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Original Research Paper

Seasonal Periodic Autoregressive Processes with Values in Hilbert Spaces

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Abstract. Time series analysis is a widely used technique in data analytics. This paper introduces a new model, the first-order seasonal periodic autoregressive Hilbertian process, designed for functional time series analysis. This model integrates elements of both first-order periodic autoregressive Hilbertian and seasonal Hilbertian autoregressive models. The paper outlines key properties of this process, including its autocovariance operators, and discusses its alignment with the law of large numbers and the central limit theorem.

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1 Introduction

Time series with seasonal or periodic characteristics are widely applied in various domains, including climatology, economics, and hydrology. The classical Box-Jenkins Seasonal Autoregressive Moving Average (SARMA) model was proposed by Box et al. in 1994. It addresses the dependencies of consecutive observations within a period and across different periods, while maintaining constant parameters irrespective of the season. In 1997, Reinsel enhanced this approach with the introduction of Multiplicative Seasonal Vector Autoregressive Moving Average (SVARMA) models. These models, which are stationary, exhibit large norms in their autocorrelation matrices at lags that are multiples of the season, T, and maintain consistent lag h autocorrelations independent of T for $h = 1, \cdots, T$. Despite extensive research on these models, [11] identified numerous applications requiring seasonally varying model parameters.

To address time series data that exhibit periodic structures, scholars have developed and explored periodic time series models with time-varying parameters. Key contributors in this field include [5], [8], [14], [21], [27, 28], [29], and [11, 13]. These processes, often referred to as periodically correlated, are typically nonstationary yet harmonizable and have found extensive application in areas such as signal processing and economics. In the multivariate case, [22, 23] introduced the concept of multivariate periodic ARAM processes. Besides, [4] examined the class of vector autoregressive (VAR) models with periodically varying parameters.

The stochastic nature of autocorrelation in seasonal models distinctly sets them apart from periodic time series models, which display a deterministic autocorrelation pattern, as noted by [24]. In pursuit of a model that integrates both characteristics, it is advantageous to combine seasonal and periodic time series. To this end, [1] pioneered the development of first-order seasonal periodic autoregressive processes, abbreviated SPAR(1,1).

In recent years, statistical inference for processes in abstract spaces has attracted increased attention from researchers. This field focuses primarily on the theoretical foundations of operatorial statistics and the analysis of functional data, [9] and [7]. Among these, Hilbertian autoregressive models, as defined by [2], have gained significant popularity in the analysis of time series in Hilbert spaces. These models, which extend real-valued autoregressive processes, are widely utilized despite their limitations. Consequently, there has been a concerted effort to adapt other well-established real-valued time series models to Hilbertian processes. Notable developments include the study of functional ARMA models by [10], the presentation of periodic autoregressive Hilbertian processes by [18, 19], and the introduction of functional versions of ARCH models by [6]. Furthermore, [30] introduced the concept of pure seasonal functional autoregressive processes of order 1 with seasonality T, abbreviated $\text{SARH}(1)_T$.

In this paper, we aim to expand the concept of first-order seasonal periodic autoregressive processes to include Hilbertian processes. To illustrate the relevance and potential of this extension, lets consider some motivating examples. Consider the analysis of S&P 500 data. In 2022, Maïnassara and Amir applied seasonal periodic autoregressive moving average (SPARMA) models to analyze the daily log returns of the S&P 500 (New York) stock market index from January 4, 1999, to November 20, 2020. While their study focused on the closing values of the index, an alternative approach could involve analyzing intraday 5-minute S&P 500 indexes, as explored by [17]. This approach enables viewing daily S&P 500 indexes as functional data, allowing the analysis of entire functions rather than single closing values. Similarly, functional data analysis has been applied in other contexts, such as electricity demand forecasting, [26]. In 2022, Shah et al. highlighted the complexity of forecasting in such data due to factors like "multiple periodicities reflecting cyclical variations over days, weeks, or seasons." The proposed SPARH models offer the potential to enhance forecasting accuracy in these scenarios by capturing intricate seasonal and periodic dependencies.

The rest of this paper is as organized as follows. Section 2 introduces preliminary notations and definitions. In Section 3, we introduce Seasonal Hilbertian Autoregressive Processes with Periodically Varying Parameters (SPARH), along with their autocovariance operators. Section 4 is dedicated to examining the limiting properties of these processes. Finally, the conclusion and some ideas on future works are presented in Section 5.

2 Preliminary Notations and Definitions

Consider H as a real separable Hilbert space equipped with the Borel σ-algebra, $\mathcal{B}$. This space is furnished with an inner product, denoted as $\langle \cdot, \cdot \rangle_H$, and a corresponding norm, $\|\cdot\|_H$. Furthermore, the space of bounded linear operators acting on H is represented by $\mathcal{L}(H)$. For any operator A within $\mathcal{L}(H)$, the notation A^* is used to signify the adjoint of A.

A random variable, denoted as X, possessing values within a Hilbert space H, is defined as a measurable function that maps from a sample space Ω into H. This mapping adheres to the measurability criteria established by the $\mathcal{F}/\mathcal{B}$ sigma-algebra, where $\mathcal{F}$ represents the Borel field associated with the probability space $(\Omega, \mathcal{F}, P)$. Furthermore, the random variable X is categorized as strongly second-order if the expected value of its squared Hilbert space norm, denoted as $E\|X\|_H^2$, is finite. In the context of this paper, for the sake of brevity and clarity, random variables that are strongly second-order and possess values in H will be referred to as H-valued random variables.

In the realm of zero-mean H-valued random variables X and Y, the covariance and the cross-covariance operators are defined, respectively, as:

$$C_X(x) := E[(X \otimes X)x] = E\langle X, x\rangle_H X, \tag{1}$$

$$C_{X,Y}(x) := E[(X \otimes Y)x] = E\langle X, x\rangle_H Y, \tag{2}$$

where x belongs to H. For any two elements u and v within H, the expression $u \otimes v$ denotes the tensorial products of u and v. It is imperative to note that the expectations in equations (1) and (2) are computed via the Bochner integral.

We define a sequence of T-periodic operators as follows:

Definition 2.1. *A sequence of operators $\{\phi_n, n \in \mathbb{Z}\}$ in the space $\mathcal{L}(H)$ is said to be T-periodic if condition $\phi_n = \phi_{n+T}$ is true for each $n \in \mathbb{Z}$.*

The investigation of time series data crucially involves the study of noise processes. In the context of Hilbertian processes, the concepts of H-white noise and H-strong white noise processes were introduced by [2]. Subsequently, we defined the periodic H-white noise (PHWN) processes.

Definition 2.2. *A zero-mean H-valued process $\{\varepsilon_n, n \in \mathbb{Z}\}$ is characterized as periodic Hilbertian white noise (PHWN) if, for $n = r + kT$, $r = 0, 1, \ldots, T-1$ and $k = 0, 1, \ldots,$ it satisfies the following conditions:*

- $0 < E\left\|\varepsilon_n\right\|_H^2 = \sigma_r^2 < \infty$,
- *for all $n \in \mathbb{Z}$, it holds that $C_{\varepsilon_n} = C_{\varepsilon_r}$,*
- $C_{\varepsilon_n, \varepsilon_{n'}} = 0$, *for $n \neq n'$.*

3 SPARH(1,1) Processses

Consider the H-valued time series $\{X_n,\ n \in \mathbb{Z}\}$ satisfying the seasonal autoregressive difference equation articulated as follows:

$$X_n = \phi_n X_{n-1} + \alpha_n X_{n-T} - \alpha_n \phi_n X_{n-T-1} + \varepsilon_n, \tag{3}$$

where the parameters ϕ_n and α_n represent T-periodic operators within the space $\mathcal{L}(H)$, and the series $\{\varepsilon_n\}$ is characterized as PHWN. We designate the model described by (3) as a first-order seasonal periodic autoregressive Hilbertian process (SPARH(1,1)) with period T. It is easy to show that the solution to Eq. (3) satisfies the subsequent pair of equations:

$$X_n = \phi_n X_{n-1} + Z_n \qquad \textit{and} \qquad Z_n = \alpha_n Z_{n-T} + \varepsilon_n. \tag{4}$$

If $\phi_n \equiv 0$, $\alpha_n \equiv \alpha$, and $C_{\varepsilon_n} \equiv C_0$ for all $n \in \mathbb{Z}$, the SPARH(1,1) model simplifies to the SARH$(1)_T$ model as described by [30]. Note that $\phi_n = 0$ if and only if $\langle \phi_n(x), x \rangle = 0$ for all $x \in H$, [20]. Alternatively, setting $\alpha_n \equiv 0$ for all $n \in \mathbb{Z}$ leads to the formulation of a PCARH(1) model, as introduced by [18]. Consequently, both the SARH$(1)_T$ and PCARH(1) models can be regarded as particular instances of the SPARH(1,1) framework.

Furthermore, the difference equation characterizing the SPARH(1,1) model can be equivalently represented within the framework of a PCARH(p) model, where $p = T + 1$, as presented below:

$$X_n = \sum_{i=1}^{T+1} \phi_{n,i} X_{n-i} + \varepsilon_n, \tag{5}$$

with the conditions $\phi_{n,1} = \phi_n$, $\phi_{n,i} = 0$ for $1 < i < T$, $\phi_{n,T} = \alpha_n$, and $\phi_{n,T+1} = -\alpha_n \phi_n$.

In this context, lets define the vector $\mathbf{X}_n = (X_{nT+1}, \ldots, X_{nT+T})'$ and the error vector $\boldsymbol{\varepsilon}_n = (\varepsilon_{nT+1}, \ldots, \varepsilon_{nT+T})'$. Given these definitions, the SPARH(1,1) model can be expressed as an autoregressive model with values in the Hilbert space H^T :

$$\boldsymbol{\Phi}_0 \mathbf{X}_n = \boldsymbol{\Phi}_1 \mathbf{X}_{n-1} + \boldsymbol{\Phi}_2 \mathbf{X}_{n-2} + \boldsymbol{\varepsilon}_n, \tag{6}$$

where $\boldsymbol{\Phi}_0$, $\boldsymbol{\Phi}_1$ and $\boldsymbol{\Phi}_2$ are $T \times T$ matrices of operators defined as follows:

$$\boldsymbol{\Phi}_0 = \begin{pmatrix} I & 0 & \cdots & 0 & 0 \\ -\phi_2 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -\phi_T & I \end{pmatrix},$$

$$\boldsymbol{\Phi}_1 = \begin{pmatrix} \alpha_1 & 0 & \cdots & 0 & \phi_1 \\ -\alpha_2\phi_2 & \alpha_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -\alpha_T\phi_T & \alpha_T \end{pmatrix},$$

and

$$\boldsymbol{\Phi}_2 = \begin{pmatrix} 0 & 0 & \cdots & 0 & \alpha_1\phi_1 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Lemma 3.1. *Consider the time series $\{X_n,\ n \in \mathbb{Z}\}$, characterized as a SPARH(1,1) process with a periodicity of T, as defined in (3). Then, this process can be equivalently formulated as an ARH^T(2) model:*

$$\mathbf{X}_n = \boldsymbol{\Phi}_0^{-1}\boldsymbol{\Phi}_1 \mathbf{X}_{n-1} + \boldsymbol{\Phi}_0^{-1}\boldsymbol{\Phi}_2 \mathbf{X}_{n-2} + \boldsymbol{\varepsilon}'_n, \tag{7}$$

where the term $\boldsymbol{\varepsilon}'_n := \boldsymbol{\Phi}_0^{-1}\boldsymbol{\varepsilon}_n$ denotes a mean-zero white noise process with covariance operator $\boldsymbol{\Phi}_0^{-1} diag\left(C_{\varepsilon_1}, \cdots, C_{\varepsilon_T}\right)\left(\boldsymbol{\Phi}_0^{-1}\right)^*$.

Proof. The proof is an easy consequence of equation (6). □

Remark 3.2. *The invertibility of the matrix operator* $\mathbf{\Phi}_0$ *is assured due to its bidiagonal unit structure, as explained in [25]. As an example, consider the case where* $T = 3$*. Here, the inverse of* $\mathbf{\Phi}_0$*, denoted as* $\mathbf{\Phi}_0^{-1}$*, is explicitly given by:*

$$\mathbf{\Phi}_0^{-1} = \begin{pmatrix} I & 0 & 0 \\ \phi_2 & I & 0 \\ \phi_3\phi_2 & \phi_3 & I \end{pmatrix}.$$

Define the projection function $\pi(x_1, x_2) = x_1$, where $x_1, x_2 \in H^T$. The following theorem presents a condition required for the existence and uniqueness of the sequence $\{X_n;\ n \in \mathbb{Z}\}$.

Theorem 3.3. *Consider the matrix*

$$\boldsymbol{\rho} = \begin{pmatrix} \mathbf{\Phi}_0^{-1}\mathbf{\Phi}_1 & \mathbf{\Phi}_0^{-1}\mathbf{\Phi}_2 \\ \mathbf{I} & \mathbf{0} \end{pmatrix}.$$

Suppose there exists an integer $j_0 \geq 1$ *such that*

$$\left\|\boldsymbol{\rho}^{j_0}\right\|_{\mathcal{L}(H^{2T})} < 1, \tag{8}$$

then equation (7) possesses a unique stationary solution, expressed as

$$\mathbf{X}_n = \boldsymbol{\mu} + \sum_{j=0}^{\infty} \left(\pi\boldsymbol{\rho}^j\right)\left(\mathbf{\Phi}_0^{-1}\boldsymbol{\varepsilon}_{n-j}\right),\ n \in \mathbb{Z}, \tag{9}$$

where the series converges in $L^2_H(\Omega, \mathcal{A}, P)$ *and with probability 1.*

Proof. The process $\mathbf{X}_n$ is conceptualized as an $\text{ARH}^T(2)$ process as established in Lemma 3.4. The proof of this theorem follows analogously to the methodology employed in Theorem 5.1 of [2]. □

It would desirable to reformulate Equation (8) utilizing the specific parameters of our model, namely $\alpha_1, \ldots, \alpha_T$ and $\phi_1, \ldots, \phi_p$. To ensure the validity of Equation (8), the subsequent lemma proposes a condition derived from these parameters.

Lemma 3.4. *If* $\sum_i \|\alpha_i\|_{\mathcal{L}(H)} + \sum_i \|\alpha_i\|_{\mathcal{L}(H)}\|\phi_i\|_{\mathcal{L}(H)} + \|\phi_1\|_{\mathcal{L}(H)} < 1$*, then (8) holds.*

Proof. It can be shown that under the condition $\sum_{j=1}^{2} \|\mathbf{\Phi}_0^{-1}\mathbf{\Phi}_j\|_{\mathcal{L}(H^T)} < 1$, as per [2], Equation (8) is indeed satisfied. Further, it is demonstrable that

$$\|\mathbf{\Phi}_0^{-1}\mathbf{\Phi}_1\|_{\mathcal{L}(H^T)} \leq \|\mathbf{\Phi}_0^{-1}\|_{\mathcal{L}(H^T)}\|\mathbf{\Phi}_1\|_{\mathcal{L}(H^T)},$$
$$\|\mathbf{\Phi}_0^{-1}\mathbf{\Phi}_2\|_{\mathcal{L}(H^T)} \leq \|\mathbf{\Phi}_0^{-1}\|_{\mathcal{L}(H^T)}\|\mathbf{\Phi}_2\|_{\mathcal{L}(H^T)}.$$

By invoking the definition of the operatorial norm, the ensuing inequalities are established:

$$\begin{aligned}&\|\mathbf{\Phi}_0^{-1}\|_{\mathcal{L}(H^T)} \leq 1,\\ &\quad\|\mathbf{\Phi}_1\|_{\mathcal{L}(H^T)} \leq \sum_{i=1}^{T}\|\alpha_i\|_{\mathcal{L}(H)} + \sum_{i=2}^{T}\|\alpha_i\|_{\mathcal{L}(H)}\|\phi_i\|_{\mathcal{L}(H)} + \|\phi_1\|_{\mathcal{L}(H)},\\ &\quad\|\mathbf{\Phi}_2\|_{\mathcal{L}(H^T)} \leq \|\alpha_1\|_{\mathcal{L}(H)}\|\phi_1\|_{\mathcal{L}(H)}.\end{aligned}$$

Thus, these considerations collectively complete the proof. □

The next theorem deals with a result concerning the projection of SPARH(1,1) process.

Theorem 3.5. *Let $\{X_n,\ n \in \mathbb{Z}\}$ be a zero-mean SPARH(1,1) process. Suppose that there exists an element $e \in H$ and scalar values $\lambda_{1,n}$ and $\lambda_{2,n} \in \mathbb{R}$, fulfilling the conditions $\phi_n^*(e) = \lambda_{1,n}e$ and $\alpha_n^*(e) = \lambda_{2,n}e$, in conjunction with $E\langle\varepsilon_0, e\rangle_H^2 > 0$. Under these conditions, $\{\langle X_t, e\rangle_H ; t \in \mathbb{Z}\}$ constitutes a SPAR(1,1) process, which adheres to the following relation:*

$$\begin{aligned}\langle X_n, e\rangle_H =&\lambda_{1,n}\langle X_{n-1}, e\rangle_H + \lambda_{2,n}\langle X_{n-T}, e\rangle_H\\ &-\lambda_{1,n}\lambda_{2,n}\langle X_{n-T-1}, e\rangle_H + \langle\varepsilon_n, e\rangle_H.\end{aligned} \tag{10}$$

Proof. The derivation of this theorem is straightforward, resulting from the fundamental properties of operators and inner products. □

3.1 The Autocovariance Operators

Let $\boldsymbol{\mu} := E(\mathbf{X}_n)$. Consider $\mathbf{Y}_n = \mathbf{X}_n - \boldsymbol{\mu}$, which constitutes a mean zero ARH^T(2) process as mentioned in Lemma 3.4. If the condition mentioned in Eq. (8) holds, $\mathbf{Y}_n$ is designated as a standard ARH^T(2) process.

The autocovariance operator for this process, denoted as $\{\mathbf{C}_h;\ h \in \mathbb{Z}\}$, represents a bounded linear operator within $\mathcal{L}\left(H^T\right)$ and is defined by:

$$\mathbf{C}_h = \mathbf{C}_{\mathbf{Y}_0,\mathbf{Y}_h}.$$

It is evident that $\mathbf{C}_{-h} = \mathbf{C}_h^*$, $h \in \mathbb{Z}$.

Theorem 3.6. *Assuming $\mathbf{Y}_n$ as the standard $ARH^T(2)$ processes, which are derived from the standard SPARH(1,1) process as specified in Lemma 3.4, the following relations hold:*

$$\mathbf{C}_h = \mathbf{\Phi}_0^{-1}\mathbf{\Phi}_1\mathbf{C}_{h-1} + \mathbf{\Phi}_0^{-1}\mathbf{\Phi}_2\mathbf{C}_{h-2}, \tag{11}$$

and

$$\mathbf{C}_0 = \mathbf{\Phi}_0^{-1}\mathbf{\Phi}_1\mathbf{C}_1 + \mathbf{\Phi}_0^{-1}\mathbf{\Phi}_2\mathbf{C}_2 + \mathbf{\Phi}_0^{-1}\mathbf{C}_\varepsilon\left(\mathbf{\Phi}_0^{-1}\right)^*, \tag{12}$$

where $\mathbf{C}_\varepsilon := diag\left(C_{\varepsilon_1}, \cdots, C_{\varepsilon_T}\right)$ is the covariance operator of the process $\{\boldsymbol{\varepsilon}_n,\ n \in \mathbb{Z}\}$.

Proof. Utilizing the definition of the tensorial product, it can be shown that for a bounded linear operator A, the relation $x \otimes Ay = A\left(x \otimes y\right)$ holds. Applying this property, we derive:

$$\begin{aligned}
\mathbf{Y}_0 \otimes \mathbf{Y}_h &= \mathbf{Y}_0 \otimes \mathbf{\Phi}_0^{-1}\mathbf{\Phi}_1\mathbf{Y}_{h-1} + \mathbf{Y}_0 \otimes \mathbf{\Phi}_0^{-1}\mathbf{\Phi}_2\mathbf{Y}_{h-2} \\
&\quad + \mathbf{Y}_0 \otimes \mathbf{\Phi}_0^{-1}\boldsymbol{\varepsilon}_h - \mathbf{Y}_0 \otimes \left(\mathbf{I} - \mathbf{\Phi}_0^{-1}\mathbf{\Phi}_1 - \mathbf{\Phi}_0^{-1}\mathbf{\Phi}_2\right)\boldsymbol{\mu} \\
&= \mathbf{\Phi}_0^{-1}\mathbf{\Phi}_1\left(\mathbf{Y}_0 \otimes \mathbf{Y}_{h-1}\right) + \mathbf{\Phi}_0^{-1}\mathbf{\Phi}_2\left(\mathbf{Y}_0 \otimes \mathbf{Y}_{h-2}\right) \\
&\quad + \mathbf{\Phi}_0^{-1}\left(\mathbf{Y}_0 \otimes \boldsymbol{\varepsilon}_h\right) - \left(\mathbf{I} - \mathbf{\Phi}_0^{-1}\mathbf{\Phi}_1 - \mathbf{\Phi}_0^{-1}\mathbf{\Phi}_2\right)\left(\mathbf{Y}_0 \otimes \boldsymbol{\mu}\right).
\end{aligned} \tag{13}$$

Given the properties of the Bochner integral, it follows that $E\left(\mathbf{Y}_0 \otimes \boldsymbol{\mu}\right) = 0$. Since $\{\boldsymbol{\varepsilon}_n,\ n \in \mathbb{Z}\}$ constitutes the innovation process of $\mathbf{Y}_n$, we can concluded that

$$E\left(\mathbf{Y}_0 \otimes \boldsymbol{\varepsilon}_h\right) = E\left(\boldsymbol{\varepsilon}_0 \otimes \boldsymbol{\varepsilon}_h\right) = \begin{cases} \mathbf{C}_\varepsilon & h = 0 \\ 0 & h \neq 0 \end{cases}. \tag{14}$$

The remainder of this proof seamlessly follows from the integration of equations (13) and (14). □

4 Limit Theorems

This section introduces a lemma presenting the Law of Large Numbers as it applies to a standard SPARH(1,1) process.

Theorem 4.1. *Consider $\mathbf{Y}_t$ representing the standard ARH^T(2) processes, which are derivatives of the standard SPARH(1,1) process as outlined in Lemma 3.4. As $n \to \infty$, the following relationship holds:*

$$\frac{n^{1/4}}{(Log\,(n))^{\beta}} \frac{S_n}{n} \to 0, \quad \beta > 0.5, \tag{15}$$

where $S_n = \mathbf{Y}_1 + \cdots + \mathbf{Y}_n$.

Proof. This theorem's proof parallels the methodology employed in the proof of Theorem 5.6, as explained in [2]. □

The subsequent theorem articulates the Central Limit Theorem specific to a standard SPARH(1,1) process.

Theorem 4.2. *Let $\mathbf{Y}_t$ be identified as the standard ARH^T(2) processes, initially introduced in Lemma 3.4, with a strong white noise $\boldsymbol{\varepsilon}_t$. Assume that the matrix $\mathbf{I}-\boldsymbol{\Phi}_0^{-1}\boldsymbol{\Phi}_1-\boldsymbol{\Phi}_0^{-1}\boldsymbol{\Phi}_2$ is invertible. Under these conditions:*

$$\frac{S_n}{n} \to N\left(\mathbf{0}, \boldsymbol{\Gamma}\right), \tag{16}$$

where $\boldsymbol{\Gamma}$ is defined as:

$$\boldsymbol{\Gamma} = \left(\mathbf{I} - \boldsymbol{\Phi}_0^{-1}\boldsymbol{\Phi}_1 - \boldsymbol{\Phi}_0^{-1}\boldsymbol{\Phi}_2\right)^{-1} \boldsymbol{\Phi}_0^{-1}\mathbf{C}_{\varepsilon}\left(\boldsymbol{\Phi}_0^{-1}\right)^{*}\left(\mathbf{I} - \left(\boldsymbol{\Phi}_0^{-1}\boldsymbol{\Phi}_1\right)^{*} - \left(\boldsymbol{\Phi}_0^{-1}\boldsymbol{\Phi}_2\right)^{*}\right)^{-1}, \tag{17}$$

and $\mathbf{C}_{\varepsilon}$ is represented as:

$$\mathbf{C}_{\varepsilon} := diag\left(C_{\varepsilon_1}, \cdots, C_{\varepsilon_T}\right).$$

Proof. The foundation of this proof lies in the established relationship between the SPARH(1,1) and ARH^T(2), as delineated in Lemma 3.4. Applying Theorem 5.9 from [2] leads to the derivation of the result. □

5 Conclusion and Future Works

This paper introduces the first-order seasonal periodic autoregressive Hilbertian process, a novel model that enhances functional time series analysis by incorporating both periodic and seasonal dynamics. The analysis of key properties, such as autocovariance operators, demonstrates the model's theoretical soundness. Additionally, its consistency with fundamental statistical principles, including the law of large numbers and the central limit theorem, highlights its robustness and potential for practical applications. As future work, we plan to develop parameter estimation methods and investigate the consistency of the estimated parameters. Additionally, implementing this model in real-world scenarios and exploring extensions to enhance its applicability in functional data analysis present promising directions for further research.

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