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Original Research Paper

Limit Summability of Functions Relative to Sequences of Polynomials of a Fixed Degree

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Abstract. The topic of limit summability of functions was introduced by Hooshmand in 2001. Thereafter, some other types of summability of functions such as analytic summability of real and complex functions were introduced and studied by him. Motivated by limit and analytic summabilities, along with fractional sums, we introduce the concept of limit summability of functions relative to a sequence of polynomials of a fixed degree. In this regard, we try to extend the relevant results of limit summability for this topic. Also, we show that this type of summability covers a wider range of functions.

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1 Introduction

The famous mathematician Leonard Euler introduced the Gamma function in 1720, which helped to develop the factorial concept from positive integers to real and complex numbers. After that, Bohr and Mollerup proved the uniqueness of theorem for the Gamma function in 1722. Later Krull investigated the solutions of the difference equation $g(x+1) - g(x) = f(x)$ in 1948 [5]. After that, the Gamma−type functions satisfying the functional equations $g(x+1) = g(x)f(x)$ were studied by Roger Webster in 1997[7]. Gamma−type functions are solutions of this equation under some unique conditions, and their unique conditions can be considered as a generalization of Bohr and Mollerup's theorem. In 2001, while M.H. Hooshmand was studying ultra power and ultra exponential functions, he introduced limit summability of functions [2]. The limit summability that he introduced has a very close relationship with the mentioned topics and consists of a wider range of functions. In that topic, all real and complex functions f whose domain includes natural numbers are considered. He showed that the logarithm of Gamma−type functions can be considered a limit summand functions in the topic of limit summability. Furthermore he proved a generalization of Bohr−Mollerup and Webster's theorems in the uniqueness theorem for limit summand functions (see [2, 4]). In 2010, Muller and Schneider introduced the concept of fractional sums and semi−Euler factorization [6]. In 2016 and 2017, Hooshmand introduced two other types of summability named analytic summability and trigonometric summability [3]. From 2018 to the present, he and his students have conducted further studies on the different types of summability and related topics such as limit summability of order two, derivative and integral of limit summand functions, Euler-type constants, etc.

1.1 Limit summability of functions

Here, we give a summary of limit summability from [2]. Let f be a real or complex function with $\mathbb{Z}^+ \subseteq D_f$. The summand set of D_f is defined by

$$\Sigma_f := \{x : x + \mathbb{Z}^+ \subseteq D_f\} = \{x : \{x+1, x+2, ...\} \subseteq D_f\}.$$

For function f, the functional sequence f_{σ_n} is defined as follows

$$f_{\sigma_n}(x) := xf(n) + \sum_{k=1}^{n} \Big(f(k) - f(x+k) \Big); \quad x \in \Sigma_f.$$

by setting

$$R_n(f,x) := R_n(x) = f(n) - f(x+n),$$

we have

$$f_{\sigma_n}(x) = xf(n) + \sum_{k=1}^{n} R_k(x).$$

w We obtain

$$f_{\sigma_n}(x) - f_{\sigma_{n-1}}(x) = R_n(x) - xR_{n-1}(1); \quad x \in \Sigma_f, \ n > 1, \tag{1}$$

and

$$f_{\sigma_n}(x) - f_{\sigma_n}(x-1) = f(x) + R_n(x); \quad x \in \Sigma_f + 1 = \Sigma_f \cap D_f. \tag{2}$$

The limit function of $f_{\sigma_n}(x)$ and $R_n(f,x)$ are denoted by $f_\sigma(x)$ and $R(f,x)$ (or $R(x)$) and $f_\sigma(x)$ is called the limit summand function of $f(x)$. Note that if $0 \in D_f$, then $f_\sigma(0) = 0$, $f_\sigma(-1) = -f(0)$, and $1 \in D_{f_\sigma}$ if and only if $R_n(1)$ is convergent. The necessary condition for limit summability of f at x is

$$\lim_{n\to\infty} (R_n(x) - xR_{n-1}(1)) = 0.$$

This shows that if $1 \in D_{f_\sigma}$, then $R(x) = R(1)x$, for all $x \in D_{f_\sigma}$. Function f is a limit summable at $x_0 \in \Sigma_f$ (resp. on $S \subseteq \Sigma_f$) if the functional sequence $f_{\sigma_n}(x)$ is convergent at $x_0 \in \Sigma_f$ (resp. on S). Hence

$$D_{f_\sigma} = \{x \in \Sigma_f; \quad f \text{ is limit summable at } x\}.$$

The function f is called uniformly limit summable on $S \subseteq \Sigma_f$ if $f_{\sigma_n}(x)$ is uniformly convergent on S. If $R(1) = 0$, then

$$f_\sigma(x) = f(x) + f_\sigma(x-1); \quad x \in D_{f_\sigma} + 1, \tag{3}$$

so

$$f_\sigma(m) = \sum_{j=1}^{m} f(j); \quad m \in \mathbb{Z}^+.$$

Example 1.1. The complex function f defined by $f(z) = b^z$ for $\mid b \mid < 1$ is limit summable and $f_\sigma(z) = \frac{b}{b-1}(b^z - 1)$.

The real function $h(x) = \log x$ with $D_h = \mathbb{R}^+$ is limit summable and $h_\sigma(x) = \log\Gamma(x+1)$ and the function $g(x) = \frac{1}{x}$ is not summable.

Fix $s \in \mathbb{R}$ and consider the function $f(x) := x^s$ with the restricted domain

$$D_f = \begin{cases} [0, +\infty) & s > 0 \\ (0, +\infty) & s < 0 \end{cases}.$$

Then, we have

$$\Sigma_f = \begin{cases} [-1, +\infty) & s > 0 \\ (-1, +\infty) & s < 0 \end{cases}$$

This function, for any negative real number s is limit summable and we have

$$f_\sigma(x) = \begin{cases} \sum_{n=1}^{\infty}(n^s - (n+x)^s) & -1 < s < 0 \\ \psi(x+1) + \gamma & s = -1 \\ \zeta(-s) - \zeta(-s, x+1) & s < -1 \end{cases}$$

where $\psi(x)$ is the di-gama function, $\zeta(x)$ is the Riemann zeta function, $\zeta(a, x)$ is the Hurwitz zeta function, and γ is the Euler$-$Mascheroni constant (See [1]).

1.2 Bernoulli polynomials and analytic sumability of functions

We recall some definitions and properties of Bernoulli polynomials and numbers from [1]:
The Bernoulli polynomials $B_k(t)$ for $k = 0, 1, 2, ...$ are defined by

$$\frac{\theta e^{t\theta}}{e^\theta - 1} = \sum_{n=0}^{\infty} B_k(t)\frac{\theta^k}{k!} \quad (\mid \theta \mid < 2\pi,\ t \in \mathbb{C}). \tag{4}$$

The numbers $B_k = B_k(0)$ are the first Bernoulli numbers and the second Bernoulli numbers are denoted by $\mathbf{b}_k = B_k(1)$. The first few polynomials are

$$B_0(t) = 1,\ \ B_1(t) = t - \frac{1}{2},\ \ B_2(t) = t^2 - t + \frac{1}{6},\ \ B_3(t) = t^3 - \frac{3}{2}t^2 + \frac{1}{2}t, \tag{5}$$

and related numbers are as follows

$$B_0 = 1, \quad B_1 = \frac{-1}{2}, \quad B_2 = \frac{1}{6}, \quad B_3 = 0, \quad B_4 = \frac{-1}{30}, \quad B_5 = 0. \tag{6}$$

All Bernoulli numbers are rational. Explicit formula for $B_k(t)$ is given by

$$B_k(t) = \sum_{i=0}^{k} \binom{k}{i} B_i t^{k-i} \tag{7}$$

Also, Bernoulli polynomials follow many relations, such as the different functional equation below

$$B_k(t+1) - B_k(t) = kt^{k-1}. \tag{8}$$

Some other basic relations are

$$S_k(t) := \frac{B_{k+1}(t+1) - \mathbf{b}_{k+1}}{k+1}; \quad t \in \mathbb{C}, k \geqslant 0, \tag{9}$$

and

$$S_k(t) = t^k + S_k(t-1); \quad t \in \mathbb{C}, \tag{10}$$

hence

$$S_k(r) = 1^k + 2^k + \ldots + r^k; \quad r \in \mathbb{Z}^+.$$

The topic of analytic summability studies the convergence and properties of $\sum_{n=0}^{\infty} c_n S_n(z)$, for analytic functions $f(z) = \sum_{n=0}^{\infty} c_n z^n$ defined on an open domain D.

2 Limit Summability of Functions Relative to Sequences of Polynomials

One limitations of the limit summability of functions is that some important functions, such as polynomials were not limit summable. In the topic of analytic summability of functions with a new approach, those problems were solved for a wide range of functions, including polynomials and trigonometric functions. moreover some new types of summable functions were introduced. Here, by introducing p-limit summability of functions, where $p(x) = p_{m,n}(x)$ is a sequence of given polynomials. Not only limitation will be removed for polynomials, but also the topic of limit summability is generalized, in this way, we create a much wider space of desired functions.

2.1 p-Limit summability

One of the motivations for introducing p-limit summability of functions (in addition to normal limit summability) is the fractional sum of functions, which was introduced in [6]. In one of the next Sections of the article, we will discuss the expressions and connections between the two topics p-limit summability and the fractional sum of functions.

Let $m \geq 0$ be a fixed integer and $\{a_0(n)\}_{n=1}^{\infty}, \cdots, \{a_m(n)\}_{n=1}^{\infty}$ be sequences of (real or complex) numbers. Consider the sequence of polynomials $p(x) = p_m(x) = p_{m,n}(x)$ defined by

$$p_{m,n}(x) := \sum_{k=0}^{m} a_k(n)\ x^k; \quad x \in \mathbb{C}.$$

Now, define

$$\begin{aligned} P(x) = P_{m,n}(x) &:= \sum_{k=0}^{m} a_k(n) S_k(x) \\ &= \sum_{k=0}^{m} \frac{a_k(n)}{k+1} (B_{k+1}(x+1) - B_{k+1}(1)) \qquad (11) \\ &= a_0(n)x + a_1(n)\frac{x^2+x}{2} + ... + a_m(n)\frac{B_{m+1}(x+1) - B_{m+1}(1)}{m+1} \end{aligned}$$

There are some interesting relationships between $p_{m,n}$ and $P_{m,n}$ induced by the properties of $S_k(n)$, with the two most important identities as follows.

$$P(x) = p(x) + P(x-1); \quad x \in \mathbb{C} \qquad (12)$$

$$P(r) = \sum_{j=1}^{r} p(j) = p(1) + ... + p(r); \quad r \in \mathbb{Z}^+. \tag{13}$$

Let $p = p_{m,n}$ be a sequence of polynomials of the fixed degree m (as mentioned above). For every function $f : D_f \to \mathbb{C}$ with $\mathbb{Z}^+ \subseteq D_f$, we assign the functional sequence $f_{\sigma_n}(p)$ defined by

$$\begin{aligned} f_{\sigma_n}(p,x) : &= R_n(-P,x) + \sum_{k=1}^{n} R_k(f,x) \\ &= P(n+x) - P(n) + \sum_{k=1}^{n} (f(k) - f(k+x)); \quad x \in \Sigma_f. \end{aligned}$$

Now we can introduce the main definition.

Definition 2.1. Let f be a real or complex function with $\mathbb{Z}^+ \subseteq D_f$ and $p = p_{m,n}$ a sequence of polynomials of the fixed degree m. We call f limit summable relative to p at $x \in \Sigma_f$ (or p-limit summable at x) if the functional sequence $f_{\sigma_n}(p,x)$ is convergent. Also, we put

$$f_\sigma(p,x) := \lim_{n\to\infty} f_{\sigma_n}(p,x)$$

and

$$D_{f_\sigma(p)} := \{x \in \Sigma_f; \quad f_{\sigma_n}(p,x) \ \text{ is convergent}\}.$$

The function $f_\sigma(p,x)$ is called the p-limit sammand function of f.

Note that the domain of $f_\sigma(p)$ is $D_{f_\sigma(p)}$, $\{0\} \subseteq D_{f_\sigma(p)} \subseteq \Sigma_f$ and $f_\sigma(p,0) = 0$ for every p.

Theorem 2.2. *If $deg(p) = 0$ (i.e., m= 0), then $p(x) = p_0(x) = a_0(n)$. That is independent from x, $P(x) = P_0(x) = a_0(n)x$, and*

$$f_{\sigma_n}(p_0,x) = xa_0(n) + \sum_{k=1}^{n} (f(k) - f(k+x)); \quad x \in \Sigma_f.$$

In particular, by setting $a_0(n) := f(n)$, we get $f_{\sigma_n}(p_0,x) = f_{\sigma_n}(x)$ which is the same functional sequence in the topic of limit summability introduced in [2].

Example 2.3. Let $p(x) = \frac{1}{n!}x + \frac{1}{n}$, $f(x) = b^x$ (as a complex function) where |b|< 1. Then $\Sigma_f = \mathbb{C}$ and

$$\begin{aligned} f_\sigma(p,x) &= \lim_{n\to\infty}\left(\frac{1}{n}x + \frac{1}{n!}(nx + \frac{x^2+x}{2}) + (1-b^x)\sum_{k=1}^{n} b^k\right) \\ &= (1-b^x)\frac{b}{1-b}; \quad x \in \mathbb{C}. \end{aligned}$$

Thus f is limit summable related to p which is a sequence of polynomials of degree one on $\mathbb{C}$.

From now on, we consider p as $p_{m,n}$ (a sequence of polynomials with the degree m) and $P = P_{m,n}$ as mentioned in (2.1).

Theorem 2.4. *For every $x \in \Sigma_f$, the following statements hold.*

(a) *A necessary condition for p-limit summability of f at x is*

$$\lim_{n\to\infty}\left(R_n(p-f,x) + S_{n,m}(p,x)\right) = 0 \tag{14}$$

where $S_{n,m}(p,x)$ is defined by

$$S_{n,m}(p,x) := \sum_{k=0}^{m} \frac{1}{k+1} R_{n-1}(a_k,1) R_n(B_{k+1},x).$$

(*note that*

$$\begin{aligned} R_n(B_{k+1},x) &:= B_{k+1}(n) - B_{k+1}(n+x), \\ R_{n-1}(a_k,1) &:= a_k(n-1) - a_k(n)). \end{aligned}$$

(b) *A necessary and sufficient condition for the p-limit summability of f at x is the convergence of the functional series*

$$\sum_{n=2}^{\infty}\left(R_n(f-p,x) + S_{n,m}(p,x)\right)$$

and we have

$$f_\sigma(p,x) = f_{\sigma_1}(p,x) + \sum_{n=2}^{\infty} R_n(f-p,x) + S_{n,m}(p,x).$$

Also if $0 \in D_f$, *then*

$$f_\sigma(p,x) = xf(0) + \sum_{n=1}^{\infty} R_n(f-p,x) + S_{n,m}(p,x).$$

Proof. First, a calculation shows that

$$\begin{aligned}
&f_{\sigma_n}(p,x) - f_{\sigma_{n-1}}(p,x) \\
&= R_n(p-f,x) + \sum_{k=0}^{m} \frac{1}{k+1} R_{n-1}(a_k,1) R_n(B_{k+1},k),
\end{aligned}$$

because

$$\begin{aligned}
&f_{\sigma_n}(p,x) - f_{\sigma_{n-1}}(p,x) \\
&= -R_n(P_{m,n},x) + \sum_{k=1}^{n} R_k(f,x) + R_{n-1}(P_{m,n-1},x) - \sum_{k=1}^{n-1} R_k(f,x) \\
&= R_n(f,x) + P_{m,n}(n+x) - P_{m,n}(n) + P_{m,n-1}(n-1) \\
&- P_{m,n-1}(n-1+x) \\
&= R_n(f,x) + \sum_{k=0}^{m} a_k(n) \frac{B_{k+1}(n+x+1) - B_{k+1}(n+1)}{k+1} \\
&- \sum_{k=0}^{m} a_k(n-1) \frac{B_{k+1}(n+x) - B_{k+1}(n)}{k+1} \\
&= R_n(f,x) + \sum_{k=0}^{m} \frac{a_k(n)}{k+1} \Big(B_{k+1}(n+x) - B_{k+1}(n) \Big) \\
&- \sum_{k=0}^{m} \frac{a_k(n-1)}{k+1} \Big(B_{k+1}(n+x) - B_{k+1}(n) \Big) \\
&+ \sum_{k=0}^{m} \frac{a_k(n)}{k+1} \Big((k+1)((n+x)^k - n^k) \Big)
\end{aligned}$$

$$= R_n(f,x) - \sum_{k=0}^{m} \frac{a_k(n-1)-a_k(n)}{k+1}\Big(B_{k+1}(n+x) - B_{k+1}(n)\Big)$$
$$+ \sum_{k=0}^{m} a_k(n)((n+x)^k - n^k)$$
$$= R_n(f,x) + \sum_{k=0}^{m} \frac{1}{k+1} R_{n-1}(a_k,1)R_n(B_{k+1},x)$$
$$+ \sum_{k=0}^{m} a_k(n)(S_k(n+x) - S_k(n))$$
$$= R_n(f,x) + \sum_{k=0}^{m} \frac{1}{k+1} R_{n-1}(a_k,1)R_n(B_{k+1},x) + p(n+x)$$
$$- p(n) = R_n(f-p,x) + \sum_{k=0}^{m} \frac{1}{k+1} R_{n-1}(a_k,1)R_n(B_{k+1},x)$$
$$= R_n(f-p,x) + S_{n,m}(p,x).$$

Thus we obtain (2.4) if $x \in D_{f_\sigma(p)}$.
On the other hand, we have

$$\sum_{n=2}^{N}(f_{\sigma_n}(p,x) - f_{\sigma_{n-1}}(p,x)) = f_{\sigma_N}(p,x) - f_{\sigma_1}(p,x)$$

and so

$$\begin{aligned} f_{\sigma_N}(p,x) - f_{\sigma_1}(p,x) &= \sum_{n=2}^{N}(f_{\sigma_n}(p,x) - f_{\sigma_{n-1}}(p,x)) \\ &= \sum_{n=2}^{N}(R_n(f-p,x) + S_{n,m}(p,x)), \end{aligned}$$

for every integer $N \geqslant 2$. Therefore

$$f_{\sigma_N}(p,x) = f_{\sigma_1}(p,x) + \sum_{n=2}^{N}(R_n(f-p,x) + S_{n,m}(p,x)).$$

for every $x \in \Sigma_f$. Also

$$f_{\sigma_N}(p,x) = xf(0) + \sum_{n=2}^{N}(R_n(f-p,x) + S_{n,m}(p,x)) \ ; x \in \Sigma_f,$$

if $0 \in D_f$ and $N \geq 1$.
Therefore, $x \in D_{f_\sigma(p)}$ if and only if the series (2.5) (similarly (2.6)) is convergent. $\square$

Remark 2.5. By putting $m = 0$ and $a_0(n) = f(n)$ in the above identity we obtain

$$f_{\sigma_n}(x) = f_{\sigma_1}(x) + \sum_{k=1}^{n} R_k(f,x) - xR_{k-1}(f,1)$$

which is statement $(*_5)$, page 78, section 2, Note, in [4].

Corollary 2.6. *If $x \in D_{f_\sigma(p)}$ and $R_n(f-p,x)$ is convergent, then*

$$S_m(p,x) = \lim_{n\to\infty} S_{n,m}(p,x) = R(p-f,x).$$

For every function g and $x \in \Sigma_g$, we put $g_\infty(x) := \lim_{n\to\infty} g(n+x)$ (if the limit exists).

Theorem 2.7. *Let f be a function such that $\mathbb{Z}^+ \subseteq D_f$.*
(a) The functional sequence $p(n+x) - f(n+x)$ is convergent on $D_{f_\sigma(p)} \cap (D_{f_\sigma(p)}+1)$ and for $x \in D_{f_\sigma(p)} \cap (D_{f_\sigma(p)}+1)$ we have

$$f_\sigma(p,x) = f(x) + f_\sigma(p,x-1) + (p-f)_\infty(x) \tag{15}$$

(b) $1 \in D_{f_\sigma(p)}$ if and only if the sequence $(p-f)(n)$ is convergent, and we have

$$R(f-p,x) = (p-f)_\infty(x) + (f-p)_\infty(1); \ \ x \in D_{f_\sigma}(p), \tag{16}$$

Thus if $p(n) - f(n) \to 0$ as $n \to \infty$, then

$$f_\sigma(p,x) = f(x) + f_\sigma(p,x-1) + R\,(f-p,x); \quad x \in D_{f_\sigma}(p) \tag{17}$$

Proof. (a) If $x \in D_{f_\sigma(p)} \cap (D_{f_\sigma(p)}+1)$, then $x \in \Sigma_f + 1$ (so $x, x-1 \in \Sigma_f$) and we have

$$\begin{aligned}
f_{\sigma_n}(p,x) - f_{\sigma_n}(p,x-1) &= \sum_{k=1}^{n} R_k(f,x) - R_n(P,x) \\
&+ R_n(P,x-1) - \sum_{k=1}^{n} R_k(f,x-1) \\
&= P(n+x) - P(n+x-1) \\
&+ \sum_{k=1}^{n} f(x+k-1) - f(x+k) \\
&= P(n+x) - P(n+x-1) + f(x) - f(x+n) \\
&= p(n+x) + f(x) - f(x+n).
\end{aligned}$$

Thus

$$f_{\sigma_n}(p,x) = f(x) + f_{\sigma_n}(p,x-1) + p(n+x) - f(n+x); \quad x \in \Sigma_f + 1. \tag{18}$$

Now, according to the assumption, if $n \to \infty$, then we obtain

$$\begin{aligned}
f_\sigma(p,x) &= \lim_{n\to\infty} (f(x) + f_{\sigma_n}(p,x-1) + p(n+x) - f(n+x)) \\
&= f(x) + f_\sigma(p,x-1) + (p-f)_\infty(x).
\end{aligned}$$

On the other hand, by setting $x = 1$ in (18), we obtain

$$f_{\sigma_n}(1) - f_{\sigma_n}(0) = f(1) + p(n+1) - f(n+1).$$

Hence

$$f_{\sigma_n}(1) = f(1) + (p-f)(n+1).$$

Therefore, $1 \in D_{f_{\sigma_n}(p)}$ if and only if the sequence $(p-f)(n)$ is convergent, and

$$f_\sigma(1) = f(1) + (p-f)_\infty(1). \tag{19}$$

□

Theorem 2.8. *Note that if $m = 0$ in the above theorem, then $p(n) = p(n+1) = a_0(n)$ and*

$$p(n+1) - f(n+1) = a_0(n) - f(n+1).$$

Thus, by putting $a_0(n) = f(n)$, we obtain

$$(p-f)_\infty(1) = \lim_{n\to\infty}\Big(f(n) - f(n+1)\Big) = \lim_{n\to\infty} R_n(f,1) = R(f,1)$$

and arrive at $f_\sigma(1) = f(1) + R(f,1)$ which is mentioned in [2, p. 464] *(where $n \to \infty$).*

The following theorem is a generalization of [2, Theorem1.2].

Theorem 2.9. *The sequence $(p-f)(n)$ is convergent (equivalently, $1 \in D_{f_\sigma(p)}$), and the functional sequence $S_{n,m}(p,x)$ converges on $D_{f_\sigma(p)} \cap D_f$ if and only if $D_f \cap D_{f_\sigma(p)} = D_{f_\sigma(p)} + 1$.*
Moreover, if this holds, then $S_m(p,x) = R(p-f,x)$, where $S_m(p,x) := \lim\limits_{n\to\infty} S_{n,m}(p,x)$ for all $x \in D_{f_\sigma(p)} + 1$(i.e., $x \in D_f \cap D_{f_\sigma(p)}$).

Proof. If $x \in D_f \cap D_{f_\sigma(p)}$, then the assumptions implies that $R_n(f-p,x)$ is convergent and by Theorem (2.4)(b) we obtain $p(n+x) - f(n+x)$ is convergent and so is $f_{\sigma_n}(x-1)$, by (18). Hence $x-1 \in D_{f_\sigma(p)}$.
Conversely, if $x \in D_{f_\sigma(p)} + 1$, then $x \in D_f$ and $x - 1 \in D_{f_\sigma(p)}$. Now (17) and the hypothesis imply that $R_n(f-p,x-1)$ and $p(n+x) - f(n+x)$ are convergent. Thus (15) guarantees the convergence of $f_{\sigma_n}(p,x)$, so $x \in D_f \cap D_{f_\sigma(p)}$.
Now, let $D_f \cap D_{f_\sigma(p)} = D_{f_\sigma(p)} + 1$. Clearly $1 \in D_{f_\sigma(p)}$ and so the sequence $(p-f)(n)$ is convergent. If $x \in D_f \cap D_{f_\sigma(p)}$, then $x-1 \in D_{f_\sigma(p)}$ and so $p(n+x) - f(n+x)$ and $R_n(f-p,x)$ are convergent. On the other hand, since $x \in D_{f_\sigma(p)}$, the left hand of (17) tends to zero, as $n \to \infty$. Regarding the convergence of $R_n(f-p,x)$, we conclude that

$$S_m(p,x) = \lim_{n\to\infty} S_{n,m}(p,x) = -R(f-p,x) = R(p-f,x).$$

□

Example 2.10. Let $0 <$|b|< 1, $f(x) = b^x$, and $p(x) = \frac{1}{n!}x + \frac{1}{n}$. Then f is limit summable related to p and we have $f_\sigma(p,x) - f_\sigma(p,x-1) = f(x) + \frac{3e}{2}$. Thus the function $f_\sigma(p,x)$ satisfies the functional equation $\lambda(x) - \lambda(x-1) = f(x) + c$, where $c = \frac{3e}{2}$.

The next example is important because the considered function is not a limit summable function with [2. Definitions 1.1], but is a $p-$limit summable.

Example 2.11. The square function $f(x) = x^2$ is limit summable only at $x = 0, -1$ (i.e., $D_{f_\sigma} = \{0, -1\}$). But, if we set $p_{2,n}(x) = x^2$ (i.e., $a_0(n) = a_1(n) = 0$ and $a_2(n) = 1$), then f is limit summable related to p on $\mathbb{C}$ and we obtain $f_\sigma(p, x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 + \frac{1}{6}$. Also, it satisfies the related difference functional equation.

The next theorem and corollary state some important equivalent conditions that enable us to state p-limit summable functions.

Theorem 2.12. *The following statements are equivalent.*

(*a*) $f_\sigma(p, x) = f(x) + f_\sigma(p, x - 1) + (p - f)_\infty(x)$ *for all* $x \in D_f$.

(*b*) $D_f \subseteq D_{f_{\sigma(p)}}$ *and* $S_{n,m}(p, x)$ *is convergent on* D_f
(*and* $S_m(p, x) = R(p - f, x)$)

(*c*) $D_{f_{\sigma(p)}} = \Sigma_f$ *and* $D_f \subseteq D_f - 1$

Proof. $(a) \Rightarrow (b)$. The assumption (a) implies that $D_f \subseteq D_{f_{\sigma(p)}} \cap (D_{f_\sigma(p)} + 1)$ and $1 \in D_{f_\sigma(p)}$. Therefore Theorem (2.9) implies that the sequence $(p - f)_n$ is convergent. Now if $x \in D_f$, then $x \in D_{f_\sigma(p)}$. Thus Theorem 2.3 illustrates the functional sequence $S_{n,m}(p, x)$ is convergent on D_f and

$$S_m(p, x) = -R(f - p, x) = R(p - f, x); \;\; x \in D_f.$$

$(b) \Rightarrow (c)$. The first part of the assumption requires $D_f \subseteq \Sigma_f$. Hence $D_f = \Sigma_f \cap D_f = \Sigma_f + 1$. So $\Sigma_f = D_f - 1$ and $D_f \subseteq D_f - 1$. Since $1 \in D_{f_\sigma(p)}$, we have

$$\Sigma_f = D_f - 1 = (D_f \cap D_{f_\sigma(p)}) - 1 = D_{f_\sigma(p)},$$

by (2.9).
$(c) \Rightarrow (a)$. First $D_f \subseteq D_f - 1$ requires $\Sigma_f = D_f - 1$. Thus

$$D_{f_\sigma}(p) = \Sigma_f \supseteq D_f = \Sigma_f + 1 = D_{f_\sigma}(p) + 1.$$

Then Theorem (2.7) (a) impllies that (15) holds for each x belongs to $D_f = D_{f_\sigma}(p) + 1$. $\square$

Corollary 2.13. *The following statements are equivalent.*

(a) $f_\sigma(p,x) = f(x) + f_\sigma(p,x-1)$ *for all* $x \in D_f$.

(b) $S_m(p,x) = (p-f)_\infty(1) = 0$ *for all* $x \in D_f$ *and* $D_f \subseteq D_{f_\sigma}$.

(c) $S_m(p,x) = (p-f)_\infty(1) = 0$ *for all* $x \in D_f$, $D_f \subseteq D_{f_\sigma} - 1$, *and* $D_{f_\sigma(p)} = \Sigma_f$.

Proof. This is a direct result of Theorem (2.9) and (2.12). $\square$

Definition 2.14. Let $p = p_{m,n}$ be a sequence of polynomials of the fixed degree m. The function f is called p$-$limit summable if it is p$-$limit summable on its domain D_f and $S_m(p,x) = (p-f)_\infty(1) = 0$ on D_f. In this case, the function $f_\sigma(p)$ is referred to as the p-limit summand function of f.

The p-limit summand function of f (id exists) satisfies the difference functional equation $\lambda(x) = f(x) + \lambda(x-1)$, and so $f_\sigma(p,r) = \sum_{k=1}^{r} f(k)$ for all $r \in \mathbb{Z}_+$.
One of the appropriate conditions for p-limit summability is that $S_{n,m}(p,x)$ is convergent. Below we state and prove a simple criteria that states sufficient conditions for the convergence of $S_{n,m}(p,x)$.

Theorem 2.15. *If the sequences* $(a_k(n) - a_k(n-1))n^k$ *are convergent for* $k = 0,1,...,m$, *then* $S_{n,m}(p,x)$ *converges on* $\mathbb{C}$.

Proof. First, we have

$$\begin{aligned} R_n(B_{k+1},x) &= B_{k+1}(n+x) - B_{k+1}(n) \\ &= \sum_{j=0}^{k+1} \binom{k+1}{j} B_{k+1-j}((n+x)^j - n^j). \end{aligned}$$

Therefore

$$\begin{aligned} S_{n,m}(p,x) &= \sum_{k=0}^{m} \big(a_k(n-1) - a_k(n)\big) \frac{B_{k+1}(n+x) - B_{k+1}(n)}{k+1} \\ &= \sum_{k=0}^{m} \sum_{j=0}^{k+1} \binom{k+1}{j} \frac{B_{k+1-j}}{k+1} \big(a_k(n-1) - a_k(n)\big)\big((n+x)^j - n^j\big). \end{aligned}$$

Since the sequences

$(a_0(n)-a_0(n-1)), (a_1(n)-a_1(n-1))n), ..., (a_{m+1}(n)-a_{m+1}(n-1))n^m$

are convergent, we conclude that the sequences $(a_k(n)-a_k(n-1))\big((n+x)^j - n^j\big)$ are convergent for all $0 \leq k \leq m$ and $j \in \mathbb{Z}^+$. Therefore the sequences $S_{n,m}(p,x)$ are convergent as $n \to \infty$. $\square$

Corollary 2.16. *Fix an integer $m \geq 0$ and suppose that the sequences $b_0(n), \cdots, b_m(n)$ are convergent. Put*

$$a_k(n) := \sum_{j=0}^{n} \frac{b_k(j)}{j^k} \ ; \ k = 0, 1, ..., m$$

Then, the sufficient conditions of Theorem 2.5 hold and so $S_{n,m}(p,x)$ is convergent.

2.2 Relations to the fractional sums.

There are some connections between the p-limit summability and fractional sums, introduced by Muller and Schneider in [6]. For instance, if $m = 0$ and $a_0(n) = 0$ for all n (i.e., $p_{0,n}$ is the zero constant polynomial), then we have

$$f_\sigma(p,x) = \sum_{k=1}^{\infty} (f(k) - f(k+x))$$

which is the fractional sum of degree $-\infty$ in [6]. If $m = 1$, $a_0(n) = f(n)$ for all n then

$$f_\sigma(p,x) = \lim_{n\to\infty} (xf(n) + \sum_{k=1}^{n} f(k) - f(k+x))$$

that is the fractional sum of degree 0 in it. For $m = 1$, $a_0(n) = f(n)$, and $a_1(n) = f(n+1) - f(n) = -R_n(f,1)$ (for all n) we arrive at

$$\lim_{n\to\infty} \Big(xf(n) + \frac{x(x+1)}{2}(f(n+1) - f(n)) + \sum_{k=1}^{n} f(k) - f(k+x)\Big)$$

that is the fractional sum of degree 1 in it. But no general form or recurrence was observed in that paper to continue this process. However, we have arrived at the following conjecture for it.

Conjecture. Define $R_n^{(0)}(f,1) := f(n)$, and let $R_n^{(j)}(f,1)$ be defined by the recurrence relation

$$R_n^{(j)}(f,1) \;=\; R_n^{(j-1)}(f,1) - R_{n+1}^{(j-1)}(f,1) \;;\; j = 1,2,3,\cdots .$$

For every integer $m \geq 0$, the fractional sum of degree m (mentioned above) is of the form

$$\psi_{n,m}(f,x) + \sum_{k=0}^{n}(f(k) - f(k+x)) \;\; ; x \in \sum\nolimits_f . \tag{20}$$

where

$$\psi_{n,m}(f,x) \;=\; xf(n) + \sum_{j=1}^{m}(-1)^j R_n^{(j)}(f,1)\binom{x+1}{j+1}.$$

Note that $\binom{x+1}{j+1} = \frac{(x+1)\cdots(x-j+1)}{(j+1)!}$, $\sum_{j=1}^{0} a_j = 0$, and the form is valid for $m = -\infty$ if we define $\psi_{n,-\infty}(f,x) := 0$. Additionally, the above form is equal to

$$f_{\sigma_n}(x) + \sum_{j=1}^{m}(-1)^j R_n^{(j)}(f,1)\binom{x+1}{j+1},$$

for all $m \geq 0$, which shows its relation to the limit summability introduced by Hooshmand.

References

[1] T. M. Apostol, *Introduction to Analytic Number Theory*, Springer, Berlin (1976)

[2] M. H. Hooshmand, Limit Summability of Real Functions, *Real Anal. Exch*, 27 (2001), 463 – 472.

[3] M. H. Hooshmand, Analytic Summability of Real and complex Functions, *J. Contemp. Math. Anal.*, 5 (2016), 262 – 268.

[4] M. H. Hooshmand, Another Look at the Limit Summability of Real Functions, *J. Math. Ext.*, 4 (2009), $73-89$.

[5] W. Krull, Bemerkungen Zur Differenzengleichung $g(x+1)-g(x)=F(x)$, *I. Math. Nachr.*, 1 (1984), $365-376$.

[6] M. Muller, D. Schleicher, Fractional Sums and Euler-Like Identities, *Ramanujan J.*, 21 (2) (2010), $123-143$.

[7] R.J. Webster, Log-Convex Solutions to the Functional Equation $f(x+1)=g(x)f(x)$: Γ -Type Functions, *J. Math. Anal. Appl.*, 209 (1997), $605-623$.

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