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Original Research Paper

On Solvability of Some Impulsive Initial Value Fractional Differential Equations Via Generalized Weak Wardowski Contractions

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Abstract. In this paper, we investigate the solvability of some impulsive initial value fractional differential equations of arbitrary order via providing a generalized weak Wardowski contraction. Our results introduce a model for solving impulsive initial value fractional differential equations of arbitrary order. An illustrative example is given to show the usability and usefulness of our main result.

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1 Introduction

In many technical fields of science, including chemistry, mechanics, and physics, fractional integro-differential operators have many applications for investigating mathematical modeling of natural phenomena. For several published articles in this field, we refer the reader to [14, 8, 19, 9, 10, 5, 13, 1, 3, 22, 17, 23]. Among these works, the fractional Riemann-Liouville and Caputo operators have been the most widely used. A

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special type of differential equations, called impulsive differential equations, has played an important role in modeling phenomena, especially in describing the dynamics of populations and their sudden changes, as well as other phenomena such as crop harvests, diseases, etc. Over the past century, some authors have used impulsive differential systems to describe this model. For example, many physical situations are modeled by problems of this kind: problems in optimal control theory and problems in threshold theory in biology. The past ten years or so have witnessed major developments in the field of impulse differential equations. For an introduction to the basic theory of impulsive differential equations, the reader can refer to the monographs of Burton and Simeonov [2], Lakshmikantham et al. [16] and Benchohra et al. [4]. In [5], Benchohra and Slimani studied existence and uniqueness of solutions for the following impulsive initial value problem (IVP for short) for fractional order differential equation:

$$\begin{gathered} {}^{\mathfrak{C}}\mathfrak{D}^{\gamma} y(t) = f(t, y(t)),\ t \in J = [0, T],\ t \neq t_k, \\ \triangle y|_{t=t_k} = I_k(y(t_k^-)), \\ y(0) = y_0, \end{gathered} \tag{1}$$

where $m \in \mathbb{N}$, $k = 1, ..., m, 0 < \gamma \leq 1$, ${}^{\mathfrak{C}}\mathfrak{D}^{\gamma}$ is the Caputo fractional derivative, $f : J \times \mathbb{R} \to \mathbb{R}$ is a given function, $I_k : \mathbb{R} \to \mathbb{R}$ and $y_0 \in \mathbb{R}, 0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $\triangle y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, and $y(t_k^+) = \lim_{h \to 0^+} y(t_k + h)$, $y(t_k^-) = \lim_{h \to 0^-} y(t_k + h)$ represent the right and left limits of $y(t)$ at $t = t_k$, respectively. Afterwards, several researchers have studied on this type of equations considering other conditions and derivative operators (see for example [11, 12, 7, 21, 26, 24]). To the author's knowledge, no study has been done on this type of fractional differential equations with arbitrary order.

Inspired with this vacuity, in this paper, we investigate solvability of the following IVP:

$$\begin{gathered} {}^{\mathfrak{C}}\mathfrak{D}^{\gamma} y(t) = f(t, y(t)),\ t \in J = [0, T],\ t \neq t_k, \\ \triangle y^{(j)}|_{t=t_k} = I_{kj}(y^{(j)}(t_k^-)), \\ y^{(j)}(0) = y_j, I_{kj}(y^{(j)}(t_k^-)) = I_{k0}(y(t_k^-)) \end{gathered} \tag{2}$$

using a generalized weak Wardowski contraction introduced in the sequel while $f : J \times \mathbb{R} \to \mathbb{R}$ is a given function which does not apprise necessarily a Banach type contraction, where $k = 1, ..., m, n-1 < \gamma \leq n$, ${}^{\mathfrak{C}}\mathfrak{D}^{\gamma}$ is the Caputo fractional derivative, $I_k : \mathbb{R} \to \mathbb{R}$, and $y_j \in \mathbb{R}$, $j = 0, 1, 2, \cdots, n-1$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $\triangle y^{(j)}|_{t=t_k} = y^{(j)}(t_k^+) - y^{(j)}(t_k^-), y^{(j)}(t_k^+) = \lim_{h\to 0^+} y^{(j)}(t_k + h)$ and $y^{(j)}(t_k^-) = \lim_{h\to 0^-} y^{(j)}(t_k + h)$ represent the right and left limits of $y^{(j)}(t)$ at $t = t_k$, respectively. Our results open a way to model physical situations that obey impulsive fractional differential equations of arbitrary order. An example is given to show the usability of our new results.

2 Preliminaries and Auxiliary Notiones

In 2012, Wardowski introduced a new proper generalization of Banach contraction as follows. Suppose that $\mathcal{F}$ represents the collection of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ satisfying the following conditions:

(F1) F is strictly increasing,

(F2) For each sequence $\{\alpha_n\}$ in $(0, +\infty)$, $\lim\limits_{n\to\infty} \alpha_n = 0$ if and only if $\lim\limits_{n\to\infty} F(\alpha_n) = -\infty$,

(F3) There exists $k \in (0, 1)$ such that $\lim\limits_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

Definition 2.1. [25] *Let (Ω, ρ) be a complete metric space. A mapping $T : \Omega \to \Omega$ is said to be an F-contraction if there exist $\tau \in \mathbb{R}^+$ and $F \in \mathcal{F}$ such that for all $\delta, \xi \in \Omega$,*

$$\rho(T\delta, T\xi) > 0 \Longrightarrow \tau + F(\rho(T\delta, T\xi)) \leq F(\rho(\delta, \xi)). \quad (3)$$

Theorem 2.2. *[25] Let (Ω, ρ) be a complete metric space and let $T : \Omega \to \Omega$ be an F-contraction. Then T has a unique fixed point δ^* in Ω and for any point $\delta \in \Omega$ the sequence $\{T^n\delta\}$ converges to δ^*.*

In order to obtain a new generalization of Banach contraction principle, in this paper we replace the positive constant τ with a function θ and to change some properties of function F as follows: Denote by Ξ the set of all functions $\Im : [0, \infty] \to [-\infty, \infty]$ such that:

(δ_1) $\Im$ is increasing and continuous,

(δ_2) $\Im(s) = 0 \quad \Leftrightarrow \quad s = 1.$

As examples of elements of Ξ:

(i) $\Im_1(t) = \begin{cases} \ln(t), & t \in (0, \infty), \\ -\infty, & t = 0, \\ \infty, & t = \infty, \end{cases}$

(ii) $\Im_2(t) = \begin{cases} t + \ln(t) - 1, & t \in (0, \infty), \\ -\infty, & t = 0, \\ \infty, & t = \infty, \end{cases}$

(iii) $\Im_3(t) = \begin{cases} -\frac{1}{\sqrt{t}} + 1, & t \in (0, \infty), \\ -\infty, & t = 0, \\ 1, & t = \infty, \end{cases}$

(iv) $\Im_4(t) = \begin{cases} -\frac{1}{t} + 1, & t \in (0, \infty), \\ -\infty, & t = 0, \\ 1, & t = \infty. \end{cases}$

Denote by Θ the collection of all functions $\vartheta : \mathbb{R} \to (0, \infty)$ such that ϑ is continuous.

As examples of elements of Θ:

(i) $\vartheta_1(s) = \tau$, $\tau > 0$,

(ii) $\vartheta_2(s) = \tau e^{-s}$, $\tau > 0$,

(iii) $\vartheta_3(s) = \tau + e^{-s}$, $\tau > 0$,

(iv) $\vartheta_4(s) = \tau + s^2$, $\tau > 0$,

(v) $\vartheta_5(s) = s^2 + s + 1$.

Definition 2.3. [20]*Let Ω be a non-empty set and $\eta : \Omega \times \Omega \to [0, \infty)$ be a function. A mapping $\Lambda : \Omega \to \Omega$ is said to be η-admissible if for all $\delta, \xi \in \Omega$, $\eta(\delta, \xi) \geq 1$ implies $\eta(\Lambda\delta, \Lambda\xi) \geq 1$.*

A mapping $\Lambda : \Omega \to \Omega$ is called triangular η-admissible, if Λ is η-admissible and for all $\delta, \xi, \varrho \in \Omega$, $\eta(\delta, \xi) \geq 1$ and $\eta(\xi, \varrho) \geq 1$ imply that $\eta(\delta, \varrho) \geq 1$.

Througout this paper for any $\delta, \xi \in \Omega$, where (Ω, ρ) is a metric space, take

$$M_\rho(\delta, \xi) = \max\left\{\rho(\delta, \xi),\ \rho(\delta, \Lambda\delta),\ \rho(\xi, \Lambda\xi),\ \frac{\rho(\delta, \Lambda\xi) + \rho(\xi, \Lambda\delta)}{2}\right\}.$$

3 Main Results

Now, we introduce a new generalization of Banach' contraction in this section.

Definition 3.1. *Let (Ω, ρ) be a metric space and $\Lambda : \Omega \to \Omega$ be a self-mapping. We say Λ is a generalized weak Wardowski η-contraction whenever there exist functions $\eta : \Omega \times \Omega \to [0, \infty)$, $\vartheta \in \Theta$ and $\mathfrak{F} \in \Xi$ such that*

$$\mathfrak{F}(\rho(\Lambda\delta, \Lambda\xi)) \leq \mathfrak{F}(M_\rho(\delta, \xi)) - \vartheta(\mathfrak{F}(M_\rho(\delta, \xi))) \tag{4}$$

for all $\delta, \xi \in \Omega$ with $\eta(\delta, \xi) \geq 1$ and $\Lambda\delta \neq \Lambda\xi$.

Theorem 3.2. *Let (Ω, ρ) be a complete metric space and $\Lambda : \Omega \to \Omega$ be a triangular η-admissible generalized weak Wardowski η-contraction self-mapping. Moreover, let*

(i) there exists $\delta_0 \in \Omega$ such that $\eta(\delta_0, \Lambda\delta_0) \geq 1$,

(ii) for each sequence $\{\delta_n\}$ in Ω with $\eta(\delta_n, \delta_{n+1}) \geq 1$ for all n and $\delta_n \to \delta$, then $\eta(\delta_n, \delta) \geq 1$ for all n.

Then, Λ has a fixed point. Moreover, if $\eta(\delta^*, \xi^*) \geq 1$ for any fixed points δ^*, ξ^*, then the fixed point of Λ is unique.

Proof. Construct the sequence $\{\delta_n\}$ by $\delta_n = T\delta_{n-1}$ for all $n \in \mathbb{N}$. If $\delta_n = \delta_{n-1}$ for some $n \in \mathbb{N}$, then δ_{n-1} is a fixed point of T. So, we may assume $\delta_n \neq \delta_{n-1}$ for all $n \in \mathbb{N}$. Then, from (4), for any $n \in \mathbb{N}$, we have

$$\begin{aligned} \mathfrak{F}(\rho(\delta_{n+1}, \delta_n)) &= \mathfrak{F}(\rho(T\delta_n, T\delta_{n-1})) \\ &\leq \mathfrak{F}(M_\rho(\delta_n, \delta_{n-1})) - \vartheta(\mathfrak{F}(M_\rho(\delta_n, \delta_{n-1}))), \end{aligned} \tag{5}$$

where

$$\begin{aligned} &\max\left\{\rho(\delta_{n-1}, \delta_n), \rho(\delta_n, \delta_{n+1})\right\} \leq M(\delta_{n-1}, \delta_n) \\ &= \max\left\{\rho(\delta_{n-1}, \delta_n), \rho(\delta_{n-1}, T\delta_{n-1}), \rho(\delta_n, T\delta_n) \right. \\ &\left., \tfrac{\rho(\delta_n, T\delta_{n-1}) + \rho(\delta_{n-1}, T\delta_n)}{2}\right\}, \\ &\leq \max\left\{\rho(\delta_{n-1}, \delta_n), \rho(\delta_n, \delta_{n+1})\right\}. \end{aligned}$$

Thus

$$M(\delta_{n-1}, \delta_n) = \max\left\{\rho(\delta_{n-1}, \delta_n), \rho(\delta_n, \delta_{n+1})\right\}.$$

If

$$\max\left\{\rho(\delta_{n-1}, \delta_n), \rho(\delta_n, \delta_{n+1})\right\} = \rho(\delta_n, \delta_{n+1}),$$

then, by (5), we have

$$\mathfrak{F}(\rho(\delta_n, \delta_{n+1})) \leq \mathfrak{F}(\rho(\delta_n, \delta_{n+1})) - \vartheta(\mathfrak{F}(\rho(\delta_n, \delta_{n+1})) < \mathfrak{F}(\rho(\delta_n, \delta_{n+1}))$$

which gives a contradiction. Thus,

$$\max\left\{\rho(\delta_{n-1}, \delta_n), \rho(\delta_n, \delta_{n+1})\right\} = \rho(\delta_{n-1}, \delta_n).$$

Therefore, from (5),

$$\mathfrak{F}(\rho(\delta_n, \delta_{n+1})) \leq \mathfrak{F}(\rho(\delta_{n-1}, \delta_n)) - \vartheta(\mathfrak{F}(\rho(\delta_{n-1}, \delta_n)) < \mathfrak{F}(\rho(\delta_{n-1}, \delta_n)). \tag{6}$$

Since $\mathfrak{F}$ is increasing, thus we have

$$\rho(\delta_n, \delta_{n+1}) < \rho(\delta_{n-1}, \delta_n), \quad for\ each\ n \geq 0.$$

So $\{\rho(x_{n-1}, x_n)\}$ is a decreasing sequence in $[0, \infty)$ and so there is $r \geq 0$ such that $\rho(\delta_{n-1}, \delta_n) \to r^+$. Now, we show that $r = 0$. Suppose to the contrary $r > 0$. Passing to the limit through (6),

$$\mathfrak{F}(r) \leq \mathfrak{F}(r) - \vartheta(\mathfrak{F}(r) < \mathfrak{F}(r),$$

which is a contradiction. So $\lim_{n\to\infty} \rho(\delta_{n-1}, \delta_n) = r = 0$. We claim that $\{\delta_n\}$ is Cauchy. If $\{\delta_n\}$ is not Cauchy, then there are $\varepsilon > 0$ and subsequences $\{\delta_{m_i}\}$ and $\{\delta_{n_i}\}$ of $\{\delta_n\}$

$$n_i > m_i > i,\ \rho(\delta_{m_i}, \delta_{n_i}) \geq \varepsilon \tag{7}$$

and

$$\rho(\delta_{m_i}, \delta_{n_i-1}) < \varepsilon.$$

Using (7), we get

$$\varepsilon \leq \rho(\delta_{m_i}, \delta_{n_i}) \leq \rho(\delta_{m_i}, \delta_{n_i-1}) + \rho(\delta_{n_i-1}, \delta_{n_i}) < \varepsilon + \rho(\delta_{n_i-1}, \delta_{n_i}).$$

As $i \to \infty$, we find

$$\lim_{i\to\infty} \rho(\delta_{m_i}, \delta_{n_i}) = \varepsilon.$$

Also, we have

$$\begin{aligned}
&\rho(\delta_{m_i}, \delta_{n_i}) - \rho(\delta_{m_i}, \delta_{m_i+1}) - \rho(\delta_{n_i}, \delta_{n_i+1}) \\
&\leq \rho(\delta_{m_i+1}, \delta_{n_i+1}) \\
&\leq \rho(\delta_{m_i}, \delta_{m_i+1}) + \rho(\delta_{m_i}, \delta_{n_i}) + \rho(\delta_{n_i}, \delta_{n_i+1}).
\end{aligned}$$

As $i \to \infty$, we find

$$\lim_{i\to\infty} \rho(\delta_{m_i+1}, \delta_{n_i+1}) = \varepsilon.$$

By triangular α-admissibility of T, we find $\alpha(\delta_{m_i}, \delta_{n_i}) \geq 1$, for all $i \in \mathbb{N}$. From (4), we get

$$\mathfrak{F}(\rho(\delta_{m_i+1}, \delta_{n_i+1})) \leq \mathfrak{F}(M_\rho(\delta_{m_i}, \delta_{n_i})) - \vartheta(\mathfrak{F}(M_\rho(\delta_{m_i}, \delta_{n_i}))). \tag{8}$$

On the other hand

$$\begin{aligned}\rho\left(\delta_{m_i},\delta_{n_i}\right) &\leq M_\rho\left(\delta_{m_i},\delta_{n_i}\right)\\ &= \max\{\rho\left(\delta_{m_i},\delta_{n_i}\right),\rho\left(\delta_{m_i},T\delta_{m_i}\right),\rho\left(\delta_{n_i},T\delta_{n_i}\right)\\ &,\frac{\rho\left(\delta_{m_i},T\delta_{n_i}\right)+\rho\left(\delta_{n_i},T\delta_{m_i}\right)}{2}\}\\ &\leq \max\{\rho\left(\delta_{m_i},\delta_{n_i}\right),\rho\left(\delta_{m_i},\delta_{m_i+1}\right),\rho\left(\delta_{n_i},\delta_{n_i+1}\right),\\ &\frac{2\rho\left(\delta_{m_i},\delta_{n_i}\right)+\rho\left(\delta_{n_i},\delta_{n_i+1}\right)+\rho\left(\delta_{m_i},\delta_{m_i+1}\right)}{2}\}.\end{aligned}$$

Taking the limit as $i\to\infty$ in the above inequality, we have

$$\lim_{i\to\infty} M_\rho\left(\delta_{m_i},\delta_{n_i}\right)=\varepsilon.$$

Taking limit in both sides of (8),

$$\mathfrak{F}(\varepsilon)\leq\mathfrak{F}(\varepsilon)-\vartheta(\mathfrak{F}(\varepsilon))<\mathfrak{F}(\varepsilon),$$

a contradiction.

Thus, $\{\delta_n\}$ is a Cauchy sequence in the complete metric space (Π,ρ). Hence there is $x\in X$ such that

$$\lim_{n\longrightarrow\infty}\delta_n=\delta^*.$$

Finally, we claim that $T\delta^*=\delta^*$. To show this, we have two cases:

(1) There is $N\in\mathbb{N}$ such that $T\delta_n\neq T\delta^*$ for each $n\geq N$,

(2) There is a subsequence $\{\delta_{n_k}\}$ of $\{\delta_n\}$ such that $T\delta_{n_k}=T\delta^*$ for each $k\geq 0$.

In the case (1), if $\rho\left(\delta^*,T\delta*\right)\neq 0$, we have

$$\begin{aligned}\mathfrak{F}(d(\delta_{n+1},T\delta^*)) &= \mathfrak{F}(d(T\delta_n,T\delta^*))\\ &\leq \mathfrak{F}(M_\rho(\delta_n,\delta^*))-\vartheta(\mathfrak{F}(M_\rho(\delta_n,\delta^*))).\end{aligned} \tag{9}$$

On the other hand,

$$\begin{aligned}\rho\left(\delta*, T\delta^*\right) &\leq M_\rho\left(\delta_n, \delta^*\right)\\ &= \max\{\rho\left(\delta_n, \delta^*\right), \rho\left(\delta_n, T\delta_n\right), \rho\left(\delta^*, T\delta^*\right)\\ &, \frac{\rho\left(\delta_n, T\delta^*\right) + \rho\left(\delta^*, T\delta_n\right)}{2}\}\\ &\leq \max\{\rho\left(\delta_n, \delta^*\right), \rho\left(\delta_n, \delta_{n+1}\right), \rho\left(\delta^*, T\delta^*\right),\\ &\frac{\rho\left(\delta_n, \delta^*\right) + \rho\left(\delta^*, T\delta^*\right) + \rho\left(\delta^*, \delta_{n+1}\right)}{2}\}.\end{aligned}$$

Taking limit as $i \to \infty$ in the above inequality, we have

$$\lim_{i\to\infty} M_\rho\left(\delta_n, \delta^*\right) = \rho\left(\delta^*, T\delta^*\right).$$

Passing to the limit through (9), we obtain

$$\begin{aligned}\mathfrak{F}(\rho(\delta^*, T\delta^*)) &\leq \mathfrak{F}(\rho(\delta^*, T\delta^*)) - \vartheta(\mathfrak{F}(\rho(\delta^*, T\delta^*)))\\ &< \mathfrak{F}(\rho(\delta^*, T\delta^*)),\end{aligned}$$

a contradiction. Thus $\rho\left(\delta^*, T\delta^*\right) = 0$ and so $\delta^* = T\delta^*$. In the case (2),

$$\rho(\delta^*, T\delta^*) = \lim_{n\to\infty} d(\delta_{n_k+1}, T\delta^*) = \lim_{n\to\infty} \rho(T\delta_n, T\delta^*) = 0.$$

We deduce that $T\delta^* = \delta^*$. To show the uniqueness of fixed point, suppose that δ^*, ξ^* are two distinct fixed points of T. Using (4), we have

$$\begin{aligned}\mathfrak{F}(\rho(\delta^*, \xi^*)) &= \mathfrak{F}(\rho(T\delta^*, T\xi^*))\\ &\leq \mathfrak{F}(\rho(\delta^*, \xi^*)) - \vartheta(\mathfrak{F}(\rho(\delta^*, \xi^*)))\\ &< \mathfrak{F}(\rho(\delta^*, \xi^*)),\end{aligned}$$

a contradiction. Thus $\delta^* = \xi^*$. □

If we take $\mathfrak{F}(t) = -\frac{1}{t}$, $\vartheta(t) = \frac{1}{3}$ and $\eta(\delta, \xi) = 1$ for all $\delta, \xi \in \Omega$ in (4) and Theorem 3.2, since $\rho(\delta, \xi) \leq M_\rho(\delta, \xi)$ and the function $f(t) = \frac{3t}{3+t}$ is increasing, we have the following result:

Corollary 3.3. *Let (Ω, ρ) be a complete metric space and $\Lambda : \Omega \to \Omega$ be a self-mapping satisfying*

$$\rho(\Lambda\delta, \Lambda\xi) \leq \frac{3\rho(\delta, \xi)}{3 + \rho(\delta, \xi)}$$

for all $\delta, \xi \in \xi^*$. Then, Λ has a unique fixed point.

Example 3.4. Let $\Omega = \{\frac{1}{n} > n \in \mathbb{N}\} \cup \{0\}$. Consider the usual metric $\rho(\delta, \xi) = |\delta - \xi|$ on Ω. Then (Ω, ρ) is a complete metric space. Define $\Lambda : \Omega \to \Omega$ by

$$\Lambda(\delta) = \begin{cases} \frac{1}{n+1}, & \delta = \frac{1}{n}, \\ 0, & \delta = 0. \end{cases}$$

Then, it is easy to check that the conditions of Corollary 3.3 hold. Thus Λ has a unique fixed point 0. Note that this mapping does not satisfy the Banach contraction since

$$\sup_{\delta,\xi\in\Omega} \frac{\rho(\Lambda\delta, \Lambda\xi)}{\rho(\delta, \xi)} \geq \sup_{n\in\mathbb{N}} \frac{\rho(\Lambda(\frac{1}{n}), \Lambda(0))}{\rho(\frac{1}{n}, 0)} = \sup_{n\in\mathbb{N}} \frac{\frac{1}{n+1}}{\frac{1}{n}} = 1.$$

Remark 3.5. Taking $\mathfrak{F}(t) = \ln(t)$, $\vartheta(t) = \tau$ and $\eta(\delta, \xi) = 1$ for all $\delta, \xi \in \Omega$ in (4) and Theorem 3.2, this theorem reduces to the Banach contraction principle. So, this theorem is a generalization of Banach contraction principle. However, Example 3.4 shows that this generalization is real.

Now, let us recall some introductive definitions of fractional differential equations (see [18], [15]):

Given a closed interval $[a, b]$, for a continuous function $f : [a, b] \to \mathbb{R}$, the Riemann-Liouville integral of fractional order γ is defined by

$$\mathcal{I}_a^\gamma f(t) = \frac{1}{\Gamma(r)} \int_a^t (t - \tau)^{\gamma-1} f(\tau) d\tau.$$

The Caputo-derivative of fractional order γ is defined by

$${}^{\mathfrak{C}}\mathfrak{D}_{a^+}^\gamma f(t) = \frac{1}{\Gamma(n - \gamma)} \int_a^t (t - \tau)^{n-\gamma-1} f^{(n)}(\tau) d\tau \ \ (n - 1 < \gamma \leq n),$$

where as the Riemann-Liouville fractional derivative of order γ is defined by

$${}^{\mathfrak{RL}}\mathfrak{D}_{a^+}^\gamma f(t) = \frac{1}{\Gamma(n - \gamma)} (\frac{d}{dt})^n \int_a^t (t - \tau)^{n-\gamma-1} f(\tau) d\tau \ \ (n - 1 < \gamma \leq n).$$

By $AC^n[a, b]$ we denote the set of all functions with an absolutely continuous $(n - 1)$st derivative and with nst derivative integrable on $[a, b]$. The composition rules for the Caputo-derivative of fractional order and the Riemann-Liouville integral of fractional order are recalled in the following lemma:

Theorem 3.6. *[15, 6] Assume that $\gamma \geq 0$, $n = \lceil \gamma \rceil$, and $f \in AC^n[a, b]$. Then*

$$\mathcal{I}_a^\gamma \, {}^{\mathfrak{C}}\mathfrak{D}_{a^+}^\gamma f(x) = f(x) - \sum_{j=0}^{n-1} \frac{f^{(j)}(a^+)}{j!}(x-a)^j.$$

Consider a closed interval $[a, b]$ and the partition $a = t_0 < t_1 < ... < t_m < t_{m+1} = b$, where $m \in \mathbb{N}$. Let $\gamma > 0$ and $n - 1 < \gamma < n$, $n \in \mathbb{N}$. From now on, assume that $\Omega = PC^n([a, b], \mathbb{R})$ is the set of nonnegative real valued functions y which $y \in AC^n((t_i, t_{i+1}], \mathbb{R})$ for $i = 1, 2, ..., m$ and $y \in AC^n([t_0, t_1], \mathbb{R})$ and $y^{(j)}(t_i^+) = \lim_{h\to 0^+} y^{(j)}(t_i + h)$ and $y^{(j)}(t_i^-) = \lim_{h\to 0^-} y^{(j)}(t_i + h)$ exists and finite at each t_i for $i = 1, 2, ...m$. Define a norm on Ω with

$$\|y\| = \sup_{t\in[a,b]} |y(t)|.$$

Define $\rho(y_1, y_2) = \|y_1 - y_2\|$ for all $y_1, y_2 \in \Omega$. Then (Ω, ρ) is a complete metric space.

Lemma 3.7. *Let $\gamma > 0$, $n = \lceil \gamma \rceil$ and $m \in \mathbb{N}$. Given $T > 0$, for a function $h \in L^1([0, T], \mathbb{R})$, a function $y \in PC^n([0, T], \mathbb{R})$ is a solution of the equation*

$$\begin{cases} {}^{\mathfrak{C}}\mathfrak{D}^\gamma y(t) = h(t);\ t \in [0, T] - \{t_1, ..., t_m\} = J'; \\ \triangle y^{(j)}|_{t=t_i} = I_i(y^{(j)}(t_i^-)), j = 0, 1, 2, ..., n-1, i = 1, 2, ..., m; \\ y^{(j)}(0) = y_j, j = 0, 1, 2, ..., n-1 \end{cases} \tag{10}$$

if and only if

$$y(t) = \left\{ \begin{array}{l} \sum_{j=0}^{n-1} \frac{y_j}{j!} t^j \\ + \sum_{j=0}^{n-1} \frac{1}{j!\Gamma(\gamma-j)} \sum_{i=1}^{p} (t-t_i)^j \int_{t_{i-1}}^{t_i} (t_i - s)^{\gamma-j-1} h(s)ds \\ + \sum_{j=0}^{n-1} \sum_{i=1}^{p} I_k(y^{(j)}(t_k^-))(t-t_i)^j \\ + \frac{1}{\Gamma(\gamma)} \int_{t_p}^{t} (t-s)^{\gamma-1} h(s)ds, \quad t \in (t_p, t_{p+1}], \quad p = 1, 2, \ldots, m \\ \sum_{j=0}^{n-1} \frac{y_j}{j!} t^j + \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} h(s)ds, \quad t \in [0, t_1] \end{array} \right\} \tag{11}$$

Proof. If $t \in [0, t_1]$, then by Theorem 3.6,

$$\begin{aligned} y(t) &= \textstyle\sum_{j=0}^{n-1} \frac{y^{(j)}(0)}{j!} t^j + \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} h(s) ds \\ &= \textstyle\sum_{k=0}^{n-1} \frac{y^{(k)}(0)}{k!} t^k + \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} h(s) ds. \end{aligned} \tag{12}$$

Now let $t \in (t_1, t_2]$. Then applying $\mathcal{I}_{t_1}^{\gamma}$ on equation (10), and using Theorem 3.6 on $t \in (t_1, t_2]$, we have

$$y(t) = \sum_{j=0}^{n-1} \frac{y^{(j)}(t_1^+)}{j!} (t-t_1)^j + \frac{1}{\Gamma(\gamma)} \int_{t_1}^t (t-s)^{\gamma-1} h(s) ds.$$

Since

$$y^{(j)}(t_1^+) = y^{(j)}(t_1^-) + \triangle y^{(j)}|_{t=t_1} = y^{(j)}(t_1^-) + I_j(y(t_1^-)),$$

substituting this in the above equation, we get

$$\begin{aligned} y(t) &= \textstyle\sum_{j=0}^{n-1} \frac{y^{(j)}(t_1^-)}{j!} (t-t_1)^j \\ &+ \textstyle\sum_{j=0}^{n-1} \frac{I_j(y(t_1^-))}{j!} (t-t_1)^j \\ &+ \textstyle\frac{1}{\Gamma(\gamma)} \int_{t_1}^t (t-s)^{\gamma-1} h(s) ds. \end{aligned} \tag{13}$$

Computing $y^{(j)}(t_1^-)$ from (12), we have

$$y^{(j)}(t_1^-) = \sum_{k=j}^{n-1} \frac{y^{(k)}(0)}{(k-j)!} t_1^{k-j} + \frac{1}{\Gamma(\gamma-j)} \int_0^{t_1} (t_1-s)^{\gamma-j-1} h(s) ds. \tag{14}$$

Substituting (14) in (13), we get

$$\begin{aligned} y(t) &= \textstyle\sum_{j=0}^{n-1} \sum_{k=j}^{n-1} \frac{y^{(k)}(0)}{(k-j)!j!} t_1^{k-j} (t-t_1)^j \\ &+ \textstyle\sum_{j=0}^{n-1} \frac{1}{j!} (t-t_1)^j \frac{1}{\Gamma(\gamma-j)} \int_0^{t_1} (t_1-s)^{\gamma-j-1} h(s) ds \\ &+ \textstyle\sum_{j=0}^{n-1} \frac{I_j(y(t_1^-))}{j!} (t-t_1)^j + \frac{1}{\Gamma(\gamma)} \int_{t_1}^t (t-s)^{\gamma-1} h(s) ds. \end{aligned} \tag{15}$$

By changing the order of the sums in the first expression, we have

$$\begin{aligned} &\textstyle\sum_{j=0}^{n-1} \sum_{k=j}^{n-1} \frac{y^{(k)}(0)}{(k-j)!j!} t_1^{k-j} (t-t_1)^j \\ &= \textstyle\sum_{k=0}^{n-1} \sum_{j=0}^{k} \frac{y^{(k)}(0)}{(k-j)!j!} t_1^{k-j} (t-t_1)^j \\ &= \textstyle\sum_{k=0}^{n-1} \frac{y^{(k)}(0)}{k!} \sum_{j=0}^{k} \frac{k!}{(k-j)!j!} t_1^{k-j} (t-t_1)^j \\ &= \textstyle\sum_{k=0}^{n-1} \frac{y^{(k)}(0)}{k!} t^k = \sum_{j=0}^{n-1} \frac{y^{(j)}(0)}{j!} t^j. \end{aligned} \tag{16}$$

From (15) and (16), we get

$$\begin{aligned} y(t) &= \sum_{j=0}^{n-1} \frac{y^{(j)}(0)}{j!} t^j \\ &+ \sum_{j=0}^{n-1} \frac{1}{j!} \frac{1}{\Gamma(\gamma-j)} (t-t_1)^j \int_0^{t_1} (t_1-s)^{\gamma-j-1} h(s) ds. \\ &+ \sum_{j=0}^{n-1} \frac{I_j(y(t_1^-))}{j!} (t-t_1)^j + \frac{1}{\Gamma(\gamma)} \int_{t_1}^{t} (t-s)^{\gamma-1} h(s) ds \end{aligned} \tag{17}$$

for all $t \in (t_1, t_2]$. Now let $t \in (t_2, t_3]$. Then applying $\mathcal{I}_{t_2}^{\gamma}$ on equation (10) and using Theorem 3.6 on $t \in (t_2, t_3]$, we have

$$y(t) = \sum_{j=0}^{n-1} \frac{y^{(j)}(t_2^+)}{j!} (t-t_2)^j + \frac{1}{\Gamma(\gamma)} \int_{t_2}^{t} (t-s)^{\gamma-1} h(s) ds.$$

Using $y^{(j)}(t_2^+) = y^{(j)}(t_2^-) + \triangle y^{(j)}|_{t=t_2} = y^{(j)}(t_2^-) + I_j(y(t_2^-))$ and substituting this in the above equation, we get

$$\begin{aligned} y(t) &= \sum_{j=0}^{n-1} \frac{y^{(j)}(t_2^-)}{j!} (t-t_2)^j \\ &+ \sum_{j=0}^{n-1} \frac{I_j(y(t_2^-))}{j!} (t-t_2)^j \\ &+ \frac{1}{\Gamma(\gamma)} \int_{t_2}^{t} (t-s)^{\gamma-1} h(s) ds. \end{aligned} \tag{18}$$

Computing $y^{(j)}(t_2^-)$ from (17), we have

$$\begin{aligned} y^{(j)}(t_2^-) &= \sum_{k=j}^{n-1} \frac{y^{(k)}(0)}{(k-j)!} t_2^{k-j} \\ &+ \sum_{k=j}^{n-1} \frac{1}{(k-j)!} \frac{1}{\Gamma(\gamma-k)} (t_2-t_1)^{(k-j)} \int_0^{t_1} (t_1-s)^{\gamma-k-1} h(s) ds. \\ &+ \sum_{k=j}^{n-1} \frac{I_k(y(t_1^-))}{(k-j)!} (t_2-t_1)^{(k-j)} \\ &+ \frac{1}{\Gamma(\gamma-j)} \int_{t_1}^{t_2} (t_2-s)^{\gamma-j-1} h(s) ds. \end{aligned} \tag{19}$$

Substituting (19) in (18), we get

$$\begin{aligned} y(t) &= \sum_{j=0}^{n-1} \sum_{k=j}^{n-1} \frac{y^{(k)}(0)}{(k-j)!j!} t_2^{k-j} (t-t_2)^j \\ &+ \sum_{j=0}^{n-1} \sum_{k=j}^{n-1} \frac{1}{(k-j)!j!} (t-t_2)^j (t_2-t_1)^{(k-j)} \\ &\times \frac{1}{\Gamma(\gamma-k)} \int_0^{t_1} (t_1-s)^{\gamma-k-1} h(s) ds \\ &+ \sum_{j=0}^{n-1} \sum_{k=j}^{n-1} \frac{I_k(y(t_1^-))}{j!(k-j)!} (t-t_2)^j (t_2-t_1)^{(k-j)} \\ &+ \sum_{j=0}^{n-1} \frac{1}{j!} \frac{1}{\Gamma(\gamma-j)} (t-t_2)^j \int_{t_1}^{t_2} (t_2-s)^{\gamma-j-1} h(s) ds \\ &+ \sum_{j=0}^{n-1} \frac{I_j(y(t_2^-))}{j!} (t-t_2)^j + \frac{1}{\Gamma(\gamma)} \int_{t_2}^{t} (t-s)^{\gamma-1} h(s) ds. \end{aligned} \tag{20}$$

Changing the order of sums in the first term of (20), we have

$$\begin{aligned}\sum_{j=0}^{n-1}\sum_{k=j}^{n-1} & \frac{y^{(k)}(0)}{(k-j)!j!}t_2^{k-j}(t-t_2)^j \\ & = \sum_{k=0}^{n-1}\sum_{j=0}^{k}\frac{y^{(k)}(0)}{(k-j)!j!}t_2^{k-j}(t-t_2)^j \\ & = \sum_{k=0}^{n-1}\frac{y^{(k)}(0)}{k!}\sum_{j=0}^{k}\frac{k!}{(k-j)!j!}t_2^{k-j}(t-t_2)^j \\ & = \sum_{k=0}^{n-1}\frac{y^{(k)}(0)}{k!}t^k = \sum_{j=0}^{n-1}\frac{y^{(j)}(0)}{j!}t^j.\end{aligned} \tag{21}$$

By changing the order of sums in the second term of (20), we have

$$\begin{aligned}&\sum_{j=0}^{n-1}\sum_{k=j}^{n-1}\frac{1}{(k-j)!j!}(t-t_2)^j(t_2-t_1)^{(k-j)} \\ &\times\frac{1}{\Gamma(\gamma-k)}\int_0^{t_1}(t_1-s)^{\gamma-k-1}h(s)ds \\ &= \sum_{k=0}^{n-1}\sum_{j=0}^{k}\frac{1}{(k-j)!j!}(t-t_2)^j(t_2-t_1)^{(k-j)} \\ &\times\frac{1}{\Gamma(\gamma-k)}\int_0^{t_1}(t_1-s)^{\gamma-k-1}h(s)ds \\ &= \sum_{k=0}^{n-1}\frac{1}{k!\Gamma(\gamma-k)}\int_0^{t_1}(t_1-s)^{\gamma-k-1}h(s)ds \\ &\times\sum_{j=0}^{k}\frac{k!}{(k-j)!j!}(t_2-t_1)^{k-j}(t-t_2)^j \\ &= \sum_{k=0}^{n-1}\frac{(t-t_1)^k}{k!\Gamma(\gamma-k)}\int_0^{t_1}(t_1-s)^{\gamma-k-1}h(s)ds \\ &= \sum_{j=0}^{n-1}\frac{(t-t_1)^j}{j!\Gamma(\gamma-j)}\int_0^{t_1}(t_1-s)^{\gamma-j-1}h(s)ds.\end{aligned} \tag{22}$$

A similar computation in third term of (20) gives us

$$\begin{aligned}&\sum_{j=0}^{n-1}\sum_{k=j}^{n-1}\frac{I_k(y(t_1^-))}{j!(k-j)!}(t-t_2)^j(t_2-t_1)^{(k-j)} \\ &= \sum_{j=0}^{n-1}I_j(y(t_1^-))(t-t_1)^j.\end{aligned} \tag{23}$$

Substituting (21), (22) and (23) in (20), we get

$$\begin{aligned}y(t) &= \sum_{j=0}^{n-1}\frac{y^{(j)}(0)}{j!}t^j + \sum_{j=0}^{n-1}\frac{(t-t_1)^j}{j!\Gamma(\gamma-j)}\int_0^{t_1}(t_1-s)^{\gamma-j-1}h(s)ds \\ &+ \sum_{j=0}^{n-1}I_j(y(t_1^-))(t-t_1)^j \\ &+ \sum_{j=0}^{n-1}\frac{1}{j!}\frac{1}{\Gamma(\gamma-j)}(t-t_2)^j\int_{t_1}^{t_2}(t_2-s)^{\gamma-j-1}h(s)ds \\ &+ \sum_{j=0}^{n-1}\frac{I_j(y(t_2^-))}{j!}(t-t_2)^j + \frac{1}{\Gamma(\gamma)}\int_{t_2}^{t}(t-s)^{\gamma-1}h(s)ds \\ &= \sum_{j=0}^{n-1}\frac{y^{(j)}(0)}{j!}t^j + \sum_{j=0}^{n-1}\sum_{i=1}^{2}I_j(y(t_i^-))(t-t_i)^j \\ &+ \sum_{j=0}^{n-1}\frac{1}{j!}\frac{1}{\Gamma(\gamma-j)}\sum_{i=1}^{2}(t-t_i)^j\int_{t_{i-1}}^{t_i}(t_i-s)^{\gamma-j-1}h(s)ds \\ &+ \frac{1}{\Gamma(\gamma)}\int_{t_2}^{t}(t-s)^{\gamma-1}h(s)ds\end{aligned} \tag{24}$$

for all $t \in (t_2, t_3]$. Continuing this process, we get (11), as required. The converce of the proof can be obtained easily from a inverese direct computation. $\square$

In what follows we assume the function

$$\mathfrak{F}^{-1}\Big[\mathfrak{F}(x) - \vartheta\big(\mathfrak{F}(x)\big)\Big]$$

is nondecreasing on $[0, \infty)$.

Theorem 3.8. *Let $f : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ be a continuous mapping. Suppose that there exist $\vartheta \in \Theta$ and $\mathfrak{F} \in \Xi$ such that*

$$|f(w, u) - f(w, v)| \leq \frac{1}{M}\,\mathfrak{F}^{-1}\Big[\mathfrak{F}(|u - v|) - \vartheta\Big(\mathfrak{F}(|u - v|)\Big)\Big], \tag{25}$$

for all $w \in [0, T]$ and $u, v \in \mathbb{R}$, where

$$\begin{aligned} M = {} & \sum_{j=0}^{n-1} \frac{1}{j!\,\Gamma(\gamma - j + 1)} \sum_{i=1}^{m} (T - t_i)^j (\Delta t_i)^{\gamma - j} \\ & + \sum_{j=0}^{n-1} \sum_{i=1}^{m} (T - t_i)^j + \frac{1}{\Gamma(\gamma + 1)} (\Delta t_{m+1})^{\gamma}, \end{aligned}$$

with

$$\Delta t_i = t_i - t_{i-1}, \quad i = 1, 2, \ldots, m + 1.$$

Moreover, let

$$I_{ij}\big(y^{(j)}(t_i^-)\big) = I_{i0}\big(y(t_i^-)\big)$$

and

$$|I_{i0}(u) - I_{i0}(v)| \leq \mathfrak{F}^{-1}\Big[\mathfrak{F}(|u - v|) - \vartheta\Big(\mathfrak{F}(|u - v|)\Big)\Big],$$

for all $i = 1, 2, \ldots, m$, $j = 0, 1, 2, \ldots, n - 1$ and $u, v \in \mathbb{R}$. Then, the problem (2) has a unique solution in

$$\Omega = PC^n([0, T], \mathbb{R}).$$

Proof. Define a mapping $\aleph : PC^n([0,T],\mathbb{R}) \to PC^n([0,T],\mathbb{R})$ by

$$\aleph(y)(t) = \begin{cases} \sum_{j=0}^{n-1} \frac{y_j}{j!} t^j + \sum_{j=0}^{n-1} \frac{1}{j!\,\Gamma(\gamma-j)} \sum_{i=1}^{p} (t-t_i)^j \int_{t_{i-1}}^{t_i} (t_i - s)^{\gamma-j-1} f(s,y(s))\,ds \\ \quad + \sum_{j=0}^{n-1} \sum_{i=1}^{p} I_{kj}\Big(y^{(j)}(t_k^-)\Big)(t-t_i)^j + \frac{1}{\Gamma(\gamma)} \int_{t_p}^{t} (t-s)^{\gamma-1} f(s,y(s))\,ds, \\ \qquad t \in (t_p, t_{p+1}], \quad p = 1,2,\ldots,m, \\ \sum_{j=0}^{n-1} \frac{y_j}{j!} t^j + \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} f(s,y(s))\,ds, \\ \qquad t \in [0,t_1]. \end{cases} \tag{26}$$

The existence of a unique solution for (2) is equivalent to the existence of a unique fixed point for the mapping $\aleph$. So, it is sufficient to show that $\aleph$ has a unique fixed point.

Let $y_1, y_2 \in PC^n([0,T],\mathbb{R})$. For any $t \in [0,T]$, if $t \in (t_p, t_{p+1}]$, then from (25) we have

$$\begin{aligned} |\aleph(y_1)(t) - \aleph(y_2)(t)| &\leq \frac{1}{M}\Bigg[\sum_{j=0}^{n-1} \frac{1}{j!\,\Gamma(\gamma-j+1)} \sum_{i=1}^{p} (T-t_i)^j (\Delta t_i)^{\gamma-j} \\ &\quad + \sum_{j=0}^{n-1} \sum_{i=1}^{p} (T-t_i)^j + \frac{1}{\Gamma(\gamma+1)} (\Delta t_{p+1})^{\gamma}\Bigg] \\ &\quad \times \mathfrak{F}^{-1}\Bigg[\mathfrak{F}\big(|y_1(t)-y_2(t)|\big) - \vartheta\Big(\mathfrak{F}\big(|y_1(t)-y_2(t)|\big)\Big)\Bigg] \\ &\leq \frac{1}{M}\Bigg[\sum_{j=0}^{n-1} \frac{1}{j!\,\Gamma(\gamma-j+1)} \sum_{i=1}^{m} (T-t_i)^j (\Delta t_i)^{\gamma-j} \\ &\quad + \sum_{j=0}^{n-1} \sum_{i=1}^{m} (T-t_i)^j + \frac{1}{\Gamma(\gamma+1)} (\Delta t_{m+1})^{\gamma}\Bigg] \\ &\quad \times \mathfrak{F}^{-1}\Bigg[\mathfrak{F}\big(|y_1(t)-y_2(t)|\big) - \vartheta\Big(\mathfrak{F}\big(|y_1(t)-y_2(t)|\big)\Big)\Bigg]. \end{aligned} \tag{27}$$

For $t \in [0, t_1]$, we have

$$|\aleph(y_1)(t) - \aleph(y_2)(t)| \leq \frac{1}{M}\frac{1}{\Gamma(\gamma+1)}(\Delta t_1)^\gamma \mathfrak{F}^{-1}\Big[\mathfrak{F}\big(|y_1(t) - y_2(t)|\big) - \vartheta\Big(\mathfrak{F}\big(|y_1(t) - y_2(t)|\big)\Big)\Big]. \tag{28}$$

From (27) and (28), we get □

$$\begin{aligned} |\aleph(y_1)(t)) - \aleph(y_2)(t)| &\leq \tfrac{1}{M}[\textstyle\sum_{j=0}^{n-1} \frac{1}{j!\Gamma(\gamma-j+1)} \sum_{i=1}^{m}(T-t_i)^j(\Delta t_i)^{\gamma-j} \\ &+ \textstyle\sum_{j=0}^{n-1}\sum_{i=1}^{m}(T-t_i)^j \\ &+ \tfrac{1}{\Gamma(\gamma+1)}(\Delta t_{m+1})^\gamma]\mathfrak{F}^{-1}[\mathfrak{F}(|y_1(t) - y_2(t)|) \\ &- \vartheta(\mathfrak{F}(|y_1(t) - y_2(t)|))] \\ &\leq \mathfrak{F}^{-1}[\mathfrak{F}(\|y_1 - y_2\|) - \vartheta(\mathfrak{F}(\|y_1 - y_2|))] \end{aligned}$$

for all $t \in [0, T]$. Taking sup on $t \in [0, T]$, we get

$$\|\aleph(y_1) - \aleph(y_2)\| \leq \mathfrak{F}^{-1}[\mathfrak{F}(\|y_1 - y_2\|) - \vartheta(\mathfrak{F}(\|y_1 - y_2\|)]$$

and so

$$\mathfrak{F}(\|\aleph(y_1) - \aleph(y_2)\|) \leq \mathfrak{F}(\|y_1 - y_2\|) - \vartheta(\mathfrak{F}(\|y_1 - y_2\|)).$$

Thus

$$\begin{aligned} \mathfrak{F}(\rho(\aleph y_1, \aleph y_2)) &\leq \mathfrak{F}(\rho(y_1, y_2)) - \vartheta(\mathfrak{F}(\rho(y_1, y_2))) \\ &\leq \mathfrak{F}(M_\rho(y_1, y_2)) - \vartheta(\mathfrak{F}(M_\rho(y_1, y_2))). \end{aligned}$$

Taking $\eta(x, y) = 1$ for all $x, y \in PC^n([0, T], \mathbb{R})$, the conditions of Theorem (3.2) are satisfied. Thus, the mapping $\aleph$ has a unique fixed point and so the problem (2) has a unique solution.

Example 3.9. Consider the fractional differential equation

$$\begin{aligned} &{}^{\mathfrak{C}}\mathfrak{D}^{\frac{5}{2}}y(t) = e^t + \tfrac{1}{\frac{167}{15}\sqrt{\frac{2}{\pi}}+21+\frac{8}{15}\sqrt{\pi}}\tfrac{3|y(t)|}{3+|y(t)|}, \quad t \in [0, 5] \setminus \{2\}, \\ &y(0) = 1, y'(0) = 2, y''(0) = 3, \\ &\triangle y^{(j)}|_{t=2} = I_{1j}(y^{(j)}(2^-)) = I_{10}(y(2^-)) = \tfrac{3|y(2^-)|}{3+|y(2^-)|}, j = 1, 2. \end{aligned} \tag{29}$$

Note that,

$$f(t,u) = e^t + \frac{1}{\frac{167}{15}\sqrt{\frac{2}{\pi}} + 21 + \frac{8}{15}\sqrt{\pi}} \frac{3|u|}{3+|u|}.$$

Obviously, f is continuous. Here,

$$\gamma = \frac{5}{2}, n = \lceil \frac{5}{2} \rceil = 3, T = 5, t_0 = 0, t_1 = 2, t_2 = 5, m = 1.$$

We have

$\Delta t_1 = t_1 - t_0 = 2, \Delta t_2 = t_2 - t_1 = 3$. Also

$$\begin{aligned}
M &= \sum_{j=0}^{n-1} \frac{1}{j!\Gamma(\gamma - j + 1)} \sum_{i=1}^{m} (T - t_i)^j (\Delta t_i)^{\gamma - j} \\
&+ \sum_{j=0}^{n-1} \sum_{i=1}^{m} (T - t_i)^j + \frac{1}{\Gamma(\gamma+1)} (\Delta t_{m+1})^{\gamma} \\
&= \sum_{j=0}^{2} \frac{1}{j!\Gamma(\frac{7}{2} - j)} 3^j 2^{\frac{5}{2} - j} \\
&+ \sum_{j=0}^{2} (3)^j + \frac{1}{\Gamma(\frac{7}{2})} 3^{\frac{5}{2}} = \frac{167}{15}\sqrt{\frac{2}{\pi}} + 21 + \frac{8}{15}\sqrt{\pi}.
\end{aligned}$$

For any $w \in [0,T]$ and $u, v \in \mathbb{R}$, we have

$$\begin{aligned}
|f(w,u) - f(w,v)| &= \frac{1}{\frac{167}{15}\sqrt{\frac{2}{\pi}} + 21 + \frac{8}{15}\sqrt{\pi}} \left|\frac{3|u|}{3+|u|} - \frac{3|v|}{3+|v|}\right| \\
&= \frac{1}{M} \left|\frac{9\left(\Big||u| - |v|\Big|\right)}{(3+|u|)(3+|v|)}\right| \leq \frac{1}{M} \frac{3|u-v|}{3+|u-v|} \\
&= \frac{1}{M} \mathfrak{F}^{-1}[\mathfrak{F}(|u-v|) - \vartheta(\mathfrak{F}(|u-v|))],
\end{aligned}$$

where

$$\Im(t) = \begin{cases} -\frac{1}{t} + 1, & t \in (0,\infty), \\ -\infty, & t = 0, \\ 1, & t = \infty \end{cases}$$

and $\vartheta(t) = \frac{1}{3}$. Also

$$
\begin{aligned}
|I_{i0}(u) - I_{i0}(v)| &= |\frac{3|u|}{3+|u|} - \frac{3|v|}{3+|v|}| \\
&= |\frac{9\left(\Big||u|-|v|\Big|\right.}{(3+|u|)(3+|v|)}| \leq \frac{3|u-v|}{3+|u-v|} \\
&= \mathfrak{F}^{-1}[\mathfrak{F}(|u-v|) - \vartheta(\mathfrak{F}(|u-v|))].
\end{aligned}
$$

Thus, the conditions of Theorem 3.8 are satisfied. So, by this theorem the problem (29) has a unique solution.

4 Conclusion

In this paper, we investigate the solvability of some impulsive initial value fractional differential equations of arbitrary order (which does not leads to Banach contraction necessarily) via generalized weak Wardowski contractions. Our results introduce a model for solving impulsive initial value fractional differential equations of arbitrary order. We propose to the reader in the future to establish a fixed point theorem for multi-valued mappings satisfying a generalized weak Wardowski contraction. Then, we propose to introduce a model to the solution of impulsive initial value problems for fractional differential inclusions of arbitrary order where the right hand side functions apprise a generalized weak Wardowski multi-valued contraction. It is also is proposed to the reader to investigate the solvability of impulsive differential equations for other forms of equations in literature using the model of solution which we obtained here for arbitrary order impulsive differential equations.

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