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Intuitionistic Fuzzy Filters in Sheffer Stroke Hilbert Algebras

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Abstract. Using the Atanassov's intuitionistic fuzzy set, the concept of intuitionistic fuzzy deductive system and intuitionistic fuzzy filter in Sheffer stroke Hilbert algebras is introduced, several properties are investigated. The conditions under which an intuitionistic fuzzy set can be an intuitionistic fuzzy filter are explored, and characterizations of an intuitionistic fuzzy filter are considered. The process of making an intuitionistic fuzzy filter through the collection of filters is displayed, and the union and intersection of intuitionistic fuzzy filters are discussed. Several properties are investigated in relation to the homomorphsm of Sheffer stroke Hilbert algebras, and finally the relationship between an intuitionistic fuzzy filter and an intuitionistic fuzzy deductive system is established.

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1 Introduction

Following the introduction of Hilbert algebras by L. Henkin in early 50-ties and A. Diego [11], the algebra and related concepts were developed by D. Busneag (see [5], [6], and [7]). For the general development of Hilbert algebras, the notion of deductive systems plays an important role. For example, it is known that the set of all deductive systems of a Hilbert algebra forms an algebraic lattice which is distributive (see [8]). Several authors discussed deductive systems of Hilbert algebras including their properties (see [5, 8, 9, 13, 14]).

The concept of fuzzy set was introduced by Zadeh [25] in 1965 and subsequently an extension of fuzzy set was attempted. The first extension was the L-fuzzy set which was introduced by Goguen [12] in 1967. The second extension was from Zadeh [26], who introduced the idea of interval-valued fuzzy set. The rough set was the third extension, and it was defined by Pawlak in 1981 (see [21, 22]). The fourth extension was intuitionistic fuzzy set (IFS), introduced by Atanassov in 1983 (see [1, 2, 3]).

In the late 19th century and early 20th century, Charles S. Peirce and H. M. Sheffer independently discovered that a single binary logical connective suffices to define all logical connectives, the Sheffer stroke (denoted by $|$ or $\uparrow$) and the Peirce arrow (denoted by $\downarrow$). The concept of Sheffer operation (the so-called Sheffer stroke in [4]) was first introduced by Sheffer [23] in 1913. The Sheffer stroke is defined by the truth table:

P	Q	$P \mid Q$
F	F	T
F	T	T
T	F	T
T	T	F

The Sheffer stroke has been applied to several algebraic structures, for example, Boolean algebra, MV-algebra, basic algebra, BCK-algebra, ortholattices, and Hilbert algebra, etc. (see [10, 15, 18, 19, 20]). It is also being dealt with in the fuzzy environment. In 2021, Oner et al. [17] applied the Sheffer stroke to Hilbert algebras. They introduced Sheffer stroke Hilbert algebra and investigated several properties. In [16], Oner et al. introduced the notion of deductive system and filter of Sheffer stroke Hilbert algebras, and dealt with their fuzzification.

In this paper, we handle the concepts of intuitionistic fuzzy filters within the framework of Sheffer stroke Hilbert algebras. We explore essential properties and their interconnections among these intuitionistic fuzzy filters. The primary contributions of this study are centered on introducing the notion of intuitionistic fuzzy filters through the exclusive use of the Sheffer stroke operation within Hilbert Algebras. In Section 2, we provide a review of key concepts, basic definitions, lemmas, and pertinent results that will be applied throughout this paper. In Section 3, we apply the Atanassov's intuitionistic fuzzy set to the notion of deductive system and filter of Sheffer stroke Hilbert algebras. We introduce the concept of deductive systsm and filter in Sheffer stroke Hilbert algebras, and investigate several properties. We explore the conditions under which an set can be an filter. We discuss characterizations of an filter. We show the process of making an filter through the collection of filters. We discuss the union and intersection of filters. We investigate several properties in relation to the homomorphsm of Sheffer stroke Hilbert algebras. We finally discuss the relationship between an filter and an deductive system.

2 Preliminaries

Definition 2.1 ([23])**.** Let $\mathcal{A} := (A, |)$ be a groupoid. Then the operation $|$ is said to be *Sheffer stroke* or *Sheffer operation* if it satisfies:

(s1) $(\forall a, b \in A)\ (a \mid b = b \mid a)$,

(s2) $(\forall a, b \in A)\ ((a \mid a) \mid (a \mid b) = a)$,

(s3) $(\forall a, b, c \in A)\ (a \mid ((b \mid c) \mid (b \mid c)) = ((a \mid b) \mid (a \mid b)) \mid c)$,

(s4) $(\forall a, b, c \in A)\ ((a \mid ((a \mid a) \mid (b \mid b))) \mid (a \mid ((a \mid a) \mid (b \mid b))) = a)$.

Definition 2.2 ([17])**.** A *Sheffer stroke Hilbert algebra* is a groupoid $\mathcal{H} := (H, |)$ with a Sheffer stroke that satisfies:

(sH1) $(a \mid ((b \mid (c \mid c)) \mid (b \mid (c \mid c)))) \mid (((a \mid (b \mid b)) \mid ((a \mid (c \mid c))(a \mid (c \mid c)))) \mid ((a \mid (b \mid b)) \mid ((a \mid (c \mid c)) \mid (a \mid (c \mid c))))) = a \mid (a \mid a)$,

(sH2) $a \mid (b \mid b) = b \mid (a \mid a) = a \mid (a \mid a) \ \Rightarrow\ a = b$

for all $a, b, c \in H$.

Let $\mathcal{H} := (H, |)$ be a Sheffer stroke Hilbert algebra. Then the order relation $\leq$ on H is defined as follows:

$$(\forall a, b \in H)(a \leq b \ \Leftrightarrow\ a \mid (b \mid b) = 1). \tag{1}$$

We observe that the relation $\leq$ is a partial order in a Sheffer stroke Hilbert algebra $\mathcal{H} := (H, |)$ (see [17]).

Proposition 2.3 ([17])**.** *Every Sheffer stroke Hilbert algebra $\mathcal{H} := (H, |)$ satisfies:*

$$(\forall a \in H)(a \mid (a \mid a) = 1), \tag{2}$$
$$(\forall a \in H)(a \mid (1 \mid 1) = 1), \tag{3}$$
$$(\forall a \in H)(1 \mid (a \mid a) = a), \tag{4}$$
$$(\forall a, b \in H)(a \leq b \mid (a \mid a)), \tag{5}$$
$$(\forall a, b \in H)((a \mid (b \mid b)) \mid (b \mid b) = (b \mid (a \mid a)) \mid (a \mid a)), \tag{6}$$
$$(\forall a, b \in H)\left(((a \mid b \mid b)) \mid (b \mid b)) \mid (b \mid b) = a \mid (b \mid b)\right), \tag{7}$$
$$(\forall a, b, c \in H)\left(a \mid ((b \mid (c \mid c)) \mid (b \mid (c \mid c))) = b \mid ((a \mid (c \mid c)) \mid (a \mid (c \mid c)))\right). \tag{8}$$

Definition 2.4 ([16])**.** Let $(H, |)$ be a Sheffer stroke Hilbert algebra. A subset F of H is called

- a *deductive system* of $(H, |)$ if it satisfies:

$$1 \in F, \tag{9}$$

$$(\forall a, b \in H)(a \in F,\ a \mid (b \mid b) \in F \ \Rightarrow\ b \in F). \tag{10}$$

- a *filter* of $(H, |)$ if it satisfies (9) and

$$(\forall a, b \in H)(b \in F \ \Rightarrow\ a \mid (b \mid b) \in F), \tag{11}$$

$$(\forall a, b, c \in H)(b, c \in F \ \Rightarrow\ (a \mid (b \mid c)) \mid (b \mid c) \in F). \tag{12}$$

Definition 2.5 ([16]). Let $(H, |)$ be a Sheffer stroke Hilbert algebra. A fuzzy set μ in H is called a *fuzzy filter* of $(H, |)$ if it satisfies:

$$(\forall a \in H)(\mu(1) \geq \mu(a)), \tag{13}$$

$$(\forall a, b \in H)(\mu(a \mid (b \mid b)) \geq \mu(b)), \tag{14}$$

$$(\forall a, b, c \in H)(\mu((a \mid (b \mid c)) \mid (b \mid c)) \geq \min\{\mu(b), \mu(c)\}). \tag{15}$$

Denote by $FS(H)$ the collection of all fuzzy sets in H. Define a relation "$\subseteq$" on $FS(H)$ by

$$(\forall \mu, \gamma \in FS(H))(\mu \subseteq \gamma \ \Leftrightarrow\ (\forall a \in H)(\mu(a) \leq \gamma(a))).$$

The *join* ($\vee$) and *meet* ($\wedge$) of μ and γ are defined by

$$(\mu \vee \gamma)(a) = \max\{\mu(a), \gamma(a)\},$$
$$(\mu \wedge \gamma)(a) = \min\{\mu(a), \gamma(a)\},$$

respectively, for all $a \in H$. The *complement* of μ, denoted by μ^c, is defined by

$$(\forall a \in H)(\mu^c(a) = 1 - \mu(a)).$$

As an important generalization of the notion of fuzzy sets in a non-empty set H, Atanassov [1, 2] introduced the concept of an intuitionistic fuzzy set (IFS for short) defined on H as an object having the form

$$A := \{\langle a, \mu_A(a), \gamma_A(a)\rangle \mid a \in H\},$$

where the functions $\mu_A : H \to [0, 1]$ and $\gamma_A : H \to [0, 1]$ denote the degree of membership (namely $\mu_A(a)$) and the degree of nonmembership (namely $\gamma_A(a)$) of each element $a \in H$ to A respectively, and $0 \leq \mu_A(a) + \gamma_A(a) \leq 1$ for all $a \in H$. For the sake of simplicity, we shall use the symbol $A := (H; \mu_A, \gamma_A)$ for the IFS $A := \{\langle a, \mu_A(a), \gamma_A(a)\rangle \mid a \in H\}$. Denote by $IFS(H)$ the collection of all intuitionistic fuzzy sets in H. Define a relation "$\ll$" on $IFS(H)$ by

$$A \ll B \ \Leftrightarrow\ \mu_A \subseteq \mu_B \ \&\ \gamma_B \subseteq \gamma_A$$

for all $A := (H; \mu_A, \gamma_A)$ and $B := (H; \mu_B, \gamma_B)$ in $IFS(H)$.

The *intersection* ($\Cap$) and *union* ($\Cup$) of $A := (H; \mu_A, \gamma_A)$ and $B := (H; \mu_B, \gamma_B)$ are defined by

$$A \Cap B = (H; \mu_A \wedge \mu_B, \gamma_A \vee \gamma_B),$$
$$A \Cup B = (H; \mu_A \vee \mu_B, \gamma_A \wedge \gamma_B),$$

respectively.

3 Intuitionistic fuzzy filters and deductive systems

In what follows, $\mathcal{H} := (H, |)$ stands for a Sheffer stroke Hilbert algebra, unless otherwise stated.

Definition 3.1. An IFS $A := (H; \mu_A, \gamma_A)$ in H is called

- an *intuitionistic fuzzy deductive system* of $\mathcal{H} := (H, |)$ if it satisfies:

$$(\forall x \in H)(\mu_A(1) \geq \mu_A(x),\ \gamma_A(1) \leq \gamma_A(x)), \tag{16}$$

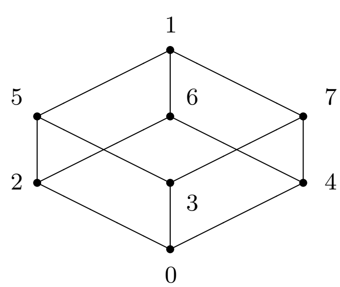
$$(\forall x, y \in H) \left(\begin{array}{l} \mu_A(y) \geq \min\{\mu_A(x), \mu_A(x \mid (y \mid y))\} \\ \gamma_A(y) \leq \max\{\gamma_A(x), \gamma_A(x \mid (y \mid y))\} \end{array} \right). \tag{17}$$

- an *intuitionistic fuzzy filter* of $\mathcal{H} := (H, |)$ if it satisfies (16) and

$$(\forall x, y \in H)(\mu_A(x \mid (y \mid y)) \geq \mu_A(y),\ \gamma_A(x \mid (y \mid y)) \leq \gamma_A(y)), \tag{18}$$

$$(\forall x, y, z \in H) \left(\begin{array}{l} \mu_A((x \mid (y \mid z)) \mid (y \mid z)) \geq \min\{\mu_A(y), \mu_A(z)\} \\ \gamma_A((x \mid (y \mid z)) \mid (y \mid z)) \leq \max\{\gamma_A(y), \gamma_A(z)\} \end{array} \right). \tag{19}$$

Example 3.2. Let $H = \{0, 1, 2, 3, 4, 5, 6, 7\}$ be a set with the following Hasse diagram:



Define a Sheffer stroke "|" on H by Table 1.
Then $\mathcal{H} := (H, |)$ is a Sheffer stroke Hilbert algebra (see [17]).

(i) Define an IFS $A := (H; \mu_A, \gamma_A)$ in H as follows:

$$A := (H; \mu_A, \gamma_A) : H \to [0,1] \times [0,1],\ x \mapsto \begin{cases} (0.69, 0.21) & \text{if } x = 1, \\ (0.63, 0.25) & \text{if } x = 5, \\ (0.55, 0.38) & \text{if } x \in \{2, 6\}, \\ (0.38, 0.52) & \text{otherwise.} \end{cases}$$

Table 1: Cayley table for the Sheffer stroke $|$

$\vert$	0	2	3	4	5	6	7	1
0	1	1	1	1	1	1	1	1
2	1	7	1	1	7	7	1	7
3	1	1	6	1	6	1	6	6
4	1	1	1	5	1	5	5	5
5	1	7	6	1	4	7	6	4
6	1	7	1	5	7	3	5	3
7	1	1	6	5	6	5	2	2
1	1	7	6	5	4	3	2	0

It is routine to verify that $A := (H; \mu_A, \gamma_A)$ is an intuitionistic fuzzy deductive system of $\mathcal{H} := (H, |)$.

(ii) Define an IFS $B := (H; \mu_B, \gamma_B)$ in H as follows:

$$B := (H; \mu_B, \gamma_B) : H \to [0,1] \times [0,1],\ x \mapsto \begin{cases} (0.67, 0.31) & \text{if } x = 1, \\ (0.61, 0.27) & \text{if } x = 7, \\ (0.46, 0.39) & \text{if } x \in \{3, 5\}, \\ (0.33, 0.53) & \text{otherwise.} \end{cases}$$

It is routine to verify that $B := (H; \mu_B, \gamma_B)$ is an intuitionistic fuzzy filter of $\mathcal{H} := (H, |)$.

Proposition 3.3. *Every intuitionistic fuzzy filter $A := (H; \mu_A, \gamma_A)$ of $\mathcal{H} := (H, |)$ satisfies:*

$$(\forall x, y \in H) \left(\begin{array}{c} \mu_A((x \mid (y \mid y)) \mid (y \mid y)) \geq \mu_A(x) \\ \gamma_A((x \mid (y \mid y)) \mid (y \mid y)) \leq \gamma_A(x)\} \end{array} \right). \tag{20}$$

$$(\forall x, y \in H) \left(x \leq y \ \Rightarrow \begin{cases} \mu_A(x) \leq \mu_A(y) \\ \gamma_A(x) \geq \gamma_A(y) \end{cases} \right). \tag{21}$$

Proof. Using (6) and (19), we get

$$\mu_A((x \mid (y \mid y)) \mid (y \mid y)) = \mu_A((y \mid (x \mid x)) \mid (x \mid x)) \geq \min\{\mu_A(x), \mu_A(x)\} = \mu_A(x),$$
$$\gamma_A((x \mid (y \mid y)) \mid (y \mid y)) = \gamma_A((y \mid (x \mid x)) \mid (x \mid x)) \leq \max\{\gamma_A(x), \gamma_A(x)\} = \gamma_A(x)$$

for all $x, y \in H$. Hence (20) is valid. Let $x, y \in H$ be such that $x \leq y$. Then $x \mid (y \mid y) = 1$. It follows from (4) and (20) that

$$\mu_A(y) = \mu_A(1 \mid (y \mid y)) = \mu_A((x \mid (y \mid y)) \mid (y \mid y)) \geq \mu_A(x)$$

and $\gamma_A(y) = \gamma_A(1 \mid (y \mid y)) = \gamma_A((x \mid (y \mid y)) \mid (y \mid y)) \leq \gamma_A(x)$. □

We raise the question, "If an IFS $A := (H; \mu_A, \gamma_A)$ in H satisfies the condition (21), then is $A := (H; \mu_A, \gamma_A)$ an intuitionistic fuzzy filter of $\mathcal{H} := (H, |)$?" And in the next example, we can see that the answer is negative.

Example 3.4. Consider a set $H = \{1, 2, 3, 0\}$, and define a Sheffer stroke "|" on H by Table 2.

Table 2: Cayley table for the Sheffer stroke |

|	1	2	3	0
1	0	3	2	1
2	3	3	1	1
3	2	1	2	1
0	1	1	1	1

Then $\mathcal{H} := (H, |)$ is a Sheffer stroke Hilbert algebra (see [17]). Define an IFS $A := (H; \mu_A, \gamma_A)$ in H as follows:

$$A := (H; \mu_A, \gamma_A) : H \to [0,1] \times [0,1],\ x \mapsto \begin{cases} (0.61, 0.19) & \text{if } x = 1, \\ (0.15, 0.63) & \text{if } x = 2, \\ (0.27, 0.46) & \text{if } x = 3, \\ (0.08, 0.88) & \text{if } x = 0. \end{cases}$$

Then $\mu_A((0 \mid (2 \mid 3)) \mid (2 \mid 3)) = 0.08 \ngeq 0.15 = \min\{\mu_A(2), \mu_A(3)\}$ and/or

$$\gamma_A((0 \mid (3 \mid 2)) \mid (3 \mid 2)) = 0.88 \nleq 0.63 = \max\{\gamma_A(3), \gamma_A(2)\}.$$

Therefore $A := (H; \mu_A, \gamma_A)$ is not an intuitionistic fuzzy filter of $\mathcal{H} := (H, |)$.

We now explore the conditions under which an IFS can be an intuitionistic fuzzy filter.

Theorem 3.5. *An IFS $A := (H; \mu_A, \gamma_A)$ in H is an intuitionistic fuzzy filter of $\mathcal{H} := (H, |)$ if and only if it satisfies the condition (21) and*

$$(\forall x, y \in H) \left(\begin{array}{l} \mu_A((x \mid y) \mid (x \mid y)) \geq \min\{\mu_A(x), \mu_A(y)\} \\ \gamma_A((x \mid y) \mid (x \mid y)) \leq \max\{\gamma_A(x), \gamma_A(y)\} \end{array} \right). \tag{22}$$

Proof. Let $A := (H; \mu_A, \gamma_A)$ be an intuitionistic fuzzy filter of $\mathcal{H} := (H, |)$. Then the condition (21) is valid by Proposition 3.3. Using (s1), (s2), (3), (4) and (19), we have $\mu_A((x \mid y) \mid (x \mid y)) = \mu_A(((1 \mid 1) \mid (x \mid y)) \mid (x \mid y)) \geq \min\{\mu_A(x), \mu_A(y)\}$ and

$$\gamma_A((x \mid y) \mid (x \mid y)) = \gamma_A(((1 \mid 1) \mid (x \mid y)) \mid (x \mid y)) \leq \max\{\gamma_A(x), \gamma_A(y)\}$$

for all $x, y \in H$.

Conversely, assume that $A := (H; \mu_A, \gamma_A)$ satisfies (21) and (22). Since $x \leq 1$ and $y \leq x \mid (y \mid y)$ for all $x, y \in H$, we have $\mu_A(1) \geq \mu_A(x)$, $\gamma_A(1) \leq \gamma_A(x)$, $\mu_A(x \mid (y \mid y)) \geq \mu_A(y)$, and $\gamma_A(x \mid (y \mid y)) \leq \gamma_A(y)$ for all $x, y \in H$ by (21). Using (5), (s2), (21) and (22), we have

$$\mu_A((x \mid (y \mid z)) \mid (y \mid z)) \geq \mu_A((y \mid z) \mid (y \mid z)) \geq \min\{\mu_A(y), \mu_A(z)\}$$

and $\gamma_A((x \mid (y \mid z)) \mid (y \mid z)) \leq \gamma_A((y \mid z) \mid (y \mid z)) \leq \max\{\gamma_A(y), \gamma_A(z)\}$ for all $x, y \in H$. Therefore $A := (H; \mu_A, \gamma_A)$ is an intuitionistic fuzzy filter of $\mathcal{H} := (H, |)$. □

As a result of Theorem 1, we give the following example.

Example 3.6. Consider a set $H = \{1, 2, 3, 0\}$, and define a Sheffer stroke "$|$" on H by Example 3.4. Define an IFS $B := (H; \mu_B, \gamma_B)$ in H as follows:

$$B := (H; \mu_B, \gamma_B) : H \to [0,1] \times [0,1],\ x \mapsto \begin{cases} (0.63, 0.72) & \text{if } x = 0, \\ (0.77, 0.27) & \text{if } x = 1, \\ (0.63, 0.35) & \text{if } x = 2, \\ (0.65, 0.72) & \text{if } x = 3. \end{cases}$$

It is routine to verify that $B := (H; \mu_B, \gamma_B)$ is an intuitionistic fuzzy filter of $\mathcal{H} := (H, |)$. Also, it satisfies the conditions (21) and (22).

Theorem 3.7. *An IFS $A := (H; \mu_A, \gamma_A)$ is an intuitionistic fuzzy filter of $\mathcal{H} := (H, |)$ if and only if the fuzzy sets μ_A and γ_A^c are fuzzy filters of $\mathcal{H} := (H, |)$.*

Proof. Assume that $A := (H; \mu_A, \gamma_A)$ is an intuitionistic fuzzy filter of $\mathcal{H} := (H, |)$. It is clear that μ_A is a fuzzy filter of $\mathcal{H} := (H, |)$. For every $x, y, z \in H$, we have $\gamma_A^c(1) = 1 - \gamma_A(1) \geq 1 - \gamma_A(x) = \gamma_A^c(x)$,

$$\gamma_A^c(x \mid (y \mid y)) = 1 - \gamma_A(x \mid (y \mid y)) \geq 1 - \gamma_A(y) = \gamma_A^c(y),$$

and

$$\begin{aligned} \gamma_A^c((x \mid (y \mid z)) \mid (y \mid z)) &= 1 - \gamma_A((x \mid (y \mid z)) \mid (y \mid z)) \\ &\geq 1 - \max\{\gamma_A(y), \gamma_A(z)\} \\ &= \min\{1 - \gamma_A(y), 1 - \gamma_A(z)\} \\ &= \min\{\gamma_A^c(y), \gamma_A(z)\}. \end{aligned}$$

Hence γ_A^c is a fuzzy filter of $\mathcal{H} := (H, |)$.

Conversely, suppose that μ_A and γ_A^c are fuzzy filters of $\mathcal{H} := (H, |)$. Then $1 - \gamma_A(1) = \gamma_A^c(1) \geq \gamma_A^c(x) = 1 - \gamma_A(x)$,

$$1 - \gamma_A(x \mid (y \mid y)) = \gamma_A^c(x \mid (y \mid y)) \geq \gamma_A^c(y) = 1 - \gamma_A(y)$$

and

$$\begin{aligned} 1-\gamma_A((x \mid (y \mid z)) \mid (y \mid z)) &= \gamma_A^c((x \mid (y \mid z)) \mid (y \mid z)) \\ &\geq \min\{\gamma_A^c(y), \gamma_A^c(z)\} \\ &= \min\{1-\gamma_A(y), \gamma_A(z)\} \\ &= 1-\max\{\gamma_A(y), \gamma_A(z)\} \end{aligned}$$

for all $x, y, z \in H$. It follows that $\gamma_A(1) \leq \gamma_A(x)$, $\gamma_A(x \mid (y \mid y)) \leq \gamma_A(y)$, and $\gamma_A((x \mid (y \mid z)) \mid (y \mid z)) \leq \max\{\gamma_A(y), \gamma_A(z)\}$ for all $x, y, z \in H$. Consequently, $A := (H; \mu_A, \gamma_A)$ is an intuitionistic fuzzy filter of $\mathcal{H} := (H, |)$. $\square$

For any $t \in [0,1]$ and a fuzzy set μ in H, the sets

$$U(\mu, t) := \{x \in H \mid \mu(x) \geq t\} \text{ and } L(\mu, t) := \{x \in H \mid \mu(x) \leq t\}$$

are called the upper t-level cut and lower t-level cut of μ, respectively.

Theorem 3.8. *An IFS $A := (H; \mu_A, \gamma_A)$ is an intuitionistic fuzzy filter of $\mathcal{H} := (H, |)$ if and only if the nonempty sets $U(\mu_A, t)$ and $L(\gamma_A, s)$ are filters of $\mathcal{H} := (H, |)$ for all $t, s \in [0, 1]$.*

Proof. Assume that $A := (H; \mu_A, \gamma_A)$ is an intuitionistic fuzzy filter of $\mathcal{H} := (H, |)$ and $U(\mu_A, t) \neq \emptyset \neq L(\gamma_A, s)$ for all $t, s \in [0, 1]$. It is clear that $1 \in U(\mu_A, t) \cap L(\gamma_A, s)$. Let $y, b \in H$ be such that $(y, b) \in U(\mu_A, t) \times L(\gamma_A, s)$. Then $\mu_A(y) \geq t$ and $\gamma_A(b) \leq s$. It follows from (18) that $\mu_A(x \mid (y \mid y)) \geq \mu_A(y) \geq t$ and $\gamma_A(a \mid (b \mid b)) \leq \gamma_A(b) \leq s$ for all $x, a \in H$. Hence $(x \mid (y \mid y), a \mid (b \mid b)) \in U(\mu_A, t) \times L(\gamma_A, s)$. Let $y, b, z, c \in H$ be such that $(y, b) \in U(\mu_A, t) \times L(\gamma_A, s)$ and $(z, c) \in U(\mu_A, t) \times L(\gamma_A, s)$. Then $\mu_A(y) \geq t$, $\mu_A(z) \geq t$, $\gamma_A(b) \leq s$, and $\gamma_A(c) \leq s$. Using (19), we get $\mu_A((x \mid (y \mid z)) \mid (y \mid z)) \geq \min\{\mu_A(y), \mu_A(z)\} \geq t$ and

$$\gamma_A((a \mid (b \mid c)) \mid (b \mid c)) \leq \max\{\gamma_A(b), \gamma_A(c)\} \leq s,$$

and so $((x \mid (y \mid z)) \mid (y \mid z), (a \mid (b \mid c)) \mid (b \mid c)) \in U(\mu_A, t) \times L(\gamma_A, s)$. Therefore $U(\mu_A, t)$ and $L(\gamma_A, s)$ are filters of $\mathcal{H} := (H, |)$.

Conversely, suppose that the nonempty sets $U(\mu_A, t)$ and $L(\gamma_A, s)$ are filters of $\mathcal{H} := (H, |)$ for all $t, s \in [0, 1]$. Let $x, a \in H$ be such that $\mu_A(1) < \mu_A(x)$ or $\gamma_A(1) > \gamma_A(a)$. Then $1 \notin U(\mu_A, \mu_A(x))$ or $1 \notin L(\gamma_A, \gamma_A(a))$, a contradiction. Hence $\mu_A(1) \geq \mu_A(x)$ and $\gamma_A(1) \leq \gamma_A(x)$ for all $x \in H$. Let $x, a, y, b \in H$ be such that $\mu_A(y) > \mu_A(x \mid (y \mid y))$ or $\gamma_A(b) < \gamma_A(a \mid (b \mid b))$. Then $y \in U(\mu_A, t)$ and $x \mid (y \mid y) \notin U(\mu_A, t)$, or $b \in L(\gamma_A, s)$ and $a \mid (b \mid b) \notin L(\gamma_A, s)$ where $t := \mu_A(x \mid (y \mid y))$ and $s := \gamma_A(a \mid (b \mid b))$. This is a contradiction, and so $\mu_A(x \mid (y \mid y)) \geq \mu_A(y)$ and $\gamma_A(x \mid (y \mid y)) \leq \gamma_A(y)$ for all $x, y \in H$. Let $x, y, z, a, b, c \in H$ be such that

$$\mu_A((x \mid (y \mid z)) \mid (y \mid z)) < \min\{\mu_A(y), \mu_A(z)\}$$

or $\gamma_A((a \mid (b \mid c)) \mid (b \mid c)) > \max\{\gamma_A(b), \gamma_A(c)\}$. If we take $t := \min\{\mu_A(y), \mu_A(z)\}$, then $y, z \in U(\mu_A, t)$ but $(x \mid (y \mid z)) \mid (y \mid z) \notin U(\mu_A, t)$. If we take $s :=$

$\max\{\gamma_A(b), \gamma_A(c)\}$, then $b, c \in L(\gamma_A, s)$ but $(a \mid (b \mid c)) \mid (b \mid c) \notin L(\gamma_A, s)$. This is a contradiction, and thus $\mu_A((x \mid (y \mid z)) \mid (y \mid z)) \geq \min\{\mu_A(y), \mu_A(z)\}$ and

$$\gamma_A((x \mid (y \mid z)) \mid (y \mid z)) \leq \max\{\gamma_A(y), \gamma_A(z)\}$$

for all $x, y, z \in H$. Consequently, $A := (H; \mu_A, \gamma_A)$ is an intuitionistic fuzzy filter of $\mathcal{H} := (H, |)$. □

Theorem 3.9. *Let $\{F_t \mid t \in \Lambda \subseteq [0,1]\}$ be a collection of filters of $\mathcal{H} := (H, |)$ such that $H = \bigcup_{t \in \Lambda} F_t$ and*

$$(\forall t, s \in \Lambda)(t < s \;\Leftrightarrow\; F_t \supset F_s). \tag{23}$$

Define an IFS $A := (H; \mu_A, \gamma_A)$ in H as follows:

$$A := (H; \mu_A, \gamma_A) : H \to [0,1] \times [0,1],$$
$$x \mapsto (\sup\{t \in \Lambda \mid x \in F_t\}, \inf\{s \in \Lambda \mid x \in F_s\}).$$

Then $A := (H; \mu_A, \gamma_A)$ is an intuitionistic fuzzy filter of $\mathcal{H} := (H, |)$.

Proof. We first show that $U(\mu_A, t)$ is a filter of $\mathcal{H} := (H, |)$ for all $t \in [0, \mu_A(1)]$. If $t = \sup\{t_a \in \Lambda \mid t_a < t\}$, then

$$x \in U(\mu_A, t) \;\Leftrightarrow\; (\forall t_a < t)(x \in F_{t_a}) \;\Leftrightarrow\; x \in \bigcap_{t_a < t} F_{t_a}.$$

Hence $U(\mu_A, t) = \bigcap_{t_a < t} F_{t_a}$, which is a filter of $\mathcal{H} := (H, |)$. Assume that $t \neq \sup\{t_a \in \Lambda \mid t_a < t\}$. If $x \in \bigcup\{F_{t_a} \mid t_a \geq t\}$, then $x \in F_{t_a}$ for some $t_a \geq t$. Hence $\mu_A(x) = \sup\{t \in \Lambda \mid x \in F_t\} \geq t_a \geq t$, and so $x \in U(\mu_A, t)$. This shows that

$$\bigcup\{F_{t_a} \mid t_a \geq t\} \subseteq U(\mu_A, t).$$

If $x \notin \bigcup\{F_{t_a} \mid t_a \geq t\}$, then $x \notin F_{t_a}$ for all $t_a \geq t$. Since $t \neq \sup\{t_a \in \Lambda \mid t_a < t\}$, there exists $\delta > 0$ such that $(t - \delta, t) \cap \Lambda = \emptyset$. Thus $x \notin F_{t_a}$ for all $t_a > t - \delta$, which means that if $x \in F_{t_a}$, then $t_a \leq t - \delta$. Hence $\mu_A(x) \leq t - \delta < t$, i.e., $x \notin U(\mu_A, t)$. Thus $U(\mu_A, t) \subseteq \bigcup\{F_{t_a} \mid t_a \geq t\}$. Therefore $U(\mu_A, t) = \bigcup\{F_{t_a} \mid t_a \geq t\}$ which is a filter of $\mathcal{H} := (H, |)$.

Now we will show that $L(\gamma_A, s)$ is a filter of $\mathcal{H} := (H, |)$ for all $s \in [\gamma_A(1), 1]$. If $s = \inf\{s_b \in \Lambda \mid s < s_b\}$, then

$$x \in L(\gamma_A, s) \;\Leftrightarrow\; (\forall s_b > s)(x \in F_{s_b}) \;\Leftrightarrow\; x \in \bigcap_{s_b > s} F_{s_b}.$$

Thus $L(\gamma_A, s) = \bigcap_{s_b > s} F_{s_b}$, which is a filter of $\mathcal{H} := (H, |)$. Suppose that $s \neq \inf\{s_b \in \Lambda \mid s < s_b\}$. Then $(s, s + \delta) \cap \Lambda = \emptyset$ for some $\delta > 0$. If $x \in \bigcup_{s \geq s_b} F_{s_b}$, then $x \in F_{s_b}$ for some $s_b \leq s$ and so $\gamma_A(x) \leq s_b \leq s$, that is, $x \in L(\gamma_A, s)$. If $x \notin \bigcup_{s \geq s_b} F_{s_b}$, then $x \notin F_{s_b}$

for all $s_b \leq s$, and thus $x \notin F_{s_b}$ for all $s_b < s + \delta$. This shows that if $x \in F_{s_b}$ then $s_b \geq s + \delta$. Hence $\gamma_A(x) \geq s + \delta > s$, i.e., $x \notin L(\gamma_A, s)$. Therefore $L(\gamma_A, s) = \bigcup_{s \geq s_b} F_{s_b}$ which is a filter of $\mathcal{H} := (H, |)$. Consequently, $A := (H; \mu_A, \gamma_A)$ is an intuitionistic fuzzy filter of $\mathcal{H} := (H, |)$ by Theorem 3.8. □

Theorem 3.10. *If $A := (H; \mu_A, \gamma_A)$ and $B := (H; \mu_B, \gamma_B)$ are intuitionistic fuzzy filters of $\mathcal{H} := (H, |)$, then so is their intersection $A \Cap B = (H;\, \mu_A \wedge \mu_B,\, \gamma_A \vee \gamma_B)$.*

Proof. Let $A := (H; \mu_A, \gamma_A)$ and $B := (H; \mu_B, \gamma_B)$ be intuitionistic fuzzy filters of $\mathcal{H} := (H, |)$. For every $x, y, z \in H$, we have

$$(\mu_A \wedge \mu_B)(1) = \min\{\mu_A(1), \mu_B(1)\} \geq \min\{\mu_A(x), \mu_B(x)\} = (\mu_A \wedge \mu_B)(x),$$

$$(\gamma_A \vee \gamma_B)(1) = \max\{\gamma_A(1), \gamma_B(1)\} \leq \max\{\gamma_A(x), \gamma_B(x)\} = (\gamma_A \vee \gamma_B)(x),$$

$$\begin{aligned}(\mu_A \wedge \mu_B)(x \mid (y \mid y)) &= \min\{\mu_A(x \mid (y \mid y)), \mu_B(x \mid (y \mid y))\} \\ &\geq \min\{\mu_A(y), \mu_B(y)\} = (\mu_A \wedge \mu_B)(y),\end{aligned}$$

$$\begin{aligned}(\gamma_A \vee \gamma_B)(x \mid (y \mid y)) &= \max\{\gamma_A(x \mid (y \mid y)), \gamma_B(x \mid (y \mid y))\} \\ &\leq \max\{\gamma_A(y), \gamma_B(y)\} = (\gamma_A \vee \gamma_B)(y),\end{aligned}$$

$$\begin{aligned}(\mu_A \wedge \mu_B)((x \mid (y \mid z)) \mid (y \mid z)) &= \min\{\mu_A((x \mid (y \mid z)) \mid (y \mid z)), \mu_B((x \mid (y \mid z)) \mid \\ &\quad (y \mid z))\} \\ &\geq \min\{\min\{\mu_A(y), \mu_A(z)\}, \min\{\mu_B(y), \mu_B(z)\}\} \\ &= \min\{\min\{\mu_A(y), \mu_B(y)\}, \min\{\mu_A(z), \mu_B(z)\}\} \\ &= \min\{(\mu_A \wedge \mu_B)(y), (\mu_A \wedge \mu_B)(z)\},\end{aligned}$$

and

$$\begin{aligned}(\gamma_A \vee \gamma_B)((x \mid (y \mid z)) \mid (y \mid z)) &= \max\{\gamma_A((x \mid (y \mid z)) \mid (y \mid z)), \gamma_B((x \mid (y \mid z)) \mid \\ &\quad (y \mid z))\} \\ &\leq \max\{\max\{\gamma_A(y), \gamma_A(z)\}, \max\{\gamma_B(y), \gamma_B(z)\}\} \\ &= \max\{\max\{\gamma_A(y), \gamma_B(y)\}, \max\{\gamma_A(z), \gamma_B(z)\}\} \\ &= \max\{(\gamma_A \vee \gamma_B)(y), (\gamma_A \vee \gamma_B)(z)\}.\end{aligned}$$

Therefore $A \Cap B = (H;\, \mu_A \wedge \mu_B,\, \gamma_A \vee \gamma_B)$ is an intuitionistic fuzzy filter of $\mathcal{H} := (H, |)$.
□

The next example shows that the union of intuitionistic fuzzy filters may not be an intuitionistic fuzzy filter.

Example 3.11. Consider the Sheffer stroke Hilbert algebra $\mathcal{H} := (H, |)$ which is given in Example 3.2. Define an IFS $A := (H; \mu_A, \gamma_A)$ in H as follows:

$$A := (H; \mu_A, \gamma_A) : H \to [0,1] \times [0,1],\ x \mapsto \begin{cases} (0.71, 0.23) & \text{if } x = 1, \\ (0.65, 0.32) & \text{if } x = 5, \\ (0.46, 0.39) & \text{if } x \in \{3, 7\}, \\ (0.34, 0.52) & \text{otherwise.} \end{cases}$$

Then $A := (H; \mu_A, \gamma_A)$ is an intuitionistic fuzzy filter of $\mathcal{H} := (H, |)$. Let's take the intuitionistic fuzzy filter $B := (H; \mu_B, \gamma_B)$ of $\mathcal{H} := (H, |)$ given in Example 3.2(ii). Then their union $A \Cup B = (H; \mu_A \vee \mu_B, \gamma_A \wedge \gamma_B)$ is given by

$$\mu_A \vee \mu_B : H \to [0,1]\ x \mapsto \begin{cases} 0.71 & \text{if } x = 1, \\ 0.46 & \text{if } x = 3, \\ 0.65 & \text{if } x = 5, \\ 0.61 & \text{if } x = 7, \\ 0.34 & \text{otherwise,} \end{cases}$$

and

$$\gamma_A \wedge \gamma_B : H \to [0,1]\ x \mapsto \begin{cases} 0.23 & \text{if } x = 1, \\ 0.39 & \text{if } x = 3, \\ 0.32 & \text{if } x = 5, \\ 0.27 & \text{if } x = 7, \\ 0.52 & \text{otherwise.} \end{cases}$$

Since

$$\begin{aligned} (\mu_A \vee \mu_B)((0 \mid (5 \mid 7)) \mid (5 \mid 7)) &= (\mu_A \vee \mu_B)(3) = 0.46 \ngeq 0.61 \\ &= \min\{(\mu_A \vee \mu_B)(5), (\mu_A \vee \mu_B)(7)\} \end{aligned}$$

and/or

$$\begin{aligned} (\gamma_A \wedge \gamma_B)((0 \mid (5 \mid 7)) \mid (5 \mid 7)) &= (\gamma_A \wedge \gamma_B)(3) = 0.39 \nleq 0.32 \\ &= \max\{(\gamma_A \wedge \gamma_B)(5), (\gamma_A \wedge \gamma_B)(7)\}, \end{aligned}$$

we know that $A \Cup B = (H; \mu_A \vee \mu_B, \gamma_A \wedge \gamma_B)$ is not an intuitionistic fuzzy filter of $\mathcal{H} := (H, |)$.

Definition 3.12 ([16]). Let $\mathcal{G} := (G, |_G)$ and $\mathcal{H} := (H, |_H)$ be Sheffer stroke Hilbert algebras. A mapping $f : G \to H$ is called a *homomorphism* if it satisfies:

$$(\forall x, y \in G)(f(x \mid_G y) = f(x) \mid_H f(y)). \tag{24}$$

It is clear that if $f : G \to H$ is a homomorphism of Sheffer stroke Hilbert algebras, then $f(1_G) = 1_H$.

Proposition 3.13. *If $f : G \to H$ is a homomorphism of Sheffer stroke Hilbert algebras $\mathcal{G} := (G, |_G)$ and $\mathcal{H} := (H, |_H)$, then*

(i) $(\forall x, y \in G)(x \leq_G y \ \Rightarrow\ f(x) \leq_H f(y))$.

(ii) *If F is a filter of $\mathcal{H} := (H, |_H)$, then the set*

$$f^{-1}(F) := \{x \in G \mid f(x) \in F\}$$

is a filter of $\mathcal{G} := (G, |_G)$.

(iii) *If f is onto and E is a filter of $\mathcal{G} := (G, |_G)$, then $f(E)$ is a filter of $\mathcal{H} := (H, |_H)$.*

Proof. Let $x, y \in G$ be such that $x \leq_G y$. Then $x |_G (y |_G y) = 1$, and so $1 = f(1) = f(x |_G (y |_G y) = f(x) |_H (f(x) |_H f(y))$. Hence $f(x) \leq_H f(y)$. Let F be a filter of $\mathcal{H} := (H, |_H)$. It is clear that $1_G \in f^{-1}(F)$. Let $x, y \in G$ be such that $y \in f^{-1}(F)$. Then $f(y) \in F$, and so $f(x |_G (y |_G y)) = f(x) |_H (f(y) |_H f(y)) \in F$, that is, $x |_G (y |_G y) \in f^{-1}(F)$. Let $x, y, z \in G$ be such that $y, z \in f^{-1}(F)$. Then $f(y) \in F$ and $f(z) \in F$. It follows that

$$f((x |_G (y |_G z)) |_G (y |_G z)) = (f(x) |_H (f(y) |_H f(z))) |_H (f(y) |_H f(z)) \in F.$$

Thus $(x |_G (y |_G z)) |_G (y |_G z) \in f^{-1}(F)$, and therefore $f^{-1}(F)$ is a filter of $\mathcal{G} := (G, |_G)$.

Assume that f is onto and let E be a filter of $\mathcal{G} := (G, |_G)$. Then $1_H = f(1_G) \in f(E)$. Let $x, y \in H$ be such that $y \in f(E)$. Then $y = f(b)$ and $x = f(a)$ for some $a \in G$ and $b \in E$. It follows that

$$x |_H (y |_H y) = f(a) |_H (f(b) |_H f(b)) = f(a |_G (b |_G b)) \in f(E).$$

Let $x, y, z \in H$ be such that $y \in f(E)$ and $z \in f(E)$. Then $f(a) = x$, $f(b) = y$ and $f(c) = z$ for some $a \in G$ and $b, c \in E$. Hence

$$\begin{aligned}(x |_H (y |_H z)) |_H (y |_H z) &= (f(a) |_H (f(b) |_H f(c))) |_H (f(b) |_H f(c)) \\ &= f((a |_G (b |_G c)) |_G (b |_G c)) \in f(E).\end{aligned}$$

Therefore $f(E)$ is a filter of $\mathcal{H} := (H, |_H)$. □

Corollary 3.14. *If $f : G \to H$ is a homomorphism of Sheffer stroke Hilbert algebras $\mathcal{G} := (G, |_G)$ and $\mathcal{H} := (H, |_H)$, then the set*

$$\ker(f) := \{x \in G \mid \mu_A(x) = 1 = \gamma_A(x)\}$$

is a filter of $\mathcal{G} := (G, |_G)$.

Theorem 3.15. *Given Sheffer stroke Hilbert algebras $\mathcal{G} := (G, |_G)$, $\mathcal{H} := (H, |_H)$ and $\mathcal{K} := (K, |_K)$, let $f : G \to H$ and $g : G \to K$ be homomorphisms. If $\ker(f) \subseteq \ker(g)$ and f is onto, then there exists a unique homomorphism $h : H \to K$ such that the diagram is commutative.*

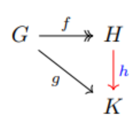


Figure 1

Proof. Assume that f is onto and $\ker(f) \subseteq \ker(g)$. For every $y \in H$, there exists $b \in G$ such that $f(b) = y$. For the element $b \in G$, put $z := g(b)$ and define

$$h : H \to K,\ y \mapsto z := g(b).$$

It is clear that the diagram in Fig. 1 is commutative. Let $y_1, y_2 \in H$ be such that $y_1 = y_2$, $y_1 = f(x_1)$ and $y_2 = f(x_2)$ for some $x_1, x_2 \in G$. Then

$$f(x_1 \,|_G\, (x_2 \,|_G\, x_2)) = f(x_1) \,|_H\, (f(x_2) \,|_H\, f(x_2)) = y_1 \,|_H\, (y_2 \,|_H\, y_2) = 1_H,$$

and so

$$x_1 \,|_G\, (x_2 \,|_G\, x_2) \in \ker(f) \subseteq \ker(g).$$

Hence

$$1_K = g(x_1 \,|_G\, (x_2 \,|_G\, x_2)) = g(x_1) \,|_K\, (g(x_2) \,|_K\, g(x_2)),$$

that is, $g(x_1) \leq_K g(x_2)$. The similar way induces $g(x_2) \leq_K g(x_1)$, and so $g(x_1) = g(x_2)$. Hence h is well-defined. Let $y_1, y_2 \in H$. For every $x_1, x_2 \in G$ with $y_1 = f(x_1)$ and $y_2 = f(x_2)$, we get

$$\begin{aligned} h(y_1 \,|_H\, y_2) &= h(f(x_1) \,|_H\, f(x_2)) \\ &= h(f(x_1 \,|_G\, x_2)) \\ &= g(x_1 \,|_G\, x_2) \\ &= g(x_1) \,|_K\, g(x_2) \\ &= h(f(x_1)) \,|_K\, h(f(x_2)) \\ &= h(y_1) \,|_K\, h(y_2). \end{aligned}$$

Hence h is a homomorphism. Since f is an onto homomorphism, the uniquence of h is straightforward. □

In what follows, if there is no fear of confusion, the Sheffer strokes "$|_G$" and "$|_H$" are simply denoted by "$|$".

Let $f : G \to H$ be a homomorphism of Sheffer stroke Hilbert algebras. For every IFS $A := (H; \mu_A, \gamma_A)$ in H, we define a new IFS $A^f := (G; \mu_A^f, \gamma_A^f)$ in G by

$$A^f := (G; \mu_A^f, \gamma_A^f) : G \to [0,1] \times [0,1],\ x \mapsto (\mu_A(f(x)), \gamma_A(f(x))). \tag{25}$$

Theorem 3.16. *Let $f : G \to H$ be a homomorphism of Sheffer stroke Hilbert algebras. If $A := (H; \mu_A, \gamma_A)$ is an intuitionistic fuzzy filter of $\mathcal{H} := (H, |)$, then $A^f := (G; \mu_A^f, \gamma_A^f)$ is an intuitionistic fuzzy filter of $\mathcal{G} := (G, |)$.*

Proof. Assume that $A := (H; \mu_A, \gamma_A)$ is an intuitionistic fuzzy filter of $\mathcal{H} := (H, |)$. Then

$$\mu_A^f(1) = \mu_A(f(1)) = \mu_A(1) \geq \mu_A(f(x)) = \mu_A^f(x),$$
$$\gamma_A^f(1) = \gamma_A(f(1)) = \gamma_A(1) \leq \gamma_A(f(x)) = \gamma_A^f(x),$$

$$\begin{aligned} \mu_A^f(x \mid (y \mid y)) &= \mu_A(f(x \mid (y \mid y))) \\ &= \mu_A(f(x) \mid (f(y) \mid f(y))) \\ &\geq \mu_A(f(y)) = \mu_A^f(y), \end{aligned}$$

$$\begin{aligned}\gamma_A^f(x \mid (y \mid y)) &= \gamma_A(f(x \mid (y \mid y)))\\ &= \gamma_A(f(x) \mid (f(y) \mid f(y)))\\ &\leq \gamma_A(f(y)) = \gamma_A^f(y),\end{aligned}$$

$$\begin{aligned}\mu_A^f((x \mid (y \mid z)) \mid (y \mid z)) &= \mu_A(f((x \mid (y \mid z)) \mid (y \mid z)))\\ &= \mu_A((f(x) \mid (f(y) \mid f(z))) \mid (f(y) \mid f(z)))\\ &\geq \min\{\mu_A(f(y)), \mu_A(f(z))\}\\ &= \min\{\mu_A^f(y), \mu_A^f(z)\}\end{aligned}$$

and

$$\begin{aligned}\gamma_A^f((x \mid (y \mid z)) \mid (y \mid z)) &= \gamma_A(f((x \mid (y \mid z)) \mid (y \mid z)))\\ &= \gamma_A((f(x) \mid (f(y) \mid f(z))) \mid (f(y) \mid f(z)))\\ &\leq \max\{\gamma_A(f(y)), \gamma_A(f(z))\}\\ &= \max\{\gamma_A^f(y), \gamma_A^f(z)\}\end{aligned}$$

for all $x, y, z \in G$. Therefore $A^f := (G; \mu_A^f, \gamma_A^f)$ is an intuitionistic fuzzy filter of $\mathcal{G} := (G, |)$. $\square$

Theorem 3.17. *Let $f : G \to H$ be an onto homomorphism of Sheffer stroke Hilbert algebras. For every IFS $A := (H; \mu_A, \gamma_A)$ in H, if $A^f := (G; \mu_A^f, \gamma_A^f)$ is an intuitionistic fuzzy filter of $\mathcal{G} := (G, |)$, then $A := (H; \mu_A, \gamma_A)$ is an intuitionistic fuzzy filter of $\mathcal{H} := (H, |)$.*

Proof. Let $f : G \to H$ be an onto homomorphism of Sheffer stroke Hilbert algebras and assume that $A^f := (G; \mu_A^f, \gamma_A^f)$ is an intuitionistic fuzzy filter of $\mathcal{G} := (G, |)$. For every $x \in H$, there exists $a \in G$ such that $f(a) = x$. Hence

$$\mu_A(1) = \mu_A(f(1)) = \mu_A^f(1) \geq \mu_A^f(a) = \mu_A(f(a)) = \mu_A(x)$$

and $\gamma_A(1) = \gamma_A(f(1)) = \gamma_A^f(1) \leq \gamma_A^f(a) = \gamma_A(f(a)) = \gamma_A(x)$. Let $x, y, z \in H$. Then $f(a) = x$, $f(b) = y$ and $f(c) = z$ for some $a, b, c \in G$. Thus

$$\begin{aligned}\mu_A(x \mid (y \mid y)) &= \mu_A(f(a) \mid (f(b) \mid f(b))) = \mu_A(f(a \mid (b \mid b)))\\ &= \mu_A^f(a \mid (b \mid b)) \geq \mu_A^f(b) = \mu_A(f(b)) = \mu_A(y),\end{aligned}$$

$$\begin{aligned}\gamma_A(x \mid (y \mid y)) &= \gamma_A(f(a) \mid (f(b) \mid f(b))) = \gamma_A(f(a \mid (b \mid b)))\\ &= \gamma_A^f(a \mid (b \mid b)) \leq \gamma_A^f(b) = \gamma_A(f(b)) = \gamma_A(y),\end{aligned}$$

$$\begin{aligned}\mu_A((x \mid (y \mid z)) \mid (y \mid z)) &= \mu_A((f(a) \mid (f(b) \mid f(c))) \mid (f(b) \mid f(c)))\\ &= \mu_A(f((a \mid (b \mid c)) \mid (b \mid c)))\\ &= \mu_A^f((a \mid (b \mid c)) \mid (b \mid c))\\ &\geq \min\{\mu_A^f(b), \mu_A^f(c)\}\\ &= \min\{\mu_A(f(b)), \mu_A(f(c))\}\\ &= \min\{\mu_A(y), \mu_A(z)\}\end{aligned}$$

and

$$\begin{aligned}\gamma_A((x \mid (y \mid z)) \mid (y \mid z)) &= \gamma_A((f(a) \mid (f(b) \mid f(c))) \mid (f(b) \mid f(c)))\\ &= \gamma_A(f((a \mid (b \mid c)) \mid (b \mid c)))\\ &= \gamma_A^f((a \mid (b \mid c)) \mid (b \mid c))\\ &\leq \max\{\gamma_A^f(b), \gamma_A^f(c)\}\\ &= \max\{\gamma_A(f(b)), \gamma_A(f(c))\}\\ &= \max\{\gamma_A(y), \gamma_A(z)\}.\end{aligned}$$

Consequently, $A := (H; \mu_A, \gamma_A)$ is an intuitionistic fuzzy filter of $\mathcal{H} := (H, |)$. □

We finally discuss the relationship between an intuitionistic fuzzy filter and an intuitionistic fuzzy deductive system.

Theorem 3.18. *An IFS $A := (H; \mu_A, \gamma_A)$ in H is an intuitionistic fuzzy filter of $\mathcal{H} := (H, |)$ if and only if it is an intuitionistic fuzzy deductive system of $\mathcal{H} := (H, |)$.*

Proof. Let $A := (H; \mu_A, \gamma_A)$ be an intuitionistic fuzzy filter of $\mathcal{H} := (H, |)$. If we replace y, z, and x in (19) with x, $x \mid (y \mid y)$, and y, respectively, we get:

$$\begin{aligned}\mu_A(y) &= \mu_A(((x \mid x) \mid (1 \mid 1)) \mid (y \mid y))\\ &= \mu_A(((x \mid x) \mid ((y \mid (y \mid y)) \mid (y \mid (y \mid y)))) \mid (y \mid y))\\ &= \mu_A((((((x \mid x) \mid y) \mid ((x \mid x) \mid y)) \mid (y \mid y)) \mid (y \mid y))\\ &= \mu_A((y \mid ((x \mid x) \mid y)) \mid ((x \mid x) \mid y))\\ &= \mu_A((((((x \mid x) \mid y) \mid y) \mid y) \mid (((x \mid x) \mid y) \mid y))\\ &= \mu_A((y \mid (x \mid (x \mid (y \mid y)))) \mid (x \mid (x \mid (y \mid y))))\\ &\geq \min\{\mu_A(x), \mu_A(x \mid (y \mid y))\}\end{aligned}$$

and

$$\begin{aligned}\gamma_A(y) &= \gamma_A(((x \mid x) \mid (1 \mid 1)) \mid (y \mid y))\\ &= \gamma_A(((x \mid x) \mid ((y \mid (y \mid y)) \mid (y \mid (y \mid y)))) \mid (y \mid y))\\ &= \gamma_A((((((x \mid x) \mid y) \mid ((x \mid x) \mid y)) \mid (y \mid y)) \mid (y \mid y))\\ &= \gamma_A((y \mid ((x \mid x) \mid y)) \mid ((x \mid x) \mid y))\\ &= \gamma_A((((((x \mid x) \mid y) \mid y) \mid y) \mid (((x \mid x) \mid y) \mid y))\\ &= \gamma_A((y \mid (x \mid (x \mid (y \mid y)))) \mid (x \mid (x \mid (y \mid y))))\\ &\leq \max\{\gamma_A(x), \gamma_A(x \mid (y \mid y))\}\end{aligned}$$

for all $x, y \in H$ by (s1), (s2), (s3), (2), (3), (4), (6), and (7). Hence $A := (H; \mu_A, \gamma_A)$ is an intuitionistic fuzzy deductive system of $\mathcal{H} := (H, |)$.

Conversely, let $A := (H; \mu_A, \gamma_A)$ be an intuitionistic fuzzy deductive system of $\mathcal{H} := (H, |)$. Let $x, y \in H$. Since $y \leq x \mid (y \mid y)$ by (5), we have

$$y \mid ((x \mid (y \mid y)) \mid (x \mid (y \mid y))) = 1.$$

It follows from (17) and (16) that

$$\begin{aligned}\mu_A(x \mid (y \mid y)) &\geq \min\{\mu_A(y), \mu_A(y \mid ((x \mid (y \mid y)) \mid (x \mid (y \mid y))))\} \\ &= \min\{\mu_A(y), \mu_A(1)\} = \mu_A(y)\end{aligned} \tag{26}$$

and

$$\begin{aligned}\gamma_A(x \mid (y \mid y)) &\leq \max\{\gamma_A(y), \gamma_A(y \mid ((x \mid (y \mid y)) \mid (x \mid (y \mid y))))\} \\ &= \max\{\gamma_A(y), \gamma_A(1)\} = \gamma_A(y)\end{aligned} \tag{27}$$

for all $x, y \in H$. Using (s2), (8) and (2), we get

$$y \mid (((y \mid z) \mid z) \mid ((y \mid z) \mid z)) = (y \mid z) \mid ((y \mid z) \mid (y \mid z)) = 1$$

for all $y, z \in H$. Hence

$$\begin{aligned}\mu_A((y \mid z) \mid z) &\geq \min\{\mu_A(y), \mu_A(y \mid (((y \mid z) \mid z) \mid ((y \mid z) \mid z)))\} \\ &= \min\{\mu_A(y), \mu_A(1)\} = \mu_A(y)\end{aligned} \tag{28}$$

and

$$\begin{aligned}\gamma_A((y \mid z) \mid z) &\leq \max\{\gamma_A(y), \gamma_A(y \mid (((y \mid z) \mid z) \mid ((y \mid z) \mid z)))\} \\ &= \max\{\gamma_A(y), \gamma_A(1)\} = \gamma_A(y)\end{aligned} \tag{29}$$

by (17) and (16). Using (s2) and (s1) leads to

$$z \mid (((y \mid z) \mid (y \mid z)) \mid ((y \mid z) \mid (y \mid z))) = (y \mid z) \mid z$$

for all $y, z \in H$. Thus

$$\begin{aligned}\mu_A((y \mid z) \mid (y \mid z)) &\geq \min\{\mu_A(z), \mu_A(z \mid (((y \mid z) \mid (y \mid z)) \mid ((y \mid z) \mid (y \mid z))))\} \\ &= \min\{\mu_A(z), \mu_A((y \mid z) \mid z)\} \\ &\geq \min\{\mu_A(z), \mu_A(y)\}\end{aligned} \tag{30}$$

and

$$\begin{aligned}\gamma_A((y \mid z) \mid (y \mid z)) &\leq \max\{\gamma_A(z), \gamma_A(z \mid (((y \mid z) \mid (y \mid z)) \mid ((y \mid z) \mid (y \mid z))))\} \\ &= \max\{\gamma_A(z), \gamma_A((y \mid z) \mid z)\} \\ &\leq \max\{\gamma_A(z), \gamma_A(y)\}\end{aligned} \tag{31}$$

for all $y, z \in H$. It follows from (s2), (26), (30), (27) and (31) that

$$\begin{aligned}&\mu_A((x \mid (y \mid z)) \mid (y \mid z)) \\ &= \mu_A((x \mid (((y \mid z) \mid (y \mid z)) \mid ((y \mid z) \mid (y \mid z)))) \mid \\ &\quad (((y \mid z) \mid (y \mid z)) \mid ((y \mid z) \mid (y \mid z)))) \text{ by (s2)} \\ &\geq \mu_A((y \mid z) \mid (y \mid z)) \text{ by (26)} \\ &\geq \min\{\mu_A(z), \mu_A(y)\} \text{ by (30)}\end{aligned}$$

and

$$\begin{aligned}
&\gamma_A((x \mid (y \mid z)) \mid (y \mid z)) \\
&= \gamma_A((x \mid (((y \mid z) \mid (y \mid z)) \mid ((y \mid z) \mid (y \mid z)))) \mid \\
&\quad (((y \mid z) \mid (y \mid z)) \mid ((y \mid z) \mid (y \mid z)))) \text{ by (s2)} \\
&\leq \gamma_A((y \mid z) \mid (y \mid z)) \text{ by (27)} \\
&\leq \max\{\gamma_A(z), \gamma_A(y)\} \text{ by (31)}
\end{aligned}$$

for all $x, y, z \in H$. Therefore $A := (H; \mu_A, \gamma_A)$ is an intuitionistic fuzzy filter of $\mathcal{H} := (H, |)$. □

By Theorem 3.18, it can be seen that the two concepts intuitionistic fuzzy filter and intuitionistic fuzzy deductive system coincide with each other. Therefore, in this paper, the properties established by intuitionistic fuzzy filter are also established by intuitionistic fuzzy deductive system.

4 Conclusion

In this paper, we present the application of Atanassov's intuitionistic fuzzy set within the context of deductive systems and filters in Sheffer stroke Hilbert algebras. We present the concept of if deductive systems and if filters in Sheffer stroke Hilbert algebras, along with an exploration of their properties. We examine the conditions that must be met for an if set to qualify as an if filter and delve into characterizations of if filters. We elucidate the process of creating an if filter by amalgamating filters. The paper also explores the union and intersection of if filters and investigates various properties related to homomorphisms in Sheffer stroke Hilbert algebras. In conclusion, we discuss the relationship between an if filter and an if deductive system.

Exploring alternative potential applications in a wide range of algebraic structures presents an exciting avenue for future research. The foundational principles of Atanassov's intuitionistic fuzzy set applied to the concepts of deductive systems and filters within Sheffer stroke Hilbert algebra can undergo comprehensive examination in various Sheffer stroke reduction algebraic frameworks. These may include Sheffer stroke basic algebras, and several others, considering state operator [24] and very-true operator interactions with these structures. This endeavor holds the promise of revealing the adaptability and versatility of these concepts in a diverse array of algebraic structures, igniting new inquiries and deepening our comprehension of their mathematical significance.

Compliance with Ethical Standards

Conflict of interest Author Arsham Borumand Saeid declares that he has no conflict of interest. Author Tahsin Oner declares that he has no conflict of interest. Author Young Bae Jun declares that he has no conflict of interest.Author Young Bae Jun declares that he has no conflict of interest.

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