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Solution and Hyperstability of Orthogonally Triple Lie Hom-derivations

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Abstract. The goal of this paper introduces a generalized concept which is called the orthogonally Jensen s-functional equation, on triple Lie algebras while preserving orthogonality and presents its additive properties with preserving orthogonality. Further, the orthogonally triple Lie hom-derivation associated with Jensen s-functional equation on orthogonally triple Lie algebras, are described. Ultimately, based on fixed point method with the orthogonally conditions, we approach to delve into stability of Hyers-Ulam sense and hyperstability of this s-functional equation and the triple Lie hom-derivation with preserving orthogonality on orthogonally triple Lie algebras.

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1 Introduction and Auxiliary Concepts

It seems that the pioneering use of ternary operations (TOs) with cubic matrices introduced by Cayley as an outstanding mathematician, in the 19th century. In fact, a TO is an n-ary operation with $n = 3$, such that for any given three elements of a set $\mathcal{B}$, mixes them to form a

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single element of $\mathcal{B}$. Generally, the non-trivial TO is formulated by the following composition rule for $\mathrm{s}, \mathrm{x}, \mathrm{w} \in \mathfrak{B}$:

$$(\mathrm{s}, \mathrm{x}, \mathrm{w})_{\mathring{\imath}\mathring{\jmath}k} = \sum_{\ell,m,n} \mathrm{s}_{n\mathring{\imath}\ell}\mathrm{x}_{\ell\mathring{\jmath}m}\mathrm{w}_{mkn}, \qquad \mathring{\imath}, \mathring{\jmath}, k = 1, 2, \ldots, \aleph,\ \aleph \in \mathbb{N}.$$

Following this innovation concept, n-ary algebraic structures have been developed and have been applied across a broad spectrum of disciplines, particularly those incorporating TOs [14, 27]. These include but are not limited to data processing, the realm of quantum mechanics, the intricate domain of mathematical physics, and notably in Nambu mechanics as evidenced by the references [25, 37, 41].

In 2006, Amyari and Moslehian introduced groundbreaking concepts related to ternary algebras [2]. A ternary algebra $\mathfrak{B}$, equipped with a TO, $(s_1, s_2, s_3) \mapsto [s_1, s_2, s_3]$ mapping from $\mathfrak{B}^3$ to $\mathfrak{B}$, constitutes a complex space where the operation is linear in the first and third arguments, and conjugate linear in the second argument, both respect to $\mathbb{C}$ [2]. It is associative in that for each $s_{\mathring{\imath}} \in \mathfrak{B}$, the equality

$$\big[[s_1, s_2, s_3], s_4, s_5\big] = \big[s_1, [s_2, s_3, s_4], s_5\big] = \big[s_1, s_2, [s_3, s_4, s_5]\big],$$

holds, according to established results in [7]. Further, it adheres to the norm conditions,

$$\big\|[s_1, s_2, s_3]\big\| \leq \|s_1\| \cdot \|s_2\| \cdot \|s_3\|, \qquad \big\|[s, s, s]\big\| = \|s\|^3,$$

and such a ternary algebra $\mathfrak{B}$ forms a B-space is referred to as a ternary Banach algebra.

On the other hand, Lie algebras are a fundamental key in scope of mathematics that are named after the Norwegian mathematician Sophus Lie (1842-1899). In 1893, the work of Scheffers compiled much of the foundational knowledge of Lie algebras based on Lie's own lectures in Leipzig [32]. A Banach algebra can be defined as a Lie algebra, with the Lie product of the form:

$$[s_1, s_2] := \tfrac{1}{2}(s_1 s_2 - s_2 s_1), \qquad s_i \in \mathcal{B}.$$

Extending this, for a triple Lie algebra, the product is defined by [29],

$$\big[[s_1, s_2], s_3\big] := \tfrac{1}{2}\big([s_1, s_2]s_3 - s_3[s_1, s_2]\big), \quad s_i \in \mathcal{B}.$$

Lie theory's application extends beyond mathematics to include fields such as Physics, Engineering, Mathematical Finance, and Economics due to its problem-solving capabilities [21]. An application of Lie groups and algebras can be seen in Lie-Poisson dynamics, which are essential to formulate basic physical equations, that vividly demonstrate Lie-Poisson dynamics of Euler's equations for rigid body dynamics [44]. These equations make use of a Lie-Poisson bracket that is derived from the Lie algebra of infinitesimal rotations which highlights how the principles of Lie algebra and Poisson brackets are applied for analyzing the dynamic behavior of rigid bodies, showcasing the interplay between mathematical structures and physical systems [8, 24, 35]. In this regard, Lo and Lui demonstrate the use of Lie algebra techniques in evaluating financial derivatives, especially those involving multiple assets in [33, 34].

In 1940, Ulam introduced the concept of stability in functional equations ($\mathsf{F}\mathbb{E}$s) which raised the question of whether functions that are approximately solutions to a $\mathsf{F}\mathbb{E}$ are also close to being true solutions of that equation [48]. This inquiry sparked a new area of study focused on investigating the stability properties of $\mathsf{F}\mathbb{E}$s, delving into the nuances in the input of a $\mathsf{F}\mathbb{E}$ that impact the proximity of the output to the true solution. In 1941, Hyers became the first to provide a positive response in Theorem 1.1, to Ulam's query within the context of B-spaces for additive functions [20].

Theorem 1.1 ([20])**.** *Assume that $\mathcal{Y}$ and $\mathcal{B}$ be a normed and a Banach spaces respectively, and $\Bbbk : \mathcal{Y} \to \mathcal{B}$ be a mapping such that for a given $\varepsilon > 0$, the inequality*

$$\left\|\Bbbk(s+\grave{s}) - \Bbbk(s) - \Bbbk(\grave{s})\right\| \leq \varepsilon, \qquad \forall s, \grave{s} \in \mathcal{Y},$$

holds. Then there exists a unique additive mapping $\mathbb{u} : \mathcal{Y} \to \mathcal{B}$, with $\|\Bbbk(s) - \mathbb{u}(s)\| \leq \varepsilon$ for each $s \in \mathcal{Y}$.

Subsequently, in 1978, Rassias expanded Hyers' work by introducing the generalized stability of Hyers-Ulam (HU) and a novel stability concept utilizing a control function for additive mappings, in Theorem 1.2.

Theorem 1.2 ([45])**.** *Let $\mathcal{Y}, \mathcal{B}$ be real normed spaces with $\mathcal{B}$ complete, and $\Bbbk : \mathcal{Y} \to \mathcal{B}$ be a mapping such that for each $s_\circ \in \mathcal{Y}$, the mapping*

$\mathbb{h}(s) = \mathbb{k}(s\, s_\circ)$ *is continuous on* $\mathbb{R}$*. If for a given* $\varepsilon \geq 0$*, the inequality,*

$$\big\|\mathbb{k}(s+\grave{s}) - \mathbb{k}(s) - \mathbb{k}(\grave{s})\big\| \leq \varepsilon\big(\|s\|^{\xi} + \|\grave{s}\|^{\xi}\big), \quad 0 \leq \xi < 1,$$

holds for $s, \grave{s} \in \mathcal{Y}$*, then there exists a unique linear mapping* $\mathbb{u} : \mathcal{Y} \to \mathcal{B}$ *such that*

$$\|\mathbb{k}(s) - \mathbb{u}(s)\| \leq \tfrac{\varepsilon\|s\|^{\xi}}{1-2^{\xi-1}}, \qquad \forall s \in \mathcal{Y}.$$

Later, in 1994, Găvruţa modified control function of Rassias to $\varphi(s, \grave{s})$ and demonstrated its stability in the next Theorem 1.3.

Theorem 1.3 ([17])**.** *Let* $(\mathcal{G}, +)$ *be an abelian group,* $(\mathcal{B}, \|\cdot\|)$ *be a B-space, and consider a mapping* $\varphi : \mathcal{G} \times \mathcal{G} \to \mathbb{R}^{\geq 0}$ *with*

$$\hat{\varphi}(s, \grave{s}) := \sum_{k=0}^{\infty} \tfrac{1}{2^k}\varphi\big(2^k s, 2^k \grave{s}\big) < \infty, \qquad s, \grave{s} \in \mathcal{G}.$$

If $\mathbb{k} : \mathcal{G} \to \mathcal{B}$ *be a mapping such that for each* $s, \grave{s} \in \mathcal{G}$*,*

$$\big\|\mathbb{k}(s+\grave{s}) - \mathbb{k}(s) - \mathbb{k}(\grave{s})\big\| \leq \varphi(s, \grave{s}),$$

then there exists a unique mapping $\mathbb{u} : \mathcal{G} \to \mathcal{B}$ *such that for* $s, \grave{s} \in \mathcal{G}$*, we have* $\mathbb{u}(s+\grave{s}) = \mathbb{u}(s) + \mathbb{u}(\grave{s})$*, and*

$$\|\mathbb{k}(s) - \mathbb{u}(s)\| \leq \tfrac{1}{2}\hat{\varphi}(s, s).$$

The investigation into the stability of $\mathsf{F}\mathbb{E}$s, fixed point ($\mathsf{F.P}$) theory, linear operators, Fibonacci numbers, and optimization theory across different spaces, has been pursued by numerous scholars [1, 13, 39]. The study of stability in the realm of $\mathsf{F}\mathbb{E}$s on B-spaces, Obloza [40], in 1993, initiated the examination of stability for linear $\mathsf{F}\mathbb{E}$s. This opened up avenues for extensive research into the stability of fractional differential equations ($\mathbb{FDE}$s), with several researchers extending UH stability to this domain [11, 19, 28, 36]. Aruchamy *et al.* in [3], studied the existence and stability results for the fractional nonlinear reaction-diffusion equation involving the Caputo fractional derivative with respect to the time variable of order $\alpha \in (0, 1)$, of the form:

$${}_0^{\mathrm{C}}D_{\xi}^{\alpha}\mathbb{u}(s, \xi) = \mathbb{D}\frac{\partial^2 \mathbb{u}(s,\xi)}{\partial s^2} + \int_0^{\xi} a(\xi - r)\frac{\partial^2 \mathbb{u}(s,r)}{\partial s^2}\,\mathrm{d}r + g\big(\xi, \mathbb{u}(s, \xi)\big),$$

for $s \in \Omega := (0, L)$, $\xi \geq 0$, with the positive kernel function $a(\xi - r)$ and

$$\begin{cases} \mathbb{u}(s,0) = \mathbb{u}_0(s), & s \in \Omega, \\ \mathbb{u}(s,\xi) = 0, & s \in \partial\Omega,\ \xi \geq 0. \end{cases}$$

Definition 1.4 ([15])**.** A binary relation $\perp \subseteq \mathfrak{B} \times \mathfrak{B}$, for the set $\mathfrak{B} \neq \varnothing$ has property orthogonally set (O-set) whenever there exists $s_\circ \in \mathfrak{B}$ such that, for each $s \in \mathfrak{B}$ either $s \perp s_\circ$ or $s_\circ \perp s$, and this O-set is denoted by $(\mathfrak{B}, \perp)$.

We will investigate some examples of O-sets based on the expressed concepts.

Example 1.5. Let $\mathfrak{B} = \mathbb{Z}$. Define $s \perp \grave{s}$ if and only if $\grave{s} \overset{s}{\equiv} 1$. Then $(\mathfrak{B}, \perp)$ is an O-set. Further, $1 \perp s$ for every s.

Example 1.6. Let $\mathfrak{B} = \mathbb{R}^{\geq 0}$. Define $s \perp \grave{s}$ if $s\,\grave{s} \leq \min\{s, \grave{s}\}$. To get the desired result, it is enough to put $s = 0$.

Example 1.7. In the inner product space $\mathfrak{B}$ with the inner product $\langle \cdot, \cdot \rangle$, we define $s \perp \grave{s}$ if and only if $\langle s, \grave{s} \rangle = 0$. One can easy check that $0 \perp s$ for each $s \in \mathfrak{B}$ and $(\mathfrak{B}, \perp)$ is an O-set.

The O-set $(\mathfrak{B}, \perp)$ is said to have property orthogonally generalized metric space (O-metric space), if $(\mathfrak{B}, d)$ be a generalized metric space and denoted by $(\mathfrak{B}, \perp, d)$. We say a sequence $\{s_n\}_{n \in \mathbb{N}}$ in O-set $(\mathfrak{B}, \perp)$ be an orthogonally sequence (O-sequence) if either $s_n \perp s_{n+1}$ or $s_{n+1} \perp s_n$ for $n \in \mathbb{N}$. Additionally, an O-sequence $\{s_n\}_{n \in \mathbb{N}} \subseteq \mathfrak{B}$ is called orthogonally Cauchy (O-Cauchy) when $\lim_{\ell, n \to \infty} d(s_\ell, s_n) = 0$. By concluding that, we say $(\mathfrak{B}, \perp, d)$ is O-complete whenever every O-Cauchy sequence converges in $\mathfrak{B}$. In O-metric space $(\mathfrak{B}, \perp, d)$, a mapping $\mathbb{k} : \mathfrak{B} \to \mathfrak{B}$ is said to be orthogonally continuous ($\perp$-continuous) in $s \in \mathcal{B}$ if, for each O-sequence $\{s_n\}_{n \in \mathbb{N}} \subseteq \mathfrak{B}$ with $s_n \to s$, we have, $\mathbb{k}(s_n) \to \mathbb{k}(s)$. Further, a mapping $\mathbb{k}$ is called orthogonal contraction ($\perp$-contraction) via Lipschitz constant $0 \leq \gamma < 1$ when, for $s, \grave{s} \in \mathcal{B}$,

$$d\big(\mathbb{k}(s), \mathbb{k}(\grave{s})\big) \leq \gamma\, d(s, \grave{s}), \qquad s \perp \grave{s}.$$

We say $\mathbb{k}$ is orthogonally preserving ($\perp$-preserving) if each $s \perp \grave{s}$ implies $\mathbb{k}(s) \perp \mathbb{k}(\grave{s})$.

Example 1.8. Let $\mathfrak{B} = [0, 1) \subset \mathbb{R}$ and d be the standard metric on $\mathfrak{B}$. Define the mapping $\Bbbk(s) = \sqrt{s}$ and $s \perp \grave{s}$ if $s\,\grave{s} \le s$. Then, $\mathfrak{B}$ is a $\perp$-contraction.

Example 1.9. Let $\mathfrak{B} = [0, 10)$ and d be the standard metric on $\mathfrak{B}$. Define the mapping $\Bbbk(s) = s^2$ and $s \perp \grave{s}$ if $s \le \grave{s} \le \frac{1}{2}$ and $s\grave{s} \le s$. Easily, one can see $\mathfrak{B}$ is a $\perp$-contraction. However, $\mathfrak{B}$ isn't a contraction because, for $s = 0.4$ and $\grave{s} = 0.9$, we have,

$$d\big(\Bbbk(s), \Bbbk(\grave{s})\big) = d\left(\tfrac{81}{100}, \tfrac{4}{25}\right) = 0.65 > \gamma\, d\left(\tfrac{9}{10}, \tfrac{4}{10}\right).$$

This notion has been extensively explored in literatures [18, 22, 30]. Chandok *et al.* proposed some weaker orthogonal mathematical equation type of contraction mappings in the setting of metric spaces endowed with an orthogonal relation, as well as certain sufficient criteria for the existence of F.Ps for this class of mapping and used their results to investigate the solution and stability of Hyers-Ulam- Rassias - Wright sense, for the following Volterra type integral equation:

$$\mathbb{u}(s) = \int_0^s g\big(s, r, \mathbb{u}(r)\big) + \digamma(s), \qquad s \in \Omega,$$

under some Lipschitz conditions [10]. Nazam *et al.* by introducing (Ψ, Φ)-orthogonal interpolative contractions, investigated different conditions on the functions $\Psi, \Phi : (0, \infty) \to \mathbb{R}$ to prove the existence of F.Ps of set-valued (Ψ, Φ)-orthogonal interpolative contractions such that, for $s, \grave{s} \in \mathcal{B}$,

$$\Psi\left(\Bbbk(s, \grave{s}) H(\mathcal{T}s, \mathcal{T}\grave{s})\right) \le \Phi\left(\big(d(s, \mathcal{T}s)\big)^{\gamma} \big(d(\grave{s}, \mathcal{T}\grave{s})\big)^{1-\gamma}\right), \tag{1}$$

where $\Bbbk$ is a strictly $\perp$-admissible mapping, $0 < \gamma < 1$, $s \notin \mathcal{T}(s)$, $\grave{s} \notin \mathcal{T}(\grave{s})$,

$$H(B_1, B_2) = \max\left\{ \sup_{p \in B_1} d(p, B_2),\ \sup_{a \in B_2} d(a, B_1) \right\} > 0,$$

B_1, B_2 in the set of all non-empty bounded and closed subsets of $\mathcal{B}$, $\mathcal{P}_{cb}(\mathcal{B})$, and to solve a $\mathbb{FDE}$ involving Caputo-Fabrizio derivative in O-complete, of the form:

$$\begin{cases} {}^{\mathrm{CF}}D_s^{\alpha} \mathbb{u}(s) = g\big(s, \mathbb{u}(s)\big), & s \in \Omega, \\ \mathbb{u}(0) = 0, \quad I^{\alpha} \mathbb{u}(1) = \mathbb{u}'(0), & \end{cases}$$

where $\mathbb{u} \in C(\Omega)$ and $g : (\Omega \times \mathbb{R}) \to \mathbb{R}$ with a few conditions [38]. In 2018, Bahraini *et al.* in [6], demonstrated the innovative use of the O-set concept to extend a generalization of Diaz-Margolis's $\mathsf{F.P}$ theorem [12]. In 2019, Park *et al.* introduced hom-derivations on Banach algebras [42]. Then, Jahedi and Keshavarz proved ternary additive-quadratic hom-derivations [23].

In this research, motivated by the mentioned works, we first introduce orthogonally Jensen s-functional ($\mathcal{J}_{s\mathsf{f}}$), $s \neq 0, \pm 1$ is a real number and orthogonally triple Lie hom-derivations on orthogonally triple Lie algebras, respectively. Then, we explore the $\mathsf{F}\mathbb{E}$ of the form,

$$\begin{aligned} 3\,\mathcal{J}_{s\mathsf{f}}\left(\tfrac{s_1+s_2+s_3}{3}\right) - \mathcal{J}_{s\mathsf{f}}(s_1) - \mathcal{J}_{s\mathsf{f}}(s_2) - \mathcal{J}_{s\mathsf{f}}(s_3) \\ = s\Big[\mathcal{J}_{s\mathsf{f}}(s_1+s_2+s_3) + \mathcal{J}_{s\mathsf{f}}(s_1) \\ - \mathcal{J}_{s\mathsf{f}}(s_1+s_2) - \mathcal{J}_{s\mathsf{f}}(s_1+s_3)\Big], \end{aligned} \tag{2}$$

for $s_{\mathring{\imath}} \in \mathfrak{B}$ ($\mathring{\imath} = 1,2,3$) with $s_1 \perp s_2$, $s_1 \perp s_3$, $s_2 \perp s_3$, as a category of orthogonally additive mappings. A $\mathbb{C}-$linear orthogonally mappings ($\perp$-mappings) $\mathbb{k}_{\mathrm{h}}, \mathbb{k}_{\mathrm{d}} : \mathfrak{B} \to \mathfrak{B}$ are called the orthogonally triple Lie homomorphism and derivation, respectively, whenever

$$\mathbb{k}_{\mathrm{h}}\big([[s_1,s_2],s_3]\big) = \Big[\big[\mathbb{k}_{\mathrm{h}}(s_1), \mathbb{k}_{\mathrm{h}}(s_2)\big], \mathbb{k}_{\mathrm{h}}(s_3)\Big],$$

and

$$\begin{aligned} \mathbb{k}_{\mathrm{d}}\big([[s_1,s_2],s_3]\big) = \big[[\mathbb{k}_{\mathrm{d}}(s_1), s_2], s_3\big] + \big[[s_1, \mathbb{k}_{\mathrm{d}}(s_2)], s_3\big] \\ + \big[[s_1,s_2], \mathbb{k}_{\mathrm{d}}(s_3)\big]. \end{aligned}$$

Definition 1.10. Consider an orthogonally triple Lie homomorphism $\mathbb{k}_{\mathrm{h}} : \mathfrak{B} \to \mathfrak{B}$. The $\mathbb{C}-$linear $\perp$-mapping $\mathbb{k}_{\mathrm{hd}} : \mathfrak{B} \to \mathfrak{B}$ is called an orthogonally triple Lie hom-derivations, when

$$\begin{aligned} \mathbb{k}_{\mathrm{hd}}\big([[s_1,s_2],s_3]\big) = \big[[\mathbb{k}_{\mathrm{hd}}(s_1), \mathbb{k}_{\mathrm{h}}(s_2)], \mathbb{k}_{\mathrm{h}}(s_3)\big] \\ + \big[[\mathbb{k}_{\mathrm{h}}(s_1), \mathbb{k}_{\mathrm{hd}}(s_2)], \mathbb{k}_{\mathrm{h}}(s_3)\big] \\ + \big[[\mathbb{k}_{\mathrm{h}}(s_1), \mathbb{k}_{\mathrm{h}}(s_2)], \mathbb{k}_{\mathrm{hd}}(s_3)\big], \end{aligned}$$

for each $s_{\mathring{\imath}} \in \mathfrak{B}$, $\mathring{\imath} = 1,2,3$.

Further, we establish the stability of HU sense for the orthogonally Jensen s-$\mathsf{F}\mathbb{E}$ and orthogonally triple Lie hom-derivations among orthogonally triple Lie algebras through the application of the next key Theorem 1.11.

Theorem 1.11 ([6]). *In O-complete metric space $(\mathfrak{B}, d, \perp)$, consider a $\perp$-preserving, $\perp$-continuous and $\perp$-contraction $\mathfrak{J}_{\mathsf{sf}} : \mathfrak{B} \to \mathfrak{B}$ with Lipschitz constant $0 < \gamma < 1$. Assume that for fixed $s_\circ \in \mathfrak{B}$ we have either $s_\circ \perp s$ or $s \perp s_c irc$, for all $s \in \mathfrak{B}$ and consider the "O-sequence of successive approximations with initial element $s_\circ$":*

$$s_\circ,\ \mathfrak{J}_{\mathsf{sf}}(s_\circ),\ \mathfrak{J}^2_{\mathsf{sf}}(s_\circ),\ \ldots,\ \mathfrak{J}^n_{\mathsf{sf}}(s_\circ), \ldots .$$

Then, either

$$d\big(\mathfrak{J}^n_{\mathsf{sf}}(s_\circ), \mathfrak{J}^{n+1}_{\mathsf{sf}}(s_\circ)\big) = \infty, \qquad \forall n \geq 0,$$

or there is $n_0 \in \mathbb{N}$ such that, $d\left(\mathfrak{J}^n_{\mathsf{sf}}(s_\circ), \mathfrak{J}^{n+1}_{\mathsf{sf}}(s_\circ)\right) < \infty$ for every $n > n_0$. If the second alternative holds, then

i) *The O-sequence of $\{\mathfrak{J}^n_{\mathsf{sf}}(s_\circ)\}$ is convergent to a $\mathsf{F.P}$, s^* of $\mathfrak{J}_{\mathsf{sf}}$.*

ii) *s^* is the unique $\mathsf{F.P}$ of $\mathfrak{J}_{\mathsf{sf}}$ in*

$$\mathfrak{B}^* = \Big\{ s \in \mathfrak{B} \,:\, d\big(\mathfrak{J}^n_{\mathsf{sf}}(s_\circ), s\big) < \infty \Big\}.$$

iii) *If $s^* \in \mathfrak{B}$, then $d(s, s^*) \leq \frac{1}{1-\gamma} d\big(s, \mathfrak{J}_{\mathsf{sf}}(s)\big)$.*

2 Main Results

In this section, let $s \neq 0, \pm 1,\ \gamma \in \mathbb{T}^1 := \big\{\gamma \in \mathbb{C} \,:\, |\gamma| = 1\big\}$ and $\mathfrak{B}$ be a orthogonally triple Lie algebra. For simplicity of work, we define

$$V_\perp := \Big\{ s_1, s_2, s_3 \in \mathfrak{B} \,:\, s_1 \perp s_2,\ s_1 \perp s_3,\ s_2 \perp s_3 \Big\}.$$

In the sequel, we investigate the orthogonally Jensen s-$\mathsf{F}\mathbb{E}$ which is an additive mapping and stability of HU for the triple Lie hom-derivations with preserving orthogonality. In Subsection 2.2, our focus shifts to examining the hyperstability of $\mathsf{F}\mathbb{E}$ (2) and the triple Lie hom-derivations with preserving orthogonality such that, this investigation is conducted through Găvruţa's control function, which serves as a pivotal tool in our analysis.

2.1 Stability of the Triple Lie Hom-Derivations

Before delving into the proof of the main theorem in this section, it is imperative to establish the additivity of the $\perp$-mapping $\mathfrak{J}_{sf}$.

Proposition 2.1. *If the proposed* $\mathsf{F}\mathbb{E}$ (2) *holds true for each* $s_{\mathring{\imath}} \in V_{\perp}$, $\mathring{\imath} = 1, 2, 3$, *then the* $\perp$*-mapping* $\mathfrak{J}_{sf} : \mathfrak{B} \to \mathfrak{B}$ *can be characterized as additive with* $\perp$*-preserving.*

Proof. Suppose that $\mathfrak{J}_{sf}$ fulfills condition $\mathsf{F}\mathbb{E}$ (2). Setting $s_{\mathring{\imath}} = 0$, $\mathring{\imath} = 1, 2, 3$, in $\mathsf{F}\mathbb{E}$ (2), we get $\mathfrak{J}_{sf}(0) = 0$. Substituting $s_2 = s_3 = 0$ into (2), we get,

$$3\,\mathfrak{J}_{sf}(s_1) = \mathfrak{J}_{sf}(3s_1). \tag{3}$$

By using (3) and putting $s_1 = s_2 = s_3$ in $\mathsf{F}\mathbb{E}$ (2), we have

$$2\,\mathfrak{J}_{sf}(s_1) = \mathfrak{J}_{sf}(2s_1).$$

Again putting $s_2 = -s_1$ and $s_3 = 0$, we get $\mathfrak{J}_{sf}(-s_1) = -\mathfrak{J}_{sf}(s_1)$. Finally, replace $s_3 = 0$ in $\mathsf{F}\mathbb{E}$ (2) and employ (3), we have

$$\mathfrak{J}_{sf}(s_1 + s_2) = \mathfrak{J}_{sf}(s_1) + \mathfrak{J}_{sf}(s_2).$$

Thus $\mathfrak{J}_{sf} : \mathfrak{B} \to \mathfrak{B}$ is an $\perp$-mapping with additive property. □

In the following proposition, we investigate $\mathfrak{J}_{sf}$ is an $\mathbb{C}-$linear $\perp$-mapping.

Proposition 2.2. *If a* $\perp$*-mapping* $\mathfrak{J}_{sf} : \mathfrak{B} \to \mathfrak{B}$ *satisfies*

$$\begin{aligned}3\,\mathfrak{J}_{sf}\left(\tfrac{\gamma[s_1+s_2+s_3]}{3}\right) - \gamma\,\mathfrak{J}_{sf}(s_1) - \gamma\,\mathfrak{J}_{sf}(s_2) - \mathbb{k}_{\mathrm{d}}\left(\mathfrak{J}_{sf}(s_3)\right)\\ = s\Big[\mathfrak{J}_{sf}\big(\gamma[s_1 + s_2 + s_3]\big) + \gamma\,\mathfrak{J}_{sf}(s_1)\\ - \gamma\,\mathfrak{J}_{sf}(s_1 + s_2) - \gamma\,\mathfrak{J}_{sf}(s_1 + s_3)\Big],\end{aligned} \tag{4}$$

for every $s_{\mathring{\imath}} \in V_{\perp}$, $\mathring{\imath} = 1, 2, 3$ *and* $\gamma \in \mathbb{T}^1$, *then the* $\perp$*-mapping* $\mathfrak{J}_{sf} : \mathfrak{B} \to \mathfrak{B}$ *is an orthogonally* $\mathbb{C}-$*linear.*

Proof. By Proposition 2.1 $\mathfrak{J}_{sf}$ is an orthogonally additive. Considering $s_2 = s_3 = 0$ in $\mathsf{F}\mathbb{E}$ (4), we get $\mathfrak{J}_{sf}(\gamma s_1) = \gamma\mathfrak{J}_{sf}(s_1)$. Following the same logic as in the proof of [43, Theorem 2.1], the mapping $\mathcal{J}_{sf}$ is established to be $\mathbb{C}-$linear $\perp$-mapping. □

Theorem 2.3 discusses regarding stability of orthogonally Jensen s-FE with Găvruţa's control function on orthogonally triple Lie algebras approach orthogonally F.P theorem. Consider $\Bbbk_{\mathrm{d}}, \psi : \mathfrak{B}^3 \to \mathbb{R}^{\geq 0}$ such that for some $0 < \lambda < 1$,

$$\Bbbk_{\mathrm{d}}\left(\tfrac{s_1}{3}, \tfrac{s_2}{3}, \tfrac{s_3}{3}\right) \leq \tfrac{\lambda}{3}\, \Bbbk_{\mathrm{d}}(s_1, s_2, s_3), \tag{5}$$

and

$$\psi\left(\tfrac{s_1}{3}, \tfrac{s_2}{3}, \tfrac{s_3}{3}\right) \leq \tfrac{\lambda}{3^3}\psi(s_1, s_2, s_3), \tag{6}$$

for each $s_{\mathring{\imath}} \in V_\perp$, $\mathring{\imath} = 1, 2, 3$. When setting $s_{\mathring{\imath}} = 0$, $\mathring{\imath} = 1, 2, 3$, we find that $\delta(0, 0, 0) = 0$ and $\psi(0, 0, 0) = 0$. From inequalities (5) and (6), it follows that,

$$\lim_{\mathring{\jmath}\to\infty} 3^{\mathring{\jmath}}\Bbbk_{\mathrm{d}}\left(\tfrac{s_1}{3^{\mathring{\jmath}}}, \tfrac{s_2}{3^{\mathring{\jmath}}}, \tfrac{s_3}{3^{\mathring{\jmath}}}\right) = 0, \qquad \lim_{\mathring{\jmath}\to\infty} 3^{3\mathring{\jmath}}\psi\left(\tfrac{s_1}{3^{\mathring{\jmath}}}, \tfrac{s_2}{3^{\mathring{\jmath}}}, \tfrac{s_3}{3^{\mathring{\jmath}}}\right) = 0. \tag{7}$$

In the upcoming theorem, we establish the stability of UH for FE (2) by employing the F.P theorem with $\perp$-preserving.

Theorem 2.3. *Let function $\Bbbk_{\mathrm{d}} : \mathfrak{B}^3 \to \mathbb{R}^{\geq 0}$ satisfies in inequality (5) and $\mathcal{J}_{sf} : \mathfrak{B} \to \mathfrak{B}$ be a mapping such that,*

$$\begin{aligned}\Bigg\|3\,\mathcal{J}_{sf}\left(\tfrac{\gamma[s_1+s_2+s_3]}{3}\right) &- \gamma\,\mathcal{J}_{sf}(s_1) - \gamma\,\mathcal{J}_{sf}(s_2) - \gamma\,\mathcal{J}_{sf}(s_3) \\ &- s\Big[\mathcal{J}_{sf}\big(\gamma[s_1 + s_2 + s_3]\big) + \gamma\,\mathcal{J}_{sf}(s_1) \\ &- \gamma\,\mathcal{J}_{sf}\big[s_1 + s_2\big] - \gamma\,\mathcal{J}_{sf}\big[s_1 + s_3\big]\Big]\Bigg\| \\ &\leq \Bbbk_{\mathrm{d}}(s_1, s_2, s_3), \end{aligned} \tag{8}$$

for every $s_{\mathring{\imath}} \in V_\perp$, $\mathring{\imath} = 1, 2, 3$ and $\gamma \in \mathbb{T}^1$. Then under these conditions, there exists a unique $\mathbb{C}-$linear $\perp$-mapping $\mathcal{T} : \mathfrak{B} \to \mathfrak{B}$ with $\perp$-preserving, for which

$$\|\mathcal{J}_{sf}(s_1) - \mathcal{T}(s_1)\| \leq \tfrac{1}{1-\lambda}\Bbbk_{\mathrm{d}}(s_1, 0, 0). \tag{9}$$

Proof. Letting $\gamma = 1$ and $s_2 = s_3 = 0$ in (8), we get

$$\left\|3\,\mathcal{J}_{sf}\left(\tfrac{s_1}{3}\right) - \mathcal{J}_{sf}(s_1)\right\| \leq \Bbbk_{\mathrm{d}}(s_1, 0, 0). \tag{10}$$

Let Υ be the set of all mappings $\mathcal{J}_{sf} : \mathfrak{B} \to \mathfrak{B}$ such that, $\mathcal{J}_{sf}(0) = 0$ and

$$\mathcal{J}_{sf}(s_1) \perp 3\,\mathcal{J}_{sf}\left(\tfrac{s_1}{3}\right) \quad \text{or} \quad 3\,\mathcal{J}_{sf}\left(\tfrac{s_1}{3}\right) \perp \mathcal{J}_{sf}(s_1).$$

Now, for every $\mathcal{J}_{sf}, \acute{\mathcal{J}}_{sf} \in \Upsilon$, we consider the metric is expressed by,

$$d(\mathcal{J}_{sf}, \acute{\mathcal{J}}_{sf}) = \inf\Big\{\lambda \in (0,\infty) \,:\, \big\|\mathcal{J}_{sf}(s_1) - \acute{\mathcal{J}}_{sf}(s_1)\big\| \\ \leq \lambda \Bbbk_{\mathrm{d}}(s_1, 0, 0),\, \forall s_1 \in V_{\perp}\Big\},$$

and define the orthogonality relation $\perp$ within Υ for each $\mathcal{J}_{sf}, \acute{\mathcal{J}}_{sf} \in \Upsilon$ in the manner described below,

$$\acute{\mathcal{J}}_{sf} \perp \mathcal{J}_{sf} \Leftrightarrow \quad \acute{\mathcal{J}}_{sf}(s_1) \perp \mathcal{J}_{sf}(s_1) \quad \text{or} \quad \mathcal{J}_{sf}(s_1) \perp \acute{\mathcal{J}}_{sf}(s_1),$$

for each $s_1 \in V_{\perp}$. Demonstrating that (Υ, d) constitutes an O-metric space is straightforward. Now, we consider the $\mathbb{C}-$linear $\perp$-mapping,

$$\begin{cases} \widehat{\Omega} : \Upsilon \to \Upsilon, \\ \widehat{\Omega}\big(\mathcal{J}_{sf}(s_1)\big) = 3\,\mathcal{J}_{sf}\left(\tfrac{s_1}{3}\right). \end{cases}$$

$\widehat{\Omega}$ maintains orthogonality. Therefore, for any $\mathcal{J}_{sf}, \acute{\mathcal{J}}_{sf} \in \Upsilon$ with $\mathcal{J}_{sf} \perp \acute{\mathcal{J}}_{sf}$ and it is given that $d(\acute{\mathcal{J}}_{sf}, \mathcal{J}_{sf}) = \varepsilon$, then,

$$\big\|\mathcal{J}_{sf}(s_1) - \acute{\mathcal{J}}_{sf}(s_1)\big\| \leq \varepsilon \Bbbk_{\mathrm{d}}(s_1, 0, 0), \qquad \forall s_1 \in V_{\perp}.$$

So

$$\big\|\widehat{\Omega}\big(\mathcal{J}_{sf}(s_1)\big) - \widehat{\Omega}\big(\acute{\mathcal{J}}_{sf}(s_1)\big)\big\| = \Big\|3\,\mathcal{J}_{sf}\left(\tfrac{s_1}{3}\right) - 3\,\acute{\mathcal{J}}_{sf}\left(\tfrac{s_1}{3}\right)\Big\| \\ \leq 3\varepsilon\, \Bbbk_{\mathrm{d}}\left(\tfrac{s_1}{3}, 0, 0\right) \leq 3\varepsilon \tfrac{\lambda}{3} \Bbbk_{\mathrm{d}}(s_1, 0, 0) = \lambda\varepsilon \Bbbk_{\mathrm{d}}(s_1, 0, 0),$$

and

$$d\big(\widehat{\Omega}(\mathcal{J}_{sf}), \widehat{\Omega}(\acute{\mathcal{J}}_{sf})\big) \leq \lambda\varepsilon, \qquad \forall\, s_1 \in V_{\perp}.$$

This means that

$$d\big(\widehat{\Omega}(\mathcal{J}_{sf}), \widehat{\Omega}(\acute{\mathcal{J}}_{sf})\big) \leq \lambda d\big(\mathcal{J}_{sf}, \acute{\mathcal{J}}_{sf}\big),$$

and so Ω acts as a strictly $\perp$-contractive self-mapping on Υ, characterized by a Lipschitz constant λ. Moreover, $\widehat{\Omega}$ exhibits $\perp$-continuity. Specifically, if $\{\mathcal{J}_{sf}^{\mathring{\jmath}}\}$ represents an O-sequence within Υ converging to some $\mathfrak{J}_{sf} \in \Upsilon$, then given any $\varepsilon > 0$, there is a $\lambda > 0$, where $\lambda < \varepsilon$, and an integer $\jmath \in \mathbb{N}$ such that,

$$\left\| \mathfrak{J}^{\jmath}_{sf}(s_1) - \mathfrak{J}_{sf}(s_1) \right\| \leq \lambda \Bbbk_{\mathrm{d}}(s_1, 0, 0).$$

According to Theorem 1.11, a mapping $\mathcal{T} : \mathfrak{B} \to \mathfrak{B}$ exists that meets the criteria below.

I) $\mathcal{T}$ serves as a F.P for $\widehat{\Omega}$, meaning,

$$\mathcal{T}(s_1) = 3\mathcal{T}\left(\tfrac{s_1}{3}\right), \qquad \forall\, s_1 \in V_\perp. \tag{11}$$

The mapping $\mathcal{T}$ represents a singular F.P of Ω. This denotes the existence of a unique $\perp$-mapping that fulfills (11), with a certain $\lambda > 0$ meeting the condition,

$$\|\mathfrak{J}_{sf}(s_1) - \mathcal{T}(s_1)\| \leq \lambda \Bbbk_{\mathrm{d}}(s_1, 0, 0), \qquad \forall\, s_1 \in V_\perp.$$

II) $d\left(\lambda^{\jmath} \mathfrak{J}_{sf}, \mathcal{T}\right) \to 0$ as $\jmath \to \infty$. This implies,

$$\lim_{\jmath \to \infty} 3^{\jmath}\, \mathfrak{J}_{sf}\left(\tfrac{s_1}{3^{\jmath}}\right) = \mathcal{T}(s_1), \qquad \forall\, s_1 \in V_\perp.$$

III) $d\big(\mathfrak{J}_{sf}, \mathcal{T}\big) \leq \frac{1}{1-\lambda} d\big(\mathfrak{J}_{sf}, \widehat{\Omega}(\mathcal{T})\big)$, which implies,

$$\left\|\mathfrak{J}_{sf}(s_1) - \mathcal{T}(s_1)\right\| \leq \tfrac{1}{1-\lambda}\, \Bbbk_{\mathrm{d}}(s_1, 0, 0), \qquad \forall\, s_1 \in V_\perp.$$

It follows from (7) and (8) that,

$$\begin{aligned}
&\Big\| 3\,\mathcal{T}\left(\tfrac{s_1+s_2+s_3}{3}\right) - \mathcal{T}(s_1) - \mathcal{T}(s_2) - \mathcal{T}(s_3) \\
&\quad - s\Big[\mathcal{T}(s_1 + s_2 + s_3) - \mathcal{T}(s_1) \\
&\quad - \mathcal{T}(s_1 + s_2) - \mathcal{T}(s_1 + s_3)\Big]\Big\| \\
&= \lim_{\jmath \to \infty} 3^{\jmath} \Big\| 3\,\mathfrak{J}_{sf}\left(\tfrac{s_1+s_2+s_3}{3^{\imath+1}}\right) - \mathfrak{J}_{sf}\left(\tfrac{s_1}{3^{\jmath}}\right) - \mathfrak{J}_{sf}\left(\tfrac{s_2}{3^{\jmath}}\right) - \mathfrak{J}_{sf}\left(\tfrac{s_3}{3^{\jmath}}\right) \\
&\quad - s\Big[\mathfrak{J}_{sf}\left(\tfrac{s_1+s_2+s_3}{3^{\jmath}}\right) + \mathfrak{J}_{sf}\left(\tfrac{s_1}{3^{\jmath}}\right) \\
&\quad - \mathfrak{J}_{sf}\left(\tfrac{s_1+s_2}{3^{\jmath}}\right) - \mathfrak{J}_{sf}\left(\tfrac{s_1+s_3}{3^{\jmath}}\right)\Big]\Big\| \\
&\leq \lim_{\jmath \to \infty} 3^{\jmath} \Bbbk_{\mathrm{d}}\left(\tfrac{s_1}{3^{\imath}}, \tfrac{s_2}{3^{\jmath}}, \tfrac{s_3}{3^{\jmath}}\right) = 0, \qquad \forall s_{\imath} \in V_\perp,\ \imath = 1, 2, 3.
\end{aligned}$$

So

$$\begin{aligned}3\mathcal{T}\left(\tfrac{s_1+s_2+s_3}{3}\right)-\mathcal{T}(s_1)-\mathcal{T}(s_2)-\mathcal{T}(s_3)&\\=s\Big[\mathcal{T}(s_1+s_2+s_3)+\mathcal{T}(s_1)&\\-\mathcal{T}(s_1+s_2)-\mathcal{T}(s_1+s_3)\Big],&\end{aligned}$$

for $s_{\mathring{\imath}}\in V_\perp$, $\mathring{\imath}=1,2,3$.

By proposition 2.2, $\mathcal{T}:\mathfrak{B}\to\mathfrak{B}$ is an $\mathbb{C}-$linear $\perp$-mapping. □

In the next theorem, we apply Găvruţa's control function to establish the stability of triple Lie hom-derivations while $\perp$-preserving through the orthogonal F.P theorem approach.

Theorem 2.4. *Let $\psi:\mathfrak{B}^3\to\mathbb{R}^{\geq 0}$ be functions such that, there exists an $0<\lambda<1$ satisfying in inequality (6) and $\acute{\eth}_{sf}$, $\eth_{sf}:\mathfrak{B}\to\mathfrak{B}$ are mappings satisfying (8) and*

$$\Big\|\acute{\eth}_{sf}\big([[s_1,s_2],s_3]\big)-\big[[\acute{\eth}_{sf}(s_1),\acute{\eth}_{sf}(s_2)],\acute{\eth}_{sf}(s_3)\big]\Big\|\leq\psi(s_1,s_2,s_3),\qquad(12)$$

$$\begin{aligned}\Big\|\eth_{sf}\big([[s_1,s_2],s_3]\big)&-\big[[\eth_{sf}(s_1),\acute{\eth}_{sf}(s_2)],\acute{\eth}_{sf}(s_3)\big]\\&-\big[[\acute{\eth}_{sf}(s_1),\eth_{sf}(s_2)],\acute{\eth}_{sf}(s_3)\big]\\&-\big[[\acute{\eth}_{sf}(s_1),\acute{\eth}_{sf}(s_2)],\acute{\eth}_{sf}(s_3)\big]\Big\|\leq\psi(s_1,s_2,s_3),\quad(13)\end{aligned}$$

for $s_{\mathring{\imath}}\in V_\perp$, $\mathring{\imath}=1,2,3$. Under these conditions, there exist unique orthogonally triple Lie homomorphism, hom-derivation $\Bbbk_{\rm h}$, $\Bbbk_{\rm hd}:\mathfrak{B}\to\mathfrak{B}$, respectively, which

$$\begin{aligned}\big\|\acute{\eth}_{sf}(s_1)-\Bbbk_{\rm h}(s_1)\big\|&\leq\tfrac{1}{1-\lambda}\psi(s_1,0,0),\\\big\|\eth_{sf}(s_1)-\Bbbk_{\rm hd}(s_1)\big\|&\leq\tfrac{1}{1-\lambda}\psi(s_1,0,0).\qquad(14)\end{aligned}$$

Proof. Following the same logic as in the proof of Theorem 2.3, for $s_1\in V_\perp$, can define the mappings,

$$\begin{cases}\Bbbk_{\rm h}:\mathfrak{B}\to\mathfrak{B},\\\Bbbk_{\rm h}(s_1)=\lim\limits_{\jmath\to\infty}3^{\jmath}\,\acute{\eth}_{sf}\left(\frac{s_1}{3^{\jmath}}\right),\end{cases}\qquad\begin{cases}\Bbbk_{\rm hd}:\mathfrak{B}\to\mathfrak{B},\\\Bbbk_{\rm h}(s_1)=\lim\limits_{\jmath\to\infty}3^{\jmath}\,\eth_{sf}\left(\frac{s_1}{3^{\jmath}}\right).\end{cases}\qquad(15)$$

Eqs. (12) and (15) imply,

$$\begin{aligned}\Big\|\Bbbk_{\mathrm{h}}\big([[s_1,s_2],s_3]\big) &- \big[[\Bbbk_{\mathrm{h}}(s_1),\Bbbk_{\mathrm{h}}(s_2)],\Bbbk_{\mathrm{h}}(s_3)\big]\Big\| \\ &= \lim_{\jmath\to\infty} 3^{3\jmath}\Big\|\acute{\mathfrak{J}}_{\mathsf{sf}}\left(\left[\left[\tfrac{s_1}{3^\jmath},\tfrac{s_2}{3^\jmath}\right],\tfrac{s_3}{3^\jmath}\right]\right) \\ &\quad - \left[\left[\acute{\mathfrak{J}}_{\mathsf{sf}}\left(\tfrac{s_1}{3^\jmath}\right),\acute{\mathfrak{J}}_{\mathsf{sf}}\left(\tfrac{s_2}{3^\jmath}\right)\right],\acute{\mathfrak{J}}_{\mathsf{sf}}\left(\tfrac{s_3}{3^\jmath}\right)\right]\Big\| \\ &\le \lim_{\jmath\to\infty} 3^{3\jmath}\,\psi\left(\tfrac{s_1}{3^\jmath},\tfrac{s_2}{3^\jmath},\tfrac{s_3}{3^\jmath}\right)=0.\end{aligned}$$

Hence, the mapping $\Bbbk_{\mathrm{h}}$ is a triple Lie homomorphism with $\perp$-preserving. It follows from (13) and (15), we have

$$\begin{aligned}\Big\|\Bbbk_{\mathrm{hd}}\big([[s_1,s_2],s_3]\big) &- \big[[\Bbbk_{\mathrm{hd}}(s_1),s_2],s_3\big] + \big[[s_1,\Bbbk_{\mathrm{hd}}(s_2)],s_3\big] \\ &\quad + \big[[s_1,s_2],\Bbbk_{\mathrm{hd}}(s_3)\big]\Big\| \\ &= \lim_{\jmath\to\infty} 3^{3\jmath}\Big\|\mathfrak{J}_{\mathsf{sf}}\left(\left[\left[\tfrac{s_1}{3^\jmath},\tfrac{s_2}{3^\jmath}\right],\tfrac{s_3}{3^\jmath}\right]\right) \\ &\quad - \left[\left[\mathfrak{J}_{\mathsf{sf}}\left(\tfrac{s_1}{3^\jmath}\right),\acute{\mathfrak{J}}_{\mathsf{sf}}\left(\tfrac{s_2}{3^\jmath}\right)\right],\acute{\mathfrak{J}}_{\mathsf{sf}}\left(\tfrac{s_3}{3^\jmath}\right)\right] \\ &\quad - \left[\left[\acute{\mathfrak{J}}_{\mathsf{sf}}\left(\tfrac{s_1}{3^\jmath}\right),\mathfrak{J}_{\mathsf{sf}}\left(\tfrac{s_2}{3^\jmath}\right)\right],\acute{\mathfrak{J}}_{\mathsf{sf}}\left(\tfrac{s_3}{3^\jmath}\right)\right] \\ &\quad - \left[\left[\acute{\mathfrak{J}}_{\mathsf{sf}}\left(\tfrac{s_1}{3^\jmath}\right),\acute{\mathfrak{J}}_{\mathsf{sf}}\left(\tfrac{s_2}{3^\jmath}\right)\right],\mathfrak{J}_{\mathsf{sf}}\left(\tfrac{s_3}{3^\jmath}\right)\right]\Big\| \\ &\le \lim_{\jmath\to\infty} 3^{3\jmath}\,\psi\left(\tfrac{s_1}{3^\jmath},\tfrac{s_2}{3^\jmath},\tfrac{s_3}{3^\jmath}\right)=0.\end{aligned}$$

Hence, the mapping $\Bbbk_{\mathrm{hd}}:\mathfrak{B}\to\mathfrak{B}$ is an orthogonally triple Lie hom-derivations. □

Theorem 2.3 generalized the result of Rassias' theorem, by the following definition

$$\Bbbk_{\mathrm{d}}(s_1,s_2,s_3):=\theta\Big[\|s_1\|^r+\|s_2\|^r+\|s_3\|^r\Big],\qquad \forall\,\theta,r\in\mathbb{R}^{>0},\ r\neq 1.$$

We have the following corollary.

Corollary 2.5. *Let $r\neq 1$ and θ be nonnegative real numbers and $\mathfrak{J}_{\mathsf{sf}}$:*

$\mathfrak{B} \to \mathfrak{B}$ be a mapping satisfying $\mathfrak{J}_{sf}(0) = 0$ and

$$\Big\| 3\,\mathfrak{J}_{sf}\left(\tfrac{\gamma[s_1+s_2+s_3]}{3}\right) - \gamma\,\mathfrak{J}_{sf}(s_1) - \gamma\,\mathfrak{J}_{sf}(s_2) - \Bbbk_{\mathrm{d}}(\mathfrak{J}_{sf}(s_3)) \\ - s\Big[\mathfrak{J}_{sf}(\gamma[s_1+s_2+s_3]) + \gamma\,\mathfrak{J}_{sf}(s_1) \\ - \gamma\,\mathfrak{J}_{sf}(s_1+s_2) - \gamma\,\mathfrak{J}_{sf}(s_1+s_3)\Big]\Big\| \\ \le \theta\Big[\|s_1\|^r + \|s_2\|^r + \|s_3\|^r\Big],$$

for each $s_{\mathring{\imath}} \in V_\perp$, $\mathring{\imath} = 1,2,3$, $\gamma \in \mathbb{T}^1$. Then, under these condition, there exists a unique $\mathbb{C}-$linear $\perp$-mapping $\mathcal{T} : \mathfrak{B} \to \mathfrak{B}$ with $\perp$-preserving, such that

$$\big\|\mathfrak{J}_{sf}(s_1) - \mathcal{T}(s_1)\big\| \le \begin{cases} \frac{2\theta}{2^r-2}\|s_1\|^r, & r > 1, \\ \frac{2\theta}{2-2^r}\|s_1\|^r, & r < 1, \end{cases} \tag{16}$$

for $s_{\mathring{\imath}} \in V_\perp$, $\mathring{\imath} = 1,2,3$.

Proof. By putting $\lambda = 2^{r-1}$, $r > 1$ and $\lambda = 2^{1-r}$, $r < 1$ in relation (16), the desired results are obtained. □

In Theorem 2.4, by taking $\lambda = 2^{r-1}$ and $\lambda = 2^{1-r}$, whenever $r > 1$, $r < 1$, respectively and

$$\psi(s_1,s_2,s_3) := \theta\big[\|s_1\|^r + \|s_2\|^r + \|s_3\|^r\big], \qquad \forall\, s_{\mathring{\imath}} \in V_\perp,\ \mathring{\imath} = 1,2,3,$$

where r and θ are nonnegative real numbers, we obtain the following result.

Corollary 2.6. *Consider two elements $r \ne 1$ and θ of $\mathbb{R}^{>0}$, let*

$$\psi(s_1,s_2,s_3) = \theta\Big[\|s_1\|^r + \|s_2\|^r + \|s_3\|^r\Big], \tag{17}$$

and assume $\acute{\mathfrak{J}}_{sf}, \mathfrak{J}_{sf} : \mathfrak{B} \to \mathfrak{B}$, are functions satisfying (8), (12) and (13). Then there exist orthogonally triple Lie homomorphism $\Bbbk_{\mathrm{h}}$ and hom-derivations $\Bbbk_{\mathrm{hd}}$ such that,

$$\begin{cases} \big\|\acute{\mathfrak{J}}_{sf}(s_1) - \Bbbk_{\mathrm{h}}(s_1)\big\| \le \frac{2\theta}{2^r-2}\|s_1\|^r, \\ \big\|\mathfrak{J}_{sf}(s_1) - \Bbbk_{\mathrm{hd}}(s_1)\big\| \le \frac{2\theta}{2^r-2}\|s_1\|^r, \end{cases}$$

and

$$\begin{cases} \left\|\acute{\mathcal{J}}_{sf}(s_1) - \Bbbk_{\rm h}(s_1)\right\| \leq \frac{2\theta}{2-2^r}\|s_1\|^r, \\ \left\|\mathcal{J}_{sf}(s_1) - \Bbbk_{\rm hd}(s_1)\right\| \leq \frac{2\theta}{2-2^r}\|s_1\|^r, \end{cases}$$

for $r > 1$ and $r < 1$, respectively.

2.2 On the Hyperstability of Orthogonally Triple Lie Hom-Derivations

The initial result on hyperstability was originally presented in [9] and focused on particular ring homomorphisms. The concept of hyperstability refers to a functional equation that is considered hyperstable if every approximate solution of the equation is, in fact, an exact solution. On the other hand, the Ulam stability problem questions the existence of an exact solution. For further details on hyperstability, see [26, 31].

In this subsection, by leveraging the orthogonally F.P technique, we investigate hyperstability of F$\mathbb{E}$ (2).

Theorem 2.7. *Let there is $\psi : \mathfrak{B}^3 \to \mathbb{R}^{\geq 0}$ such that,*

$$\lim_{\mathring{j}\to\infty} \frac{1}{3^{\mathring{j}}}\Bbbk_{\rm d}\big(0, 3^{\mathring{j}}s_2, 3^{\mathring{j}}s_3\big) = 0. \tag{18}$$

Moreover, suppose that $\mathcal{J}_{sf} : \mathfrak{B} \to \mathfrak{B}$ is mapping such that,

$$\begin{aligned} \Big\| 3\,\mathcal{J}_{sf}\left(\tfrac{\gamma[s_1+s_2+s_3]}{3}\right) &- \gamma\,\mathcal{J}_{sf}(s_1) - \gamma\,\mathcal{J}_{sf}(s_2) - \Bbbk_{\rm d}\big(\mathcal{J}_{sf}(s_3)\big) \\ &- s\Big[\mathcal{J}_{sf}\big(\gamma[s_1+s_2+s_3]\big) + \gamma\,\mathcal{J}_{sf}(s_1) \\ &- \gamma\,\mathcal{J}_{sf}(s_1+s_2) - \gamma\,\mathcal{J}_{sf}(s_1+s_3)\Big]\Big\| \\ &\leq \Bbbk_{\rm d}(0, s_2, s_3). \end{aligned} \tag{19}$$

Under these conditions, it is a $\mathbb{C}-$linear $\perp$-mapping $\mathcal{T} : \mathfrak{B} \to \mathfrak{B}$ with $\perp$-preserving.

Proof. Letting $\gamma = 1$ and $s_2 = s_3 = 0$ in (19), we get $\mathcal{J}_{sf}(3s_1) = 3\,\mathcal{J}_{sf}(s_1)$ and by using induction on $\mathring{j} \in \mathbb{N}$ we get,

$$\mathcal{J}_{sf}(s_1) = 3^{\mathring{j}}\,\mathcal{J}_{sf}\left(\tfrac{s_1}{3^{\mathring{j}}}\right).$$

Hence, we have

$$\begin{aligned}3^{\mathring{j}}\Big\|3\,\mathcal{J}_{sf}\left(\tfrac{s_1+s_2+s_3}{3^{\mathring{j}+1}}\right)-\mathcal{J}_{sf}\left(\tfrac{s_1}{3^{\mathring{j}}}\right)-\mathcal{J}_{sf}\left(\tfrac{s_2}{3^{\mathring{j}}}\right)-\mathcal{J}_{sf}\left(\tfrac{s_3}{3^{\mathring{j}}}\right)\\ -s\Big[\mathcal{J}_{sf}\left(\tfrac{s_1+s_2+s_3}{3^{\mathring{j}}}\right)+\mathcal{J}_{sf}\left(\tfrac{s_1}{3^{\mathring{j}}}\right)\\ -\mathcal{J}_{sf}\left(\tfrac{s_1+s_2}{3^{\mathring{j}}}\right)-\mathcal{J}_{sf}\left(\tfrac{s_1+s_3}{3^{\mathring{j}}}\right)\Big]\Big\|\\ \leq 3^{i}\Bbbk_{\mathrm{d}}\left(0,\tfrac{s_2}{3^{\mathring{j}}},\tfrac{s_3}{3^{\mathring{j}}}\right)=0,\end{aligned}$$

for every $s_{\mathring{\imath}}\in V_\perp$, $\mathring{\imath}=1,2,3$, $\mathring{j}\in\mathbb{N}$. So, by $\mathring{j}\to\infty$ and using (18), we get

$$\begin{aligned}3\,\mathcal{J}_{sf}\left(\frac{(s_1+s_2+s_3)}{3}\right)-\mathcal{J}_{sf}(s_1)-\mathcal{J}_{sf}(s_2)-\Bbbk_{\mathrm{d}}\big(\mathcal{J}_{sf}(s_3)\big)\\ -s\Big[\mathcal{J}_{sf}(s_1+s_2+s_3)+\mathcal{J}_{sf}(s_1)\\ -\mathcal{J}_{sf}(s_1+s_2)-\mathcal{J}_{sf}(s_1+s_3)\Big]=0,\end{aligned}$$

for $s_{\mathring{\imath}}\in V_\perp$, $\mathring{\imath}=1,2,3$. Therefore, $\mathcal{J}_{sf}$ is an $\mathbb{C}-$linear $\perp$-mapping. □

In the subsequent theorem, our focus shifts to examining the hyperstability of orthogonally triple Lie hom-derivation. This investigation is conducted through Găvruţa's control function, which serves as a pivotal tool in our analysis. By leveraging the orthogonally F.P technique, we delve into the intricacies of the equation's hyperstability properties.

Theorem 2.8. *Suppose there exist function $\psi:\mathfrak{B}^3\to\mathbb{R}^{\geq 0}$ such that,*

$$\lim_{\mathring{j}\to\infty}\tfrac{1}{3^{3\mathring{j}}}\psi\left(0,3^{\mathring{j}}s_2,3^{\mathring{j}}s_3\right)=0,\qquad \forall\, s_{\mathring{\imath}}\in V_\perp,\ \mathring{\imath}=1,2,3.$$

Moreover, suppose that $\acute{\mathcal{J}}_{sf},\mathcal{J}_{sf}:\mathfrak{B}\to\mathfrak{B}$ are mappings satisfying (19) *and*

$$\Big\|\acute{\mathcal{J}}_{sf}\big([[s_1,s_2],s_3]\big)-\big[[\zeta(s_1),\acute{\mathcal{J}}_{sf}(s_2)],\acute{\mathcal{J}}_{sf}(s_3)\big]\Big\|\leq\psi(0,s_2,s_3),$$

$$\begin{aligned}\Big\|\mathcal{J}_{sf}\big([[s_1,s_2],s_3]\big)-\big[[\mathcal{J}_{sf}(s_1),\acute{\mathcal{J}}_{sf}(s_2)],\acute{\mathcal{J}}_{sf}(s_3)\big]\\ -\big[[\acute{\mathcal{J}}_{sf}(s_1),\mathcal{J}_{sf}(s_2)],\acute{\mathcal{J}}_{sf}(s_3)\big]\\ -\big[[\acute{\mathcal{J}}_{sf}(s_1),\acute{\mathcal{J}}_{sf}(s_2)],\mathcal{J}_{sf}(s_3)\big]\Big\|\leq\psi(0,s_2,s_3),\end{aligned}$$

for $s_{\mathring{\imath}} \in V_\perp$, $\mathring{\imath} = 1, 2, 3$. Then $\acute{\mathfrak{d}}_{sf}$ and $\mathfrak{d}_{sf}$ are orthogonally triple Lie homomorphism and orthogonally triple Lie hom-derivation, respectively.

Proof. The methodology employed in the proof bears resemblance to that of Theorem 2.7, showcasing a parallel structure in the analytical approach. □

In the ensuing corollary, our focus shifts towards exploring the hyperstability of $\mathsf{F}\mathbb{E}$ (2) through control function of Rassias with ⊥-preserving.

Corollary 2.9. *Let $\theta, r \in \mathbb{R}^{>0}$ with $r \neq 1$ and $\mathfrak{d}_{sf} : \mathfrak{B} \to \mathfrak{B}$ is mapping such that,*

$$\begin{aligned} \Big\| 3\,\mathfrak{d}_{sf}\left(\tfrac{\gamma[s_1+s_2+s_3]}{3}\right) - \gamma\,\mathfrak{d}_{sf}(s_1) - \gamma\,\mathfrak{d}_{sf}(s_2) - \gamma\,\mathbb{k}_{\mathrm{d}}(\mathfrak{d}_{sf}(s_3)) \\ - s\Big[\mathfrak{d}_{sf}\big(\gamma[s_1+s_2+s_3]\big) + \gamma\,\mathfrak{d}_{sf}(s_1) \\ - \gamma\,\mathfrak{d}_{sf}(s_1+s_2) - \gamma\,\mathfrak{d}_{sf}(s_1+s_3)\Big]\Big\| \\ \leq \theta\Big[\|s_2\|^r + \|s_3\|^r\Big]. \end{aligned}$$

Under these conditions, it is a $\mathbb{C}-$linear ⊥-mapping $\mathcal{T} : \mathfrak{B} \to \mathfrak{B}$ with ⊥-preserving.

Now, in the following corollary, our focus shifts towards exploring the hyperstability of orthogonally triple Lie hom-derivation by control function of Rassias.

Corollary 2.10. *Let $\theta, r \in \mathbb{R}^{>0}$ with $r \neq 1$ and $\acute{\mathfrak{d}}_{sf}, \mathfrak{d}_{sf} : \mathfrak{B} \to \mathfrak{B}$ are mappings such that,*

$$\Big\|\acute{\mathfrak{d}}_{sf}\big([[s_1, s_2], s_3]\big) - \big[[\acute{\mathfrak{d}}_{sf}(s_1), \acute{\mathfrak{d}}_{sf}(s_2)], \acute{\mathfrak{d}}_{sf}(s_3)\big]\Big\| \leq \theta\Big[\|s_2\|^r + \|s_3\|^r\Big],$$

and

$$\begin{aligned} \Big\|\mathfrak{d}_{sf}\big([[s_1, s_2], s_3]\big) - \big[[\mathfrak{d}_{sf}(s_1), \mathfrak{d}_{sf}(s_2)], \mathfrak{d}_{sf}(s_3)\big] \\ - \big[[\mathfrak{d}_{sf}(s_1), \mathfrak{d}_{sf}(s_2)], \acute{\mathfrak{d}}_{sf}(s_3)\big] \\ - \big[\big[\acute{\mathfrak{d}}_{sf}(s_1), \acute{\mathfrak{d}}_{sf}(s_2)\big], \mathfrak{d}_{sf}(s_3)\big]\Big\| \\ \leq \theta\Big[\|s_2\|^r + \|s_3\|^r\Big]. \end{aligned}$$

Then $\acute{\mathfrak{d}}_{sf}$ and $\mathfrak{d}_{sf}$ are orthogonally triple Lie homomorphism and orthogonally triple Lie hom-derivation, respectively.

3 Conclusion

Given the significance of orthogonal sets, triple Lie and Banach algebras, triple Lie derivations, and their practical applications in various fields such as engineering and mathematical physics, mathematical finance and economics. We introduced the new concept of orthogonally Jensen equation and orthogonally triple Lie hom-derivations on orthogonally triple Lie algebras. After that, we solved some examples of the concept of O-sets that can be O-set and $\perp-$contraction.

Finally, through the utilization of the $\mathsf{F.P}$ method with $\perp$-preserving, we have successfully demonstrated that orthogonally triple Lie hom-derivations and the $\mathsf{F}\mathbb{E}$ (2) exhibit stability and hyperstability properties when applied to orthogonally triple Lie algebras. As the orthogonally Jensen s-functional equation, on triple Lie algebras while $\perp$-preserving, are applicable in different fields of science, so one can extend different models and analytic techniques by means of the generalized Jensen s-functional equation for various science processes in the next research studies [4, 5, 16, 46, 47].

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