}^{C}_{gH}\mathfrak{D}^{\alpha}_{t}\psi(\delta,\mathrm{t})=\frac{\partial^2\psi(\delta,\mathrm{t})}{\partial\delta^2}\ominus_{gH}\frac{\partial}{\partial\delta}(\exp(-\delta)\psi(\delta,\mathrm{t})) \\ \psi(\delta,0)=\Big(\exp(\delta),6.5\exp(\delta),11\exp(\delta)\Big) \end{cases} \tag{20}$$

Using the general recurrence relation (13), the (1)-Caputo differentiable solution of equation (20) is obtained by

$$\begin{aligned}
\psi_0(\delta,\mathrm{t}) &= \Big(\exp(\delta), 6.5\exp(\delta), 11\exp(\delta)\Big),\\
\psi_1(\delta,\mathrm{t}) &= \frac{\Big(\exp(\delta), 6.5\exp(\delta), 11\exp(\delta)\Big)t^{\alpha}}{\Gamma(\alpha+1)},\\
\psi_2(\delta,\mathrm{t}) &= \frac{\Big(\exp(\delta), 6.5\exp(\delta), 11\exp(\delta)\Big)t^{2\alpha}}{\Gamma(2\alpha+1)}\\
&\ \vdots\\
\psi_m(\delta,\mathrm{t}) &= \frac{\Big(\exp(\delta), 6.5\exp(\delta), 11\exp(\delta)\Big)t^{m\alpha}}{\Gamma(m\alpha+1)}
\end{aligned}$$

and

$$\begin{aligned}
\psi(\delta,\mathrm{t}) &= \Big(\exp(\delta), 6.5\exp(\delta), 11\exp(\delta)\Big)\\
&\quad \Big(1+\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\cdots+\frac{t^{m\alpha}}{\Gamma(m\alpha+1)}\Big)\\
&= \Big(\exp(\delta), 6.5\exp(\delta), 11\exp(\delta)\Big)E_{\alpha}(t^{\alpha})
\end{aligned}$$



(a). $\psi(\delta,\mathrm{t})$. (b). ${}^{C}_{gH}\mathfrak{D}_t^{\frac{1}{2}}\psi(\delta,\mathrm{t})$.

Figure 3: Plots of $\psi(\delta,\mathrm{t})$ and ${}^{C}_{gH}\mathfrak{D}_t^{\frac{1}{2}}\psi(\delta,\mathrm{t})$ for $r=\frac{1}{2}$.

To illustrate the behavior of the (1)-Caputo gH-differentiable solution of this fuzzy Cauchy problem, $\psi(\delta, \mathrm{t})$ and ${}_{gH}^{C}\mathfrak{D}_t^{\frac{1}{2}}\psi(\delta, \mathrm{t})$ are presented in Figure 3 (a) and (b) for $r = \frac{1}{2}$, respectively. As can be seen in Figure 3 (b), the position of lower cut(blue) and upper cut(red) for ${}_{gH}^{C}\mathfrak{D}_t^{\frac{1}{2}}\psi(\delta, \mathrm{t})$ does not change. It show that, $\psi(\delta, \mathrm{t})$ is (1)-Caputo gH-differentiable with respect to t.

6 Conclusions

In this article, we have considered the time-fractional Cauchy Reaction-Diffusion equation in the fuzzy concept. We have studied this equation under the generalized Hukuhara Caputo partial differentiability. To find the analytical solution of the proposed equation, we have used the fuzzy Sumudu transform method. The final results, show that the fuzzy Sumudu transform method is very efficient and more realistic to solve the time-fractional Cauchy Reaction-Diffusion equation.

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