Journal of Mathematical Extension
Vol. 18, No. 10, (2024) (4)1-19
URL: https://doi.org/10.30495/JME.2024.3177
ISSN: 1735-8299
Original Research Paper

Algebras of Toeplitz Matrices with Quaternion Entries

M. Ahsan Khan*
University of Kotli AJK

A. Yagoub
Université de Amar Telidji

Abstract. The problem of characterizing the maximal left algebras of Toeplitz matrices with quaternion entries is a complex as well as a harder problem that has not received much attention until now. In the current paper, we introduce certain families of maximal left algebras of Toeplitz matrices with entries from an algebra of quaternions that cover various classes of the left algebras of quaternion Toeplitz matrices.

AMS Subject Classification: 15B33; 15B05; 11R52
Keywords and Phrases: Quaternions, Toeplitz matrices, maximal left algebras

1 Introduction

Matrix theory and algebra comprise the theory and application of linear spaces, linear transformation, and unifying otherwise disparate topics (functional analysis, differential geometry, quantum physics, etc.). Up to now, these fields of study have been active and are used by several

Received: October 2024; Accepted: February 2025
*Corresponding Author

researchers working in different specializations as well as by every mathematician.

A square matrix is called Toeplitz if every descending diagonal from left to right is constant. In other words, the elements of the matrix are arranged in such a way that each row is a shifted version of the previous row.

These matrices arise naturally in many areas of Mathematics and are important both in theory and applications. For instance, it is well known that a large class of matrices is similar to Toeplitz matrices [7, 20]. Moreover, it is proved that every matrix can be expressed as a product of Toeplitz matrices (see [27]). Apart from this, these matrices have some of the most attractive computational properties and are amenable to a wide range of disparate algorithms. We refer the reader to [26] for a detailed study of these matrices.

Quaternions are a fascinating Mathematical construct and extend the concept of complex numbers. Instead of two real components like complex numbers, quaternions have four real components. These numbers play important roles across many areas of Mathematics generally as algebraic systems, signal processing, differential geometry, and quantum mechanics, etc.

Matrices over commutative rings received attention but, matrices having noncommuting entries (quaternion entries) have not been investigated very much yet. This is basically due to intrinsic algebraic difficulties that appear with respect to their non-commutativity. During the last decade, a large amount of research has been concentrated on Toeplitz matrices over the field of complex numbers, while their study over quaternions is quite negligible.

Hamilton first introduced the set of real quaternions (see [8, 9]), while the seminal work concerning commutative quaternions was first presented by [21]. Kosal and Tosun [17, 18] investigated some algebraic properties of commutative quaternion matrices using complex representations of commutative quaternion matrices. We refer the reader to [3, 5, 10, 16, 17, 18, 19, 23, 24] and [1, 25, 28] for a detailed study of quaternions and their matrices. In [6], the authors provide basic properties of Toeplitz and Hankel matrices over the algebra of complex numbers; most of the results therein deal with the products of these structured matrices, which, in general, are not structured over the algebra of quaternions. The most usual and basic reference for complex Toeplitz matrices is Grenander and Szego [5].

The collection of quaternion Toeplitz matrices is not closed with respect to the multiplication of matrices. So, it is interesting to find classes of quaternion Toeplitz matrices that have the structure of left vector space as well as the structure of ring, that is, left algebras of Toeplitz matrices (see [16] about the theory of Linear Algebra over Division ring). The charactarization of maximal commutative algebras of scalar Toeplitz matrices was carried out by [22]. Building on this, the authors of [11, 12, 13, 14, 15] explored the algebraic properties of block Toeplitz matrices and their maximal algebras in greater depth. In particular, they extended the characterization by considering block Toeplitz matrices whose entries all belong to a prescribed maximal commutative subalgebra of scalar matrices.

The general problem of characterizing the left algebras (or right algebras) of quaternion Toeplitz matrices is a very hard problem and no work has been done hitherto. The purpose of the present paper is to obtain the classification of maximal left algebras of quaternion Toeplitz matrices. It is probably too tough to hope for a complete classification, but the purpose is to identify possible classes of such left algebras for quaternion Toeplitz matrices.

This paper is structured as follows: By means of Section 2, we want to make sure that the reader has become familiar with quaternions and their algebraic properties required when we start the main work in upcoming sections. In Section 3, we will introduce Toeplitz matrices over the algebra of quaternions. In the last Section, we introduce a certain class of maximal left algebras of quaternion Toeplitz matrices and prove some fundamental results concerning it.

2 Quaternions and their Basic Algebraic Properties

In this section, the main object of study is the set of real quaternions and their algebraic properties. We begin with their formal definition.

Definition 2.1. The set of real quaternions denoted by $\mathbb{H}$ and is defined as

$$\mathbb{H} := \Big\{\alpha = \alpha_0 + \alpha_1 i + \alpha_2 j + \alpha_3 k : \alpha_0, \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}\Big\},$$

where $i, j, k \notin \mathbb{R}$ are versors satisfying the following multiplication rule:

$$i^2 = j^2 = k^2 = ijk = -1.$$

Since quaternion arithmetic is defined by the behavior of versors, one can also derive lots of other relations from these, for instance, $ij = k$, $ki = j$, $jk = i$, $ji = -k$, $ik = -j$, $kj = -i$ and $jki = -1$. As an additive group $\mathbb{H}$ is isomorphic to four copies of $\mathbb{R}$, i.e., $\mathbb{R}\oplus\mathbb{R}\oplus\mathbb{R}\oplus\mathbb{R}$. The map $\alpha_0 + \alpha_1 i + \alpha_2 j + \alpha_3 k \longrightarrow (\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ is clearly a group isomorphism of $\mathbb{H}$ onto $\mathbb{R}\oplus\mathbb{R}\oplus\mathbb{R}\oplus\mathbb{R}$.

For any quaternion $\alpha = \alpha_0+\alpha_1 i+\alpha_2 j+\alpha_3 k$, where $\alpha_0, \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$, we define $\Re(\alpha) = \alpha_0$, the real part of α, and $\Im(\alpha) = \alpha_1 i + \alpha_2 j + \alpha_3 k$, the vector part (or imaginary part) of α. The conjugate of α is defined by $\alpha_0 - \alpha_1 i - \alpha_2 j - \alpha_3 k = \Re(\alpha) - \Im(\alpha)$ and is denoted as $\overline{\alpha}$.

The real quaternions are the obvious generalization of complex numbers, we define their addition and multiplication as follows:
If $\alpha = \alpha_0 + \alpha_1 i + \alpha_2 j + \alpha_3 k$ and $\beta = \beta_0 + \beta_1 i + \beta_2 j + \beta_3 k$ are in $\mathbb{H}$, then

$$\alpha + \beta = (\alpha_0 + \beta_0) + (\alpha_1 + \beta_1)i + (\alpha_2 + \beta_2)j + (\alpha_3 + \beta_3)k,$$

and

$$\begin{aligned}\alpha\beta = {} & (\alpha_0\beta_0 - \alpha_1\beta_1 - \alpha_2\beta_2 - \alpha_3\beta_3)\\ & + (\alpha_0\beta_1 + \alpha_1\beta_0 + \alpha_2\beta_3 - \alpha_3\beta_2)i\\ & + (\alpha_0\beta_2 + \alpha_2\beta_0 + \alpha_1\beta_3 - \alpha_3\beta_1)j\\ & + (\alpha_0\beta_3 + \alpha_3\beta_0 + \alpha_1\beta_2 - \alpha_2\beta_1)k.\end{aligned}$$

It is easy to see that with respect to the above operations, $\mathbb{H}$ is a division algebra and also that under multiplication quaternions are not commutative. Due to this reason, one must take some care in order to perform the multiplication of quaternions. The reality that quaternions form a division algebra is among their most fundamental properties. This shows that every non zero quaternion is invertible with respect to multiplication, which is quite rare in higher dimensional algebras, just as real and complex numbers algebras do. Under the usual operation of addition and scalar multiplication, $\mathbb{H}$ is a four-dimensional vector space over $\mathbb{R}$.

The following theorem from [28] summarizes some of the main algebraic properties of quaternions.

Theorem 2.2. *Let $\alpha = \alpha_0+\alpha_1 i+\alpha_2 j+\alpha_3 k$ and $\beta = \beta_0+\beta_1 i+\beta_2 j+\beta_3 k$ in $\mathbb{H}$, then*

(i) Every α can be expressed in a unique way as $\alpha = \gamma_0+\gamma_1 j$, $\gamma_0, \gamma_1 \in \mathbb{C}$;

(ii) In general, $(\alpha+\beta)^2 \neq \alpha^2 + 2\alpha\beta + \beta^2$;

(iii) $\alpha^2 + 1 = 0$ has infinitely many roots over $\mathbb{H}$.

It is also notable that one can express any quaternion in terms of a 2×2 matrix having complex entries. Let $\alpha = \alpha_0 + \alpha_1 i + \alpha_2 j + \alpha_3 k$ is in $\mathbb{H}$. Expressing the versors $1, i, j$, and k as

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad i = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad k = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Then simple computation imply that $i^2 = j^2 = k^2 = -1$ and $ij = k$, $jk = i$, and $ki = j$. So α takes the following form

$$\begin{aligned} \alpha &= \alpha_0 + \alpha_1 j + \alpha_2 j + \alpha_3 k \\ &= \begin{pmatrix} \alpha_0 + \alpha_1 i & \alpha_2 + \alpha_3 i \\ -\alpha_2 + \alpha_3 i & \alpha_0 - \alpha_1 i \end{pmatrix}. \end{aligned}$$

We now quote from [2], the definition of the inner product on $\mathbb{H}^n$.

Definition 2.3. Let $n \in \mathbb{Z}^+$, and $x = \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n-1} \end{pmatrix}$ and $y = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{pmatrix}$ be in $\mathbb{H}^n$. The quaternion valued function $< \cdot, \cdot >: \mathbb{H}^n \times \mathbb{H}^n \longrightarrow \mathbb{H}$ given by

$$< x, y >= \sum_{k=0}^{n-1} \overline{y_k}\, x_k$$

defined an inner product on $\mathbb{H}^n$.

With this inner product, $\mathbb{H}^n$ is a left inner product space over $\mathbb{H}$ and likewise the unitary space $\mathbb{C}^n$, the set of vectors

$$e_0 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, e_1 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \cdots, e_{n-1} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

forms an orthonormal basis for $\mathbb{H}^n$(these vectors also serve as the standard Hamel Basis for $\mathbb{H}^n$).

3 Toeplitz Matrices with Quaternion Entries

Throughout we label the indices of any matrix A from 0 to $n-1$. We denote by $\mathfrak{M}_n[\mathbb{H}]$, the set of all square matrices with quaternion entries. Thus if $A \in \mathfrak{M}_n[\mathbb{H}]$ then $A = (\alpha_{rs})_{r,s=0}^{n-1}$, with $\alpha_{rs} \in \mathbb{H}$ for every $0 \leq r, s \leq n-1$.

If $A = (\alpha_{rs})_{r,s=0}^{n-1}$, $B = (\beta_{rs})_{r,s=0}^{n-1}$ are in $\mathfrak{M}_n[\mathbb{H}]$ and $\alpha \in \mathbb{H}$, then define matrix addition and scalar multiplication component-wise as follows:

$$A + B = (\alpha_{rs} + \beta_{rs})_{r,s=0}^{n-1},$$
$$\alpha A = (\alpha\alpha_{rs})_{r,s=0}^{n-1}.$$

With respect to the above defining operations, $\mathfrak{M}_n[\mathbb{H}]$ is a left vector space over $\mathbb{H}$. If one multiply $A = (\alpha_{rs})_{r,s=0}^{n-1} \in \mathfrak{M}_n[\mathbb{H}]$ by $\alpha \in \mathbb{H}$ from right then, $\mathfrak{M}_n[\mathbb{H}]$ is also a right vector space over $\mathbb{H}$. Now we define a Toeplitz matrix, whose entries all belong to $\mathbb{H}$.

Definition 3.1. A finite square matrix is called a quaternion Toeplitz matrix if its entries along each negative sloping diagonal are constant. That is, the matrix $A = (\alpha_{rs})_{r,s=0}^{n-1}$ is quaternion Toeplitz if $\alpha_{r_1,s_1} = \alpha_{r_2,s_2}$ whenever $r_1 - s_1 = r_2 - s_2$, for all $r_1, s_1, r_2, s_2 = 1-n, 2-n, \cdots, -1, 0, 1, \cdots, n-1$.

From the definition, it is clear that a quaternion Toeplitz matrix of size n^2 depends upon $2n-1$ parameters $1-n, 2-n, \cdots, -1, 0, 1, \cdots, n-$

1. The word "quaternion" refers to the fact that in the above matrix representation, the entries are from the algebra of quaternions. Thus, if $A \in \mathfrak{M}_n[\mathbb{H}]$ is Toeplitz, then it has the following structure:

$$A = \begin{pmatrix} \alpha_0 & \alpha_{-1} & \alpha_{-2} & \cdots & \alpha_{1-n} \\ \alpha_1 & \alpha_0 & \alpha_{-1} & \cdots & \alpha_{2-n} \\ \alpha_2 & \alpha_1 & \alpha_0 & \cdots & \alpha_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_{n-1} & \alpha_{n-2} & \alpha_{n-3} & \cdots & \alpha_0 \end{pmatrix}, \ \alpha_r \in \mathbb{H} \text{ for all } 1-n \leq r \leq n-1.$$

We denote by $\mathfrak{T}_n[\mathbb{H}]$ the set of all Toeplitz matrices with entries from $\mathbb{H}$. If $A = (\alpha_{r-s})_{r,s=0}^{n-1}$ is in $\mathfrak{T}_n[\mathbb{H}]$, then A is called a quaternion circulant matrix if $\alpha_r = \alpha_{r-n}$ for every $1 \leq r \leq n-1$.

Suppose $\mathfrak{S}$ be the matrix consisting of zeros except for ones below the principal diagonal, i.e.,

$$\mathfrak{S} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

Then its adjoint $\mathfrak{S}^*$ is the matrix given as

$$\mathfrak{S}^* = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

It is clear that both $\mathfrak{S}$ and $\mathfrak{S}^*$ are in $\mathfrak{T}_n[\mathbb{H}]$. Recall that if x and y are in $\mathbb{H}^n$, then the tensor product $x \otimes y$ is a square matrix of size n^2 over $\mathbb{H}$ and is defined as $x \otimes y(z) =< z, y > x, \quad \text{for every} \quad x, y \in \mathbb{H}^n$.

The following result characterized all the matrices of $\mathfrak{T}_n[\mathbb{H}]$ among the matrices of $\mathfrak{M}_n[\mathbb{H}]$. We are giving here its detailed proof.

Proposition 3.2. *$A \in \mathfrak{M}_n[\mathbb{H}]$ is in $\mathfrak{T}_n[\mathbb{H}]$ if and only if there exist vectors x and y in $\mathbb{H}^n$ such that*

$$A - \mathfrak{S}A\mathfrak{S}^* = x \otimes e_0 + e_0 \otimes y.$$

Proof. Suppose that A is in $\mathfrak{T}_n[\mathbb{H}]$, then A has the form $A = (\alpha_{r-s})_{r,s}^{n-1}$. Simple computation yields that

$$A - \mathfrak{S}A\mathfrak{S}^* = \begin{pmatrix} \alpha_0 & \alpha_{-1} & \alpha_{-2} & \cdots & \alpha_{1-n} \\ \alpha_1 & 0 & 0 & \cdots & 0 \\ \alpha_2 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_{n-1} & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

If one take $x = \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{n-1} \end{pmatrix}$ and $y = \begin{pmatrix} 0 \\ \alpha_{-1} \\ \vdots \\ \alpha_{1-n} \end{pmatrix}$, then it is easy to see that

$$A - \mathfrak{S}A\mathfrak{S}^* = x \otimes e_0 + e_0 \otimes y.$$

In order to establish converse, let $A = (\alpha_{rs})_{r,s=0}^{n-1}$ be any arbitrary $n \times n$ matrix such that for some $x = \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{n-1} \end{pmatrix}$ and $y = \begin{pmatrix} \delta_0 \\ \delta_1 \\ \vdots \\ \delta_{n-1} \end{pmatrix}$ in $\mathbb{H}^n$, the identity $A - \mathfrak{S}A\mathfrak{S}^* = x \otimes e_0 + e_0 \otimes y$ holds. In the standard basis of $\mathbb{H}^n$, one has

$$x \otimes e_0(e_r) = \begin{cases} \displaystyle\sum_{k=0}^{n-1} \alpha_k e_k & \text{for} \quad r = 0, \\ 0 & \text{for} \quad r = 1, 2, \cdots, n-1, \end{cases}.$$

and

$$e_0 \otimes y(e_r) = \delta_r e_0, \quad \text{for} \quad r = 0, 1, \cdots, n-1.$$

Therefore

$$x \otimes e_0 + e_0 \otimes y = \begin{pmatrix} \alpha_0 + \delta_0 & \delta_1 & \cdots & \delta_{n-1} \\ \alpha_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n-1} & 0 & \cdots & 0 \end{pmatrix}. \tag{1}$$

On the other hand, one note that

$$A - \mathfrak{S}A\mathfrak{S}^* = \begin{pmatrix} \alpha_{00} & \alpha_{01} & \cdots & \alpha_{0,n-1} \\ \alpha_{10} & \alpha_{11} - \alpha_{00} & \cdots & \alpha_{1,n-1} - \alpha_{0,n-2} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{1-n,0} & \alpha_{n-1,1} - \alpha_{n-2,0} & \cdots & \alpha_{n-1,n-1} - \alpha_{n-2,n-2} \end{pmatrix}. \tag{2}$$

Comparing corresponding entries of (1) and (2), one sees that $\alpha_{r,s} = \alpha_{r-1,s-1}$ for every $1 \leq r, s \leq n-1$. This shows that A is in $\mathfrak{T}_n[\mathbb{H}]$, which is what we wanted to prove. $\square$

Proposition 3.3. *$\mathfrak{T}_n[\mathbb{H}]$ is a subspace of the left vector space $\mathfrak{M}_n[\mathbb{H}]$.*

Note that $A = (\alpha_{r-s})_{r,s=0}^{n-1}$ with

$$\alpha_{r-s} = \begin{cases} i & \text{if} \quad r = s+1, \\ j & \text{if} \quad r = s-1, \\ 0 & \text{otherwise}, \end{cases}$$

is purely a quaternion Toeplitz the matrix, but its product

$$A^2 = \begin{pmatrix} k & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & -1 & \cdots & 0 \\ -1 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -k \end{pmatrix}$$

is not a quaternion Toeplitz matrix. Thus, likewise the case of complex entries, the product of two quaternion Toeplitz matrices needs not be a quaternion Toeplitz matrix.

The following proposition gives us the precise criteria in this direction.

Proposition 3.4. *Suppose that* $A = (\alpha_{r-s})_{r,s=0}^{n-1}$ *and* $B = (\beta_{r-s})_{r,s=0}^{n-1}$ *are Toeplitz matrices with quaternion entries, then* AB *is in* $\mathfrak{T}_n[\mathbb{H}]$ *if and only if*

$$\alpha_r\beta_{s-n} = \alpha_{r-n}\beta_s \quad \textit{for all} \quad r, s = 1, 2, \ldots n-1. \tag{3}$$

Proof. Suppose that AB is in $\mathfrak{T}_n[\mathbb{H}]$. Let us denote the product AB by $C = (\gamma_{r,s})_{r,s=0}^{n-1}$, then for every $r, s = 1, 2, \ldots, n-1$, we have

$$\gamma_{r,n-s} = \sum_{k=0}^{n-1} \alpha_{r-k}\beta_{k+s-n}, \tag{4}$$

$$\gamma_{r-1,n-s-1} = \sum_{k=0}^{n-1} \alpha_{r-k-1}\beta_{k+s-n+1}. \tag{5}$$

Subtracting (4) and (5) yields

$$\gamma_{r,n-s} - \gamma_{r-1,n-s-1} = \alpha_r\beta_{s-n} - \alpha_{r-n}\beta_r. \tag{6}$$

Since the product AB is a quaternion Toeplitz matrix, its elements along the negative sloping diagonals have the same value, consequently from (6)

$$\alpha_r\beta_{s-n} = \alpha_{r-n}\beta_r \quad \text{for every} \quad r, s = 1, 2, \ldots, n-1.$$

Conversely suppose that the identity (3) is true, then for every $r, s = 1, 2, \ldots, n-1$, we have

$$\begin{aligned}\gamma_{r,n-s} - \gamma_{r-1,n-s-1} &= \alpha_r\beta_{s-n} - \alpha_{r-n}\beta_r \\ &= 0.\end{aligned}$$

This shows that AB is a quaternion Toeplitz matrix. □

Proposition 3.5. *If* $A = (\alpha_{r-s})_{r,s=0}^{n-1}$ *and* $B = (\beta_{r-s})_{r,s=0}^{n-1}$ *are in* $\mathfrak{T}_n[\mathbb{H}]$ *with commuting entries, and their product* AB *is also in* $\mathfrak{T}_n[\mathbb{H}]$, *then* A *and* B *commute with each other.*

Proof. We have for every $r, s = 0, 1, \cdots, n-1$,

$$(AB)_{r,s} = \sum_{k=0}^{n-1} \alpha_{r-k}\beta_{k-s},$$
$$(BA)_{r,s} = \sum_{k=0}^{n-1} \beta_{r-k}\alpha_{k-s}.$$

Since entries are commuting, rewriting the sum by denoting $k' = r+s-k$, we obtain

$$(BA)_{r,s} = \sum_{k=0}^{n-1} \alpha_{k-s}\beta_{r-k} = \sum_{k'=r+s-(n-1)}^{r+s} \alpha_{r-k'}\beta_{k'-s}.$$

If $r+s = n-1$, then the above sum is the same as the formula for $(AB)_{r,s}$. So, $(BA)_{r,s} = (AB)_{r,s}$. Now, for instance, let us suppose that $r+s < n-1$. Then only part of the sum is the same, and from the rest, we obtain

$$\begin{aligned}(AB)_{r,s} - (BA)_{r,s} &= \sum_{k=r+s+1}^{n-1} \alpha_{r-k}\beta_{k-s} - \sum_{r+s-(n-1)}^{-1} \alpha_{r-k}\beta_{k-s} \\ &= \sum_{k=r+s+1}^{n-1} (\alpha_{r-k}\beta_{k-s} - \alpha_{r-k+n}\beta_{k-s-n}).\end{aligned}$$

Applying Proposition 3.4, we get $(AB)_{r,s} - (BA)_{r,s} = 0$. The proof for the case $r+s > n-1$ can be established in a similar fashion. $\square$

The converse of the above proposition is not true in general. This is because, if one takes

$$A = \begin{pmatrix} 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 & 0 & \cdots & k \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ k & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Then A and B are commuting quaternion Toeplitz matrices but their product

$$AB = \begin{pmatrix} k & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & k \end{pmatrix}.$$

is not a quaternion Toeplitz matrix.

4 Algebras of Toeplitz Matrices with Quaternion Entries

Fixing an algebra $\mathfrak{A}$ in $\mathbb{H}$ and $\sigma, \rho \in \mathfrak{A}'$, where $\mathfrak{A}'$ denotes the commutant of $\mathfrak{A}$ (the set of all quaternions commuting with each element of $\mathfrak{A}$). It is easy to see that $\mathfrak{A}'$ is also an algebra. We symbolize by $\mathfrak{T}_n[\mathfrak{A}]$ the left vector space of Toeplitz matrices contained in $\mathfrak{T}_n[\mathbb{H}]$. We define the family by

$$\mathfrak{G}_{\sigma,\rho}[\mathfrak{A}] := \left\{ A = (\alpha_{r-s})_{r,s=0}^{n-1} | \alpha_r \in \mathfrak{A},\ \sigma\alpha_{r-n} = \rho\alpha_r,\ \text{for all } r = 1, 2, \cdots, n-1 \right\}.$$

We will use the following simple Lemma.

Lemma 4.1. *Suppose that $\sigma, \rho \in \mathbb{H}$ be fixed. If α is any arbitrary element of $\mathbb{H}$ such that $\sigma\alpha = \rho\alpha = 0$, then $\alpha = 0$.*

Proof. Since $\mathbb{H}$ is a division algebra, the left multiplication by any nonzero element is an injective operation. This means that if $\sigma\alpha = 0$ for nonzero σ, then necessarily $\alpha = 0$. Similarly, if $\rho\alpha = 0$ for nonzero ρ, then again $\alpha = 0$. Since we assume at least one of σ or ρ is nonzero, it follows immediately that $\alpha = 0$. Thus, the desired conclusion holds. $\square$

The above Lemma enables us to prove the following main result of this section.

Theorem 4.2. *The family $\mathfrak{G}_{\sigma,\rho}[\mathfrak{A}]$ forms a left algebra in $\mathfrak{T}_n[\mathfrak{A}]$.*

Proof. A simple straightforward verification shows that $\mathfrak{G}_{\sigma,\rho}[\mathfrak{A}]$ is a left subspace of $\mathfrak{T}_n[\mathbb{H}]$. We need to only show that it is closed up to the usual multiplication of matrices. For this, let us suppose that A and B be any two arbitrary elements of $\mathfrak{G}_{\sigma,\rho}[\mathfrak{A}]$, then we must have

$$\begin{aligned} A &= (\alpha_{r-s}), \quad \sigma\alpha_{r-n} = \rho\alpha_r, \\ B &= (\beta_{r-s}), \quad \sigma\beta_{r-n} = \rho\beta_r. \end{aligned} \tag{7}$$

We have

$$\begin{aligned} (AB)_{r,s} - (AB)_{r+1,s+1} &= \sum_{k=0}^{n-1} \alpha_{r-k}\beta_{k-s} \\ &= \alpha_{r-n+1}\beta_{n-1-s} + \alpha_{r+1}\beta_{-1-s}. \end{aligned}$$

Using formulas given in (7) and multiplying with σ and ρ, it follows that

$$\begin{aligned} \sigma\left[(AB)_{r,s} - (AB)_{r+1,s+1}\right] = 0, \\ \rho\left[(AB)_{r,s} - (AB)_{r+1,s+1}\right] = 0. \end{aligned}$$

By applying Lemma 4.1, we obtained that

$$(AB)_{r,s} - (AB)_{r+1,s+1} = 0.$$

Consequently, AB is in $\mathfrak{T}_n[\mathfrak{A}]$. We denote the product AB by $C = (\gamma_{r,s})_{r,s=0}^{n-1}$, and since $\sigma, \rho \in \mathfrak{A}'$, we have

$$\rho\gamma_r = \rho(AB)_{r,0} = \rho\sum_{k=0}^{n-1}\alpha_{k-r}\beta_k = \sum_{k=0}^{n-1}\alpha_{k-r}\sigma\beta_{k-n} = \sigma\sum_{k=0}^{n-1}\alpha_{k-r}\beta_{k-n} = \sigma\gamma_{r-n}.$$

Therefore, the product AB is in $\mathfrak{G}_{\sigma,\rho}[\mathfrak{A}]$. This proves, along with the previous fact that $\mathfrak{G}_{\sigma,\rho}[\mathfrak{A}]$ is a left vector space, that $\mathfrak{G}_{\sigma,\rho}[\mathfrak{A}]$ is a left algebra in $\mathfrak{T}_n[\mathfrak{A}]$. $\square$

The following result shows that these algebras include an important general class of quaternion Toeplitz matrices algebras.

Theorem 4.3. *Suppose that $\mathfrak{A}$ be an inverse closed algebra in $\mathbb{H}$ and $\mathfrak{G}$ an algebra contained in $\mathfrak{T}_n[\mathfrak{A}]$. Let $\mathfrak{G}$ contains an element whose at least one off-diagonal entry is invertible. Then $\mathfrak{G} \subset \mathfrak{G}_{\sigma,\rho}[\mathfrak{A}]$ for some $\sigma, \rho \in \mathfrak{A}$.*

Proof. Let $A = (\alpha_{r-s})_{r,s=0}^{n-1}$ be in $\mathfrak{G}$ such that $\alpha_r \in \mathfrak{A}$ and is invertible for some nonzero r. Furthermore, suppose that $B = (\beta_{r-s})_{r,s=0}^{n-1}$ be any arbitrary element of $\mathfrak{G}$. Proposition 3.4 implies that since $\mathfrak{G}$ is an algebra included in $\mathfrak{T}_n[\mathfrak{A}]$, then the product AB is in $\mathfrak{G}$ if and only if

$$\alpha_r \beta_{s-n} = \alpha_{r-n} \beta_s \quad (s = 1, 2, \cdots n-1). \tag{8}$$

Let us assume, for instance, that $r > 0$, it follows from (8) that

$$\beta_{s-n} = \alpha_r^{-1} \alpha_{r-n} \beta_s \quad (s = 1, 2, \cdots n-1),$$

as we have assumed that α_r is invertible. Due to the inverse closeness of $\mathfrak{A}$, $\alpha_r^{-1}\alpha_{r-n} \in \mathfrak{A}$. Thus, $\mathfrak{G} \subset \mathfrak{G}_{\sigma,\rho}[\mathfrak{A}]$ gets obtained if one take $\sigma = 1$ and $\rho = \alpha_r^{-1}\alpha_{r-n}$. The proof can be completed for $r < 0$ by using a similar argument. □

Apart from that, as can be seen below, the algebra $\mathfrak{G}_{\sigma,\rho}[\mathfrak{A}]$ covers several other classes of the algebras of $\mathfrak{T}_n[\mathfrak{A}]$.

- If $\sigma = \rho = 1$, then $\mathfrak{G}_{1,1}[\mathfrak{A}]$ is the left algebra of all quaternion circulant matrices.
- If $\sigma = 0$, then $\mathfrak{G}_{0,\rho}[\mathfrak{A}]$ is the left algebra of all upper triangular quaternion Toeplitz matrices.
- Similarly, if $\rho = 0$, then $\mathfrak{G}_{\sigma,0}[\mathfrak{A}]$ is the left algebra of all lower triangular quaternion Toeplitz matrices.
- $\mathfrak{G}_{\sigma,\rho}[\mathfrak{A}]$ obviously contains the algebra of diagonal quaternion Toeplitz matrices.

The below result concerns the commutativity of $\mathfrak{G}_{\sigma,\rho}[\mathfrak{A}]$, and its proof follows directly from Proposition 3.5 in a straightforward manner.

Proposition 4.4. *If $\mathfrak{A}$ is commutative then $\mathfrak{G}_{\sigma,\rho}[\mathfrak{A}]$ is commutative.*

The following result describes when two left algebras of this type are equal.

Theorem 4.5. *Suppose that there also exist $\breve{\sigma}, \breve{\rho} \in \mathfrak{A}$, then $\mathfrak{G}_{\sigma,\rho}[\mathfrak{A}] = \mathfrak{G}_{\breve{\sigma},\breve{\rho}}[\mathfrak{A}]$ if and only if $\sigma\breve{\rho} = \breve{\sigma}\rho$.*

Proof. Suppose that $\sigma\breve{\rho} = \breve{\sigma}\rho$, and $A = (\alpha_{r-s})_{r,s=0}^{n-1}$ is in $\mathfrak{G}_{\sigma,\rho}[\mathfrak{A}]$, then $\sigma\alpha_{r-n} = \rho\alpha_r$. Multiplying the identity $\sigma\alpha_{r-n} = \rho\alpha_r$ with $\breve{\sigma}$, we have

$$\begin{aligned}\breve{\sigma}\sigma\alpha_{r-n} &= \breve{\sigma}\rho\alpha_r \\ &= \sigma\breve{\rho}\alpha_r\end{aligned}$$

or

$$\sigma(\breve{\sigma}\alpha_{r-n} - \breve{\rho}\alpha_r) = 0.$$

In the same way, multiplying both sides of $\sigma\alpha_{r-n} = \rho\alpha_r$ with $\breve{\rho}$, we get

$$\rho(\breve{\sigma}\alpha_{r-n} - \breve{\rho}\alpha_r) = 0.$$

Lemma 4.1 then imply that $\breve{\sigma}\alpha_{r-n} = \breve{\rho}\alpha_r$. Consequently, $A \in \mathfrak{G}_{\breve{\sigma},\breve{\rho}}[\mathfrak{A}]$ and therefore $\mathfrak{G}_{\sigma,\rho}[\mathfrak{A}] \subseteq \mathfrak{G}_{\breve{\sigma},\breve{\rho}}[\mathfrak{A}]$. Now for the reverse inclusion, let $A = (\alpha_{r-s})_{r,s=0}^{n-1} \in \mathfrak{G}_{\breve{\sigma},\breve{\rho}}[\mathfrak{A}]$, then we have $\breve{\sigma}\alpha_{r-n} = \breve{\rho}\alpha_r$. Multiplying the equation $\breve{\sigma}\alpha_{r-n} = \breve{\rho}\alpha_r$ by ρ and using $\sigma\breve{\rho} = \breve{\sigma}\rho$, we get $\sigma\alpha_{r-n} = \rho\alpha_r$. Therefore, $\mathfrak{G}_{\breve{\sigma},\breve{\rho}}[\mathfrak{A}] \subseteq \mathfrak{G}_{\sigma,\rho}[\mathfrak{A}]$.

Conversely suppose that $\mathfrak{G}_{\sigma,\rho}[\mathfrak{A}] = \mathfrak{G}_{\breve{\sigma},\breve{\rho}}[\mathfrak{A}]$. We will show that $\sigma\breve{\rho} = \breve{\sigma}\rho$. Since the matrix

$$\tilde{A}_{\sigma,\rho} = \begin{pmatrix} 0 & \rho & \rho \cdots & \rho \\ \sigma & 0 & \rho \cdots & \rho \\ \sigma & \sigma & 0 \cdots & \rho \\ \vdots & \vdots & \vdots \ddots & \vdots \\ \sigma & \sigma & \sigma \cdots & 0 \end{pmatrix}$$

is in $\mathfrak{G}_{\sigma,\rho}[\mathfrak{A}] = \mathfrak{G}_{\breve{\sigma},\breve{\rho}}[\mathfrak{A}]$, and therefore $\breve{\rho}\sigma = \breve{\sigma}\rho$. Finishing the proof. □

The next result regards $\mathfrak{G}_{\sigma,\rho}[\mathfrak{A}]$ maximality as a left algebra in $\mathfrak{T}_n[\mathfrak{A}]$.

Theorem 4.6. *$\mathfrak{G}_{\sigma,\rho}[\mathfrak{A}]$ is maximal left algebra in $\mathfrak{T}_n[\mathfrak{A}]$ if and only if $\{\sigma,\rho\}' = \mathfrak{A}$.*

Proof. Suppose that $\mathfrak{G}_{\sigma,\rho}[\mathfrak{A}]$ is a maximal left algebra in $\mathfrak{T}_n[\mathfrak{A}]$. We will show that $\{\sigma,\rho\}' = \mathfrak{A}$. Let $\{\sigma,\rho\}' = \breve{\mathfrak{A}}$. Since σ, ρ are in the commutant of $\mathfrak{A}$ then by definition $\{\sigma,\rho\}' \supset \mathfrak{A}$ and as a consequence, we have $\mathfrak{G}_{\sigma,\rho}[\mathfrak{A}] \subset \mathfrak{G}_{\sigma,\rho}[\breve{\mathfrak{A}}]$. This shows that $\mathfrak{G}_{\sigma,\rho}[\mathfrak{A}]$ is not a maximal left algebra, which is a contradiction to our assumption. Thus the maximality of $\mathfrak{G}_{\sigma,\rho}[\mathfrak{A}]$ imply that $\{\sigma,\rho\}' = \mathfrak{A}$.

For the converse, let us assume that $\{\sigma,\rho\}' = \mathfrak{A}$ and $\mathfrak{G}$ be any arbitrary left algebra of quaternion Toeplitz matrices such that $\mathfrak{G}_{\sigma,\rho}[\mathfrak{A}] \subseteq \mathfrak{G}$. Let $A = (\alpha_{r-s})_{r,s=0}^{n-1}$ be any arbitrary element of $\mathfrak{G}$. Since $\{\sigma,\rho\}' = \mathfrak{A}$, then the matrix $\tilde{G}_{\sigma,\rho} = (g_{r-s})_{r,s=0}^{n-1}$ given as

$$g_{r-s} = \begin{cases} \sigma & \text{if} \quad r = s+1, \\ \rho & \text{if} \quad r = 0, s = n-1, \\ 0 & \text{otherwise.} \end{cases}$$

is in $\mathfrak{G}_{\sigma,\rho}[\mathfrak{A}]$. The product

$$\tilde{G}_{\sigma,\rho}A = \begin{pmatrix} \rho\alpha_{n-1} & \rho\alpha_{n-2} & \cdots & \rho\alpha_0 \\ \sigma\alpha_0 & \sigma\alpha_{-1} & \cdots & \sigma\alpha_{1-n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma\alpha_{n-2} & \sigma\alpha_{n-3} & \cdots & \sigma\alpha_{-1} \end{pmatrix}$$

is in $\mathfrak{G}$ because $\mathfrak{G}_{\sigma,\rho}[\mathfrak{A}] \subseteq \mathfrak{G}$. Because $\mathfrak{G}$ is assumed to be a left algebra of quaternion Toeplitz matrices, the resulting matrix $\tilde{G}_{\sigma,\rho}A$ must also have a Toeplitz structure. By explicitly comparing diagonal entries of $\tilde{G}_{\sigma,\rho}A$, we obtain the identity $\sigma\alpha_{r-n} = \rho\alpha_r$ for all $r = 1, 2, \cdots, n-1$. This shows that A is in $\mathfrak{G}_{\sigma,\rho}[\mathfrak{A}]$ and hence $\mathfrak{G} = \mathfrak{G}_{\sigma,\rho}[\mathfrak{A}]$, i.e., $\mathfrak{G}_{\sigma,\rho}[\mathfrak{A}]$ is maximal in $\mathfrak{T}_n[\mathfrak{A}]$. This is what we wanted to prove. $\square$

Example 4.7. Suppose that y is one of i, j, k, then define $\mathfrak{A}_y$ as

$$\mathfrak{A}_y = \{a + by : y \in \{i, j, k\}\},$$

then it is straightforward to check that $\mathfrak{A}_y$ is a subalgebra of $\mathbb{H}$, and $\{\sigma,\rho\}' = \mathfrak{A}_y$, where $\sigma = 1 + y$ and $\rho = 2 - y$. We have

$$\mathfrak{G}_{1+y,2-y}[\mathfrak{A}_y] = \left\{ \begin{pmatrix} \alpha_0 & \alpha_1 & \cdots & \alpha_{n-1} \\ \frac{3+y}{2}\alpha_{n-1} & \alpha_0 & \cdots & \alpha_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{3+y}{2}\alpha_1 & \frac{3+y}{2}\alpha_2 & \cdots & \alpha_0 \end{pmatrix} : \alpha_r \in \mathfrak{A}_y \quad \text{for all} \quad 0 \le r \le n-1 \right\}.$$

By using the procedures described in Theorem 4.6, the reader is left with the task of verifying that $\mathfrak{G}_{1+y,2-y}[\mathfrak{A}_y]$ is, in fact, a maximal algebra inside $\mathfrak{T}_n[\mathfrak{A}_y]$.

Acknowledgements
The authors would like to express their sincere gratitude to the referee for their valuable comments and insightful suggestions, which provide important corrections and clarifications and significantly improve the work.

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Muhammad Ahsan Khan
Lecturer of Mathematics
Department of Mathematics
University of Kotli Azad Jammu & Kashmir, 11100, Azad Jammu & Kashmir, Pakistan
Kotli, AJK, Pakistan
E-mail: ahsankhan388@hotmail.com

Ameur Yagoub
Associate Professor of Mathematics
Laboratoire de mathématiques pures et appliquées
Université de Amar Telidji
Laghouat, Algeria
E-mail: a.yagoub@lagh-univ.dz